Quantum effects of interacting fields in the early Universe

B. L. Hu
The Institute for Advanced Study, Princeton, New Jersey 08540
and Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

Yuhong Zhang
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
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The effective-action formalism used by Hartle, Hu, Fischetti, and Anderson for treating particle production and back reaction of free fields near the Planck time in the early Universe is extended here to interacting fields. We consider a massive (m) self-interacting $\lambda\phi^4$ scalar field coupled (ξ) to a Friedmann-Robertson-Walker spacetime and study the effect of interaction on particle production and on the dynamics of the background geometry. A background-field splitting is introduced in the one-loop effective action which is calculated by perturbative expansions up to the second order in λ, m^2, and ξ. Ultraviolet divergences from the fluctuation field are removed by introducing counterterms via dimensional regularization. From the regularized effective action equations governing the effective geometry and the background field are derived and their solutions sought. We consider separately the conformal versus the nonconformal (ξ), and the massless versus the massive (m) cases. We also contrast the cases with or without the background field ($\mathcal{X}$) and those with or without interaction (λ). The dynamics of the scalar field and the scale factor and the probability of pair production in each case are calculated. In the discussion we present a qualitative explanation of these results, and draw the connection with related work. The classical limit of the present problem depicts the evolution of a Higgs field in curved space, its results being useful for the description of inflation and reheating processes in the grand-unified-theory epoch. The present problem is also the semiclassical limit of quantum cosmology. The results obtained here can be useful to tackling problems involving dynamical fields in curved space such as critical dynamics in the early Universe.

I. INTRODUCTION

Studies of interacting fields in curved spacetime[1] are indispensable towards understanding certain important physical processes in the early Universe from the Planck time ($t_P \sim 10^{-43}$ sec) to the grand unification time ($t_{\rm GU} \sim 10^{-34}$ sec). These processes include cosmological particle production,[2-4] phase transition,[5-7] and their effects on the dynamics of the early Universe. In the grand-unified-theory (GUT) epoch, how an interacting scalar Higgs field evolves can influence the stability of the vacuum, the extent of inflation, the rate of reheating, etc., and determine the dynamics and outcome of phase transitions.[8] In the Planck epoch, quantum processes such as particle production from the vacuum and from interactions[9,10] due to strong gravitational fields can influence the state and the fate of the Universe, such as possible singularity avoidance, horizon removal, matter creation,[3,4] etc. The methods for analyzing interacting quantum fields in curved space for these problems range from analytic (approximate) and numerical solutions of the wave equation[3,6] to the use of effective potentials.[4,7] However, as we have emphasized before,[11] the effective-potential method used in many studies of inflationary cosmologies is inconsistent with a dynamical background where the metric and the fields can vary with time and/or space. To improve on this situation, quasilocal approximations to the wave equation[12] or the effective Lagrangian[13] (via adiabatic or derivative expansion) have been devised to treat cases where the background field is slowly varying. However, for nonadiabatic processes such as particle production in dynamical spacetimes and fields, the full effective action is needed. Particle production in the reheating period of GUT inflation from the decay of scalar particles is by nature an external-field problem where quantum gravitational effects are usually ignorable. However, particle production at the Planck time involves both vacuum excitation by the (classical) background spacetime, the dynamical fields, and their interactions. In such cases one has to use quantum-field-theoretical methods in curved spacetime.

Cosmological production of interacting particles has been considered by many authors using interaction-picture canonical methods.[1,9] This is a natural extension of the canonical method via Bogolubov transformations in free-field cases originally studied by Parker and others.[2] Back-reaction studies of particle production using these methods necessitate self-consistent solutions of the Einstein equation for the background metric and the wave equation for the matter fields.[3] These have been carried out only for free fields. A more powerful method for considering the back reaction of cosmological particle production is by way of the effective-action formalism. In a series of papers, Hartle, Hu, Fischetti, and Ander-

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son[4] have considered the cosmological implications of quantum effects of free scalar fields in different classical background spacetimes. In these approaches the one-loop effective action is usually derived in orders of some small parameters, e.g., the mass of the field m, the coupling constant ξ of the field to spacetime, or the anisotropy of spacetime β, etc. In this paper we apply the same method to the study of particle production and back reaction for a self-interaction ($\lambda\phi^4$) scalar field in a Robertson-Walker spacetime. We deduce the form of the one-loop effective action by a perturbative expansion up to second order in λ, m^2, and ξ. The ultraviolet divergences in the Feynman diagrams are identified and shown to be identical to those deduced from general considerations (such as via the background-field method).[14] They are removed by introducing counterterms in dimensional regularization. The finite part has a form similar to that of the free-field case, except that the mass m^2 is replaced by an effective mass $m^2_{\rm eff}=m^2-\xi R/6+\lambda\hat{\phi}^2/2$, where R is the scalar curvature and $\hat{\phi}$ is the background field. From this the equations governing the effective geometry and the scalar field are derived and their solutions are analyzed. Note that in contradistinction to the quasilocal effective action of Hu and O'Connor,[13] which is valid for all orders in λ but only low orders in the derivatives of the background field, the present expansion in low orders of the coupling constants is not restricted to slowly varying background fields. As such, it can describe nonadiabatic processes such as particle production. The same problem ($\lambda\phi^4$ theory in Robertson-Walker spacetime) has been discussed recently by Calzetta and Hu[15] using a new "closed-time-path" (or in-in) functional formalism, by which the vacuum expectation value of the stress-energy tensor was calculated. Here, as our interest is mainly in particle production we stay for simplicity with the conventional "in-out" formalism,[4] and discuss the back reaction of particle production in detail. To get the real and casual effective action and effective geometries, our results can be generalized by the procedure outlined in Ref. 15.

This paper is organized as follows. In Sec. II we derive the one-loop effective action with perturbative expansion to second orders of m^2, ξ, and λ. The ultraviolet divergences are removed by dimensional regularization. In Sec. III we derive the equations for the effective geometry and the field and study their analytic solutions in the asymptotic time regions. We discuss the two general cases of massive conformally coupled fields and then the massless, nonconformally coupled fields, without the trace anomaly[11] and then discuss the effect due to trace anomaly. In Sec. IV we try to explain the main results in each case by qualitative considerations and draw some general conclusions.

II. EFFECTIVE ACTION

Consider a massive (m), self-interacting $\lambda\phi^4$ scalar field coupled arbitrarily (ξ) to a spatially flat Robertson-Walker (RW) universe with line element

$$ds^2=-dt^2+a^2(t)\sum_{i=1}^{3}dx_i^2 =a^2(\eta)\left[-d\eta^2+\sum_{i=1}dx_i^2\right], \tag{2.1}$$

where $a(t)$ is the scale factor and the conformal time η is related to the cosmic time t by $\eta=\int dt/a(t)$. The classical (Einstein) action S_{g0} of the background geometry ($\tilde{g}$) is given by

$$S_{g0}[\tilde{g}]=l_P^{-2}\int d^4x(-\tilde{g})^{1/2}R\ , \tag{2.2}$$

where $l_P=(16\pi G)^{1/2}=1.2\times10^{-32}$ cm is the Planck length, and $R=6a''/a$ (a prime denotes $d/d\eta$) is the scalar curvature. We assume that the Universe is filled with classical radiation with energy density ρ_r described by the action

$$S_{\rm rad}=\int d^4x(-\tilde{\rho}_r)\ . \tag{2.3}$$

Here $\tilde{\rho}_r=\rho_r a^4$ is a dimensionless number which measures the number of radiation quanta and determines the "size" of the Universe" (maximal radius in closed universes). The action for the scalar field (written in n dimensions) is given by

$$S_f[\tilde{\phi},\tilde{g}]=\int d^nx(-\tilde{g})^{-1/2}\left[\tfrac{1}{2}\tilde{\phi}(x)\Box_x\tilde{\phi}(x) - \tfrac{1}{2}[m^2+(1-\xi)\xi_nR]\tilde{\phi}^2(x) - \frac{1}{4!}\lambda\tilde{\phi}^4(x)\right]. \tag{2.4}$$

Here $\Box_x$ is the Laplace-Beltrami operator in curved space, $\xi_n=(n-2)/4(n-1)=\frac{1}{6}$ for $n=4$ and the constant $\xi=0$ and 1 correspond to conformal and minimal coupling, respectively. The scalar field $\tilde{\phi}(x)$ satisfies the equation of motion

$$\frac{\delta}{\delta\tilde{\phi}(x)}S[\tilde{\phi},\tilde{g}_{\mu\nu}]=0$$

or

$$[-\Box_x+m^2+(1-\xi)\xi_nR]\tilde{\phi}(x)+\frac{\lambda}{6}\tilde{\phi}^3(x)=0\ . \tag{2.5}$$

The vacuum persistence amplitude $\langle 0_-|0_+\rangle$ is given by the generating functional Z obtained by functionally integrating the actions over the scalar field $\tilde{\phi}$ in a background metric $\tilde{g}$:

$$\langle 0_-|0_+\rangle=Z[J,\tilde{g}] =N\int\delta\tilde{\phi}\exp\left[i\left[S[\tilde{\phi},\tilde{g}]+\int d^nx\sqrt{-g(x)}\times J(x)\tilde{\phi}(x)\right]\right] =e^{iW[J,\tilde{g}]}\ , \tag{2.6}$$

where we assume that the field $\tilde{\phi}(x)$ is coupled linearly to an external source $J(x)$. To incorporate the quantum contributions, expand S around a saddle point $\hat{\phi}(x)$ (up to $\delta^2S/\delta\hat{\phi}\delta\hat{\phi}$ terms for the one-loop approximation),

$$S[\tilde{\phi},\tilde{g}]=S[\hat{\phi}(x),\tilde{g}]+\int d^nx\sqrt{-g}\,\hat{\phi}(x)+\int d^nx\left[\frac{\delta S[\hat{\phi},g]}{\delta\hat{\phi}(x)}+\sqrt{-g}J(x)\right][\tilde{\phi}(x)-\hat{\phi}(x)]$$
$$+\tfrac{1}{2}\int d^nx\,d^nx'[\tilde{\phi}(x)-\hat{\phi}(x)]\frac{\delta^2S[\hat{\phi},g]}{\delta\hat{\phi}(x)\delta\hat{\phi}(x')}[\tilde{\phi}(x')-\hat{\phi}(x')]+\cdots\ . \tag{2.7}$$

The saddle-point field $\hat{\phi}(x)$ satisfies

$$\frac{\delta}{\delta\hat{\phi}}S[\hat{\phi},\tilde{g}]+\sqrt{-g}J=0\ , \tag{2.8}$$

which becomes the classical field in the source-free limit. The classical action for the scalar field S_{f0} is given by (2.4) with $\tilde{\phi}$ replaced by $\hat{\phi}$, the classical field. After performing the Gaussian functional integration over the fluctuation fields $\phi=\tilde{\phi}-\hat{\phi}$ we obtain

$$W[J,\hat{\phi},\tilde{g}]=-i\ln Z[J,\hat{\phi},\tilde{g}]$$
$$=S[\hat{\phi},\tilde{g}]+\int d^nx\sqrt{-g}J(x)\hat{\phi}(x)$$
$$-\frac{i}{2}\ln\det G\ , \tag{2.9}$$

where the one-loop Green's function G is

$$G(x,x')=\frac{\delta^2S[\hat{\phi},g]}{i\delta\hat{\phi}(x)\delta\hat{\phi}(x')}\ . \tag{2.10}$$

The effective field $\bar{\phi}$ is defined as the vacuum expectation value of $\tilde{\phi}$:

$$\bar{\phi}=\langle 0_+|\tilde{\phi}|0_-\rangle$$
$$=Z^{-1}[J,\hat{\phi},\tilde{g}]\int D\tilde{\varphi}\,\tilde{\varphi}\exp\left[i\left[S[\tilde{\varphi},\tilde{g}]+\int d^nx\sqrt{-g}J\tilde{\phi}\right]\right]$$
$$=\frac{\delta}{\delta J}W[J,\hat{\phi},\tilde{g}]\ . \tag{2.11}$$

It is given by

$$\bar{\phi}=\hat{\phi}+\phi_q\ ,$$

where ϕ_q is the average of all one-loop fluctuation fields

$$\phi_q=\frac{i}{2}\frac{\delta}{\delta J}\ln\det G\ .$$

Now perform a functional Legendre transformation and substitute $\hat{\phi}$ with $\bar{\phi}$ which are indistinguishable up to one loop:

$$\Gamma[\bar{\phi},\tilde{g}]=W[J,\hat{\phi},\tilde{g}]-\int d^nx\sqrt{-g}\ J\bar{\phi}(x)\ . \tag{2.12}$$

This defines the effective action $\Gamma[\bar{\phi}]$ which is equal to W at vanishing source. Equivalently

$$\langle 0_+|0_-\rangle=\exp(i\Gamma[\bar{g},\bar{\phi}])\ . \tag{2.13}$$

From its definition, Γ yields only the one-particle irreducible (1PI) Green's function.

In the above we have only considered quantum contributions of the scalar field in a given geometry $\tilde{g}$. Quantum (graviton) contributions to the gravitational field can be incorporated in a similar manner. [See Sec. I of paper I (Ref. 4).] One does this by introducing in the action (1.3) a source term $T^{\alpha\beta}$ to which the gravitons $h_{\alpha\beta}$ are coupled, i.e., of the form $\int d^nx\,h_{\alpha\beta}T^{\alpha\beta}$, and integrating over the graviton field configurations to get the generating functional. In the semiclassical approximation used here, only quantum fluctuations of the scalar field but not the gravitational field are included. The effective action which incorporates such contributions yields an effective geometry $\bar{g}$ and an effective field $\bar{\phi}$ (which is in general complex) satisfying the equations of motion

$$\frac{\delta\Gamma[\bar{\phi},\bar{g}]}{\delta\bar{g}_{\alpha\beta}}=0\quad\text{and}\quad\frac{\delta\Gamma[\bar{\phi},\bar{g}]}{\delta\bar{\phi}}=0\ . \tag{2.14}$$

They are the Einstein equation and the Klein-Gordon (KG) equation modified by quantum fluctuations of the scalar field. In our previous work we have called $\bar{g}$ and $\bar{\phi}$ the classical geometry and fields. Here we find it more appropriate to call them *effective* but save the description *classical* for the real geometry $\hat{g}$ and fields $\hat{\phi}$ satisfying the Einstein equation derived from (2.2) and the KG equation in (2.5). Since the difference of $\Gamma^{(1)}[\bar{\phi}]$ and $\Gamma^{(1)}[\hat{\phi}]$ is in higher-loop terms, we may sometimes use $\hat{\phi}$ and $\bar{\phi}$ interchangeably as functional variables.

From (2.9), in the vanishing-source limit, the one-loop effective action Γ is given by

$$\Gamma^{(1)}[\hat{\phi},\tilde{g}]=-\frac{i}{2}\operatorname{Tr}\ln G(x,x')\ . \tag{2.15}$$

The Green's function in the one-loop approximation $G(x,x')=\delta^2S/i\delta\hat{\phi}(x)\delta\hat{\phi}(x')$ satisfies the equation (in n dimensions)

$$\sqrt{-g(x)}[\Box_x+m^2+(1-\xi)\xi_nR+\tfrac{1}{2}\lambda\hat{\phi}^2]G(x,x')$$
$$=-\delta^n(x-x')\ . \tag{2.16}$$

For free, massless, conformally coupled fields ($\lambda=m=\xi=0$) in a RW spacetime, the Green's function G_0 is conformally related to the usual flat-space Feynman Green's function, i.e.,

$$G_0(x,x')=[a(\eta)]^{1-n/2}G_F(x,x')[a(\eta')]^{1-n/2}\ , \tag{2.17}$$

where

$$G_F(x,x')=\frac{-1}{(2\pi)^n}\int d^nk\frac{e^{ik\cdot(x-x')}}{k^2-i\varepsilon}\ . \tag{2.18}$$

For small m^2, ξ, and λ, one can solve (2.16) perturbatively. Defining

$$V(\hat{\phi},g)=m^2-\xi\xi_nR(x)+\tfrac{1}{2}\lambda\hat{\phi}^2(x)\equiv m^2(x)+\tfrac{1}{2}\lambda\hat{\phi}^2 \tag{2.19}$$

as the perturbation potential, G satisfies an integral equation

$$G(x,x')=G_0(x,x')+\int d^ny\, G_0(x,y)V(y)G(y,x')\ , \tag{2.20}$$

or, in operator form,

$$G=G_0+G_0VG=G_0+G_0VG_0+G_0VG_0VG_0+\cdots$$
$$=G_0(1-VG_0)^{-1}\ . \tag{2.21}$$

Substituting this into the one-loop effective action yields

$$\Gamma[\hat{\phi},\tilde{g}]=-\frac{i}{2}\mathrm{tr}\ln G_0-\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int d^nx_1\cdots\int d^nx_n V(x_1)G_0(x_1,x_2)\cdots V(x_n)G_0(x_n,x_1)+O(\hbar^2)\ . \tag{2.22}$$

If we adopt the Feynman rules for the propagator and the vertex functions as depicted in Fig. 1, i.e.,

$$\begin{aligned}&\text{(a)}\quad iG_0(x,y)\ ,\\ &\text{(b)}\quad -\frac{i}{2}\sqrt{-g(x)}m^2(x)\delta^n(x-y)\ ,\\ &\text{(c)}\quad -\frac{i\lambda}{4!}\sqrt{-g(x)}\delta^n(x_1-x_2)\delta^n(x_2-x_3)\delta^n(x_3-x_4)\ ,\end{aligned} \tag{2.23}$$

then up to one loop we need to sum over all diagrams in Figs. 2 and 3 for contributions to the first and second order of λ, respectively. We observe by power counting that

$$G_F(x_1-x_2)G_F(x_2-x_3)\cdots G_F(x_m-x_1)\sim\int d^nk\, k^{-2m}\ , \tag{2.24}$$

which contains ultraviolet divergences only for $m=1$ and 2 (in one-loop graphs) at $n=4$. We see that the lowest-order bubble diagram [Fig. 2(c)] does not contribute because it is proportional to $G_F(0)\sim\int d^nk\,(k^2-i\varepsilon)^{-1}$, which gives zero regularized value in dimensional regularization. This leaves only Fig. 2(d) to $O(\lambda)$ and 3(b)–3(d) to $O(\lambda^2)$, which need be regularized. They all contain products of propagators in the form $G_F(x-x')G_F(x'-x)$ which in the momentum-space representation is given by

$$(2\pi)^n\delta^n(p_1+p_2)\int\frac{d^nq}{(2\pi)^n}\frac{1}{q^2-i\varepsilon}\frac{1}{(q+p)^2-i\varepsilon}\ , \tag{2.25}$$

which can be evaluated by rotating both p^0 and q^0 by $\pi/2$ in the complex plane and rotating p^0 back $-\pi/2$ after integration. [See paper II (Ref. 4).] This gives

$$\begin{aligned}G_F(x-x')G_F(x'-x)=&\frac{i}{16\pi^2(n-4)}\delta^n(x-x')-\frac{i}{16\pi^2}[2+\psi(1)+\ln 4\pi-i\pi]\delta^n(x-x')\\ &+\frac{i}{16\pi^2}\int\frac{d^np}{(2\pi)^n}e^{ip(x-x')}\ln p^2+O(n-4)\ .\end{aligned} \tag{2.26}$$

The first term is the singular part. From this we can easily identify the singular part of Fig. 2(d) to be

$$-i\sqrt{-g(x)}\left[\frac{-1}{32\pi^2(n-4)}\right](m^2-\tfrac{1}{6}\xi R)\delta^2(x_1-x_2)\ , \tag{2.27}$$

which requires a term in the counter Lagrangian

$$L_1^{\rm CT}=L_{m,\xi}^{\rm CT}=-\tfrac{1}{2}(\delta m^2-\tfrac{1}{6}R\,\delta\xi)\phi^2,\ \text{ where }\ \delta m^2=\frac{\lambda m^2}{32\pi^2}\ \text{ and }\ \delta\xi=\frac{\lambda\xi}{32\pi^2}\ , \tag{2.28}$$

while the singular part from each of the three diagrams 3(b)–3(d) is

$$i\lambda\mu^{2(n-4)}\sqrt{-g(x)}\left[\frac{\lambda}{32\pi^2(n-4)}\right]\delta^2(x_1-x_2)\delta^n(x_2-x_3)\delta^n(x_3-x_4)\ , \tag{2.29}$$

which requires a combined counterterm (hence a factor of 3)

$$L_2^{\rm CT}=L_\lambda^{\rm CT}=-\frac{1}{4!}(\delta\lambda)\phi^4,\ \text{ where }\ \delta\lambda=\frac{3\lambda}{32\pi^2}\ . \tag{2.30}$$

Let us now use these results to compute the one-loop effective action up to second order (λ^2, m^4, and ξ^2):

$$\Gamma[\tilde{\varphi},a]=S[\tilde{\varphi},a]-\frac{i}{4}\int d^nx\,d^nx'a^2(x)[m^2(x)+\tfrac{1}{2}\lambda\mu^{2\epsilon}\hat{\phi}^2(x)]G_F(x-x')G_F(x'-x)a^2(x')[m^2(x')+\tfrac{1}{2}\lambda\mu^{2\epsilon}\hat{\phi}^2(x')]\ , \tag{2.31}$$

where

$$\epsilon = n - 4 \ .$$

We see that the term $a^4 m^2(x) G_F(x-x')G(x'-x)$ in (2.31) contains a divergence

$$\sqrt{-g}\left[\frac{1}{64\pi^2(n-4)}\right](m^2 - \tfrac{1}{6}\xi R)^2 \ , \tag{2.32}$$

which requires a counter Lagrangian $L_3^{\text{CT}} = L_{\Lambda,\kappa,\epsilon_1}^{\text{CT}} = \delta\Lambda + \delta\kappa R + \frac{1}{2}\delta\epsilon_1 R^2$, where

$$\delta\Lambda = m^4/64\pi^2, \quad \delta\kappa = -\frac{m^2}{32\pi^2}\left[\frac{\xi}{6}\right], \quad \text{and } \delta\epsilon_1 = \frac{\xi^2}{36\times 32\pi^2} \ .$$

This comes from the free-field two-point vacuum bubble diagram and the counterterms correspond to the renormalization of the cosmological constant Λ, Newton's constant κ, and the coefficient of the quadratic curvature ϵ_1. (See, e.g., Ref. 13.) By adding the three counteraction terms from (2.28)–(2.30) $S_i^{\text{CT}} = \int d^4x \, L_i^{\text{CT}}$ and replacing the bare coupling constants and geometric parameters with the renormalized ones, we get, upon taking the limit $n \to 4$, the effective action for the scalar field up to one-loop order

$$\Gamma_{\text{ren}}[\tilde{\varphi}, a] = \Gamma[\hat{\varphi}, a] + S_1^{\text{CT}} + S_2^{\text{CT}} + S_3^{\text{CT}} = S_{f0}[\tilde{\varphi}, a] + \Gamma_{\text{ren}}^{(1)}[\bar{\phi}, a] \ ,$$

where

$$\Gamma_{\text{ren}}^{(1)} = \int d^4x \, a^4(\eta)(m^2 - \tfrac{1}{6}\xi R + \tfrac{1}{2}\lambda\hat{\phi}^2)^2 \frac{1}{32\pi^2}\left[\ln\mu a + \frac{i\pi}{2}\right] - \int d^4x \int d^4x' a^2(\eta) a^2(\eta')(m^2 - \tfrac{1}{6}\xi R + \tfrac{1}{2}\lambda\hat{\phi}^2)$$
$$\times K(x-x')(m^2 - \tfrac{1}{6}\xi R + \tfrac{1}{2}\lambda\hat{\phi}^2) \ , \tag{2.33}$$

and

$$K(x,x') = \frac{1}{64\pi^2}\int \frac{d^4p}{(2\pi)^4} \ln p^2 e^{ip(x-x')} \ . \tag{2.34}$$

[We have redefined the renormalization constant μ to absorb the coefficient $2 + \psi(1) + \ln 4\pi$.]

This regularized effective action is the starting point for our analysis of the equation of motion, particle production and back reaction. Before ending this section, we would like to make two remarks.

(1) The divergences we identified (and counteractions introduced) from the Feynman diagrams are exactly the same as those obtained from more general considerations for arbitrary curved spacetimes, such as those from the work of Toms,[4] Bunch and Parker,[14] Hu and O'Connor,[13] and others. As shown in paper II (Ref. 4), one can deduce the Gauss-Bonnet and quadratic curvature terms from the higher-order Feynman diagrams. This leads to the trace anomaly. Since the form and the origin of the trace anomaly are well known, we shall forfeit its derivation but simply add the term (cf. paper I) to (2.33) in our consideration

$$\Gamma_{\text{TA}}^{(1)}[a] = V\int_0^\infty d\eta \left[-3\alpha\left[\frac{a''}{a}\right]^2 + \beta\left[\frac{a'}{a}\right]^4\right] \ . \tag{2.35}$$

Here, $V = L^3$ is the coordinate volume, $\alpha = \beta = 1/(2880\pi^2)$ for the scalar field.

(2) One can almost guess the form of the effective Lagrangian from the results of the free-field case (paper V) plus some experience with the background-field effective-action formalism. Note that what enters in place of the νR factor there ($\nu = -\xi\xi_n$) is the effective mass $m_{\text{eff}}^2 = m^2 - \frac{1}{6}\xi R + (\lambda/2)\hat{\phi}^2$. [The $\frac{1}{2}\lambda\hat{\phi}^2$ term, of course, comes from the one-loop approximation (1.6) of the $(\lambda/4!)\phi^4$ potential in the saddle-point expansion.] Except

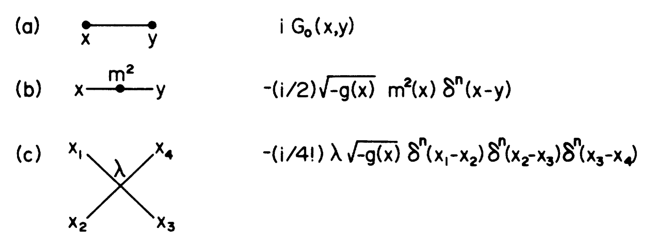

FIG. 1. A diagrammatic representation of Feynman rules in configuration space. (a) A line denotes the propagator of a massless free field. (b) A black dot denotes a two-point vertex $m^2(x) = m^2 a^2(\eta) - \xi R/6$, where $\xi = 0$ denotes conformal coupling and R is the scalar 4-curvature. (c) denotes the four-point vertex of ϕ^4 self-interaction with coupling constant λ.

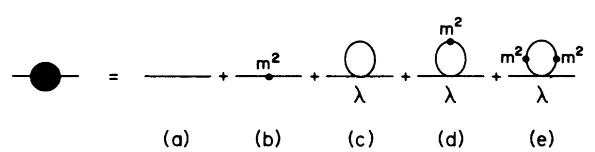

FIG. 2. One-loop expansion of a two-point dressed Green's function up to second order in λ and $m^2(X)$.

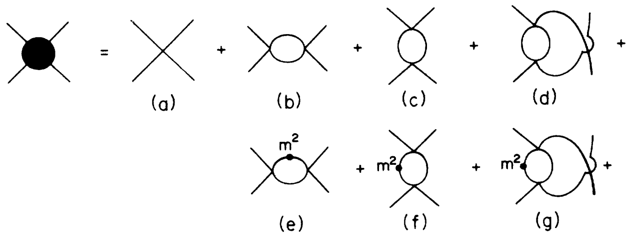

FIG. 3. One-loop expansion of a dressed four-point vertex up to second order in λ and $m^2(\chi)$.

for the additional counterterm for λ which needs to be deduced from the vertex diagrams, the other procedure is similar to the free field case. The derivation of the nonlocal parameter $K(x-x')$ in the finite contribution is exactly identical.

III. BACK REACTION AND PARTICLE PRODUCTION

In this section we consider solutions to the equations of motion (1.12) for the effective geometry and fields using the total effective action (up to one loop)

$$S=S_{\rm rad}+S_{g0}+S_{f0}+\Gamma^{(1)}_{\rm ren}+\Gamma^{(1)}_{\rm TA}\ , \tag{3.1}$$

where the five terms are given by (2.3), (2.2), (2.4), (2.30), and (2.31), respectively. Written explicitly in terms of the Robertson-Walker geometry and using the conformally related field quantities $\chi=a\bar{\phi}$, they become

$$S_{\rm rad}=V\int d\eta(-\tilde{\rho}_r)\ , \tag{3.2}$$

$$S_{g0}=-6l^{-2}V\int d\eta\, a'^2\ , \tag{3.3}$$

$$S_{f0}=\int d^4x\left[\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\chi(x)\partial_\nu\chi(x)-\tfrac{1}{2}\xi\left[\frac{a''}{a}\right]\chi^2-\tfrac{1}{2}m^2a^2\chi^2-\frac{1}{4!}\lambda\chi^4\right]\ , \tag{3.4}$$

$$\Gamma^{(1)}_{\rm ren}=\int d^4x\,(m^2_{\rm eff}a^2)(x)\frac{1}{32\pi^2}\left[\ln\mu a+\frac{i\pi}{2}\right]-\int d^4x\,d^4x'(m^2_{\rm eff}a^2)(x)K(x-x')\times(m^2_{\rm eff}a^2)(x')\ , \tag{3.5}$$

where

$$m^2_{\rm eff}a^2=m^2a^2-\xi a''/a+\lambda\chi^2/2\ , \tag{3.6}$$

and $\Gamma^{(1)}_{\rm TA}$ is given in (2.35).

We shall consider only homogeneous fields $\bar{\phi}(t)$ [or $\chi(\eta)$] and take the local truncation for $K(x-x')$, i.e., $K(x-x')\simeq\delta(x-x')$. The nonlocal effect for free fields has been studied in paper IV (Ref. 4) and found to be qualitatively similar to the local results. Since to the lowest order the effect of interacting fields amounts to changing to the effective mass while preserving the general form (3.5), we expect the local truncation here to produce a close approximation. The full result would have to come from numerical computations.

Under these assumptions, the total action can be written as

$$S[a,\chi]=V\int d\eta\mathcal{L}\ ,$$

where

$$\mathcal{L}=-\tilde{\rho}_r-\tfrac{1}{2}\tilde{a}'^2+\tfrac{1}{2}\chi'^2+\tfrac{1}{2}\tilde{m}^2\tilde{a}^2\chi^2+\tfrac{1}{2}\xi\frac{\tilde{a}''}{a}\chi^2-\frac{1}{4!}\lambda\chi^4+(\tilde{m}^2\tilde{a}^2-\xi\tilde{a}''/\tilde{a}+\lambda\chi^2/2)^2\frac{1}{32\pi^2}\times\left[\ln\tilde{\mu}\tilde{a}+\frac{i\pi}{2}\right]\ . \tag{3.7}$$

The rescaled (tilted) parameters are measured in ratio to the Planck length l_P:

$$\tilde{a}=\frac{\sqrt{12}}{l_P}a,\quad \tilde{m}=\frac{l_P}{\sqrt{12}}m,\quad \tilde{\mu}=\frac{l_P}{\sqrt{12}}\mu\ . \tag{3.8}$$

In what follows we consider back reaction in terms of solutions to the equations for the effective geometry and particle production from the effective action in separate cases. These cases are distinguished in accordance to whether or not the perturbative parameters m (mass), ξ (field coupling), χ (background field), and λ (self-interaction) are zero. From the form of the Lagrangian (3.7) we see that the effect of interaction (λ) always manifests through the background field (χ). Hence the case of zero background field leads to results equivalent to free fields ($\chi=0\Longrightarrow\lambda=0$) but the converse is not true, because the background field is also coupled to m, ξ, etc. For the treatment of massive fields the present perturbative method fails at late times because the relevant parameter ma becomes large at large a. One knows however that the classical behavior of the Universe containing massive fields at late times is given by that of a matter-dominated solution (see, e.g., Starobinsky and Anderson in Refs. 4 and 5), which we will assume for these cases. Taking into account the qualitative differences, we will first discuss massless conformal fields in case A. Being technically the simplest, this case singles out the effects of nonzero background fields χ and interaction λ. We then treat the massive conformal case B, followed by the massless, nonconformal case C. The effect of adding the trace anomaly is treated in Sec. III D. In this section we seek solutions to the dynamical equations in a straightforward manner, but the overall behavior can be understood in qualitative terms by dimensional analysis and analogy with simpler mechanical systems. This we discuss in the next section.

A. Massless, conformal fields ($m=\xi=0$)

The case of zero massless background field ($\chi=0$) has been extensively studied before. For the Robertson-Walker universe which is conformally flat, Parker's theorem[2] rules that no particle of conformally invariant field is produced. For nonzero conformal background fields, the mass term provides a coupling between the geometric scale factor a and the background scalar field χ [see Eq. (3.22)]. In the massless case a and χ will each evolve independently as $a,\chi\sim\eta$. For self-interacting, nonzero background field ($\lambda,\chi\neq 0$) the results are nontrivial. Note that the generalization of Parker's theorem (i.e., no particle production of conformally invariant fields in conformally flat spacetimes) holds for the $\lambda\phi^4$ theory only when the total (unsplit) field [we call $\tilde{\phi}$, which satisfies (2.5)] is considered. The theorem does not apply for nonzero background fields order by order, under a background-field decomposition. From the action

$$S[\tilde{a},\chi]=\int d\eta\left[-\tfrac{1}{2}\tilde{a}'^2-\tilde{\rho}_r+\tfrac{1}{2}\chi'^2-\frac{1}{4!}\lambda\chi^4+\frac{1}{32\pi^2}\frac{1}{4}\lambda^2\chi^4\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]\right] \tag{3.9}$$

we get the dynamic equations

$$\tilde{a}''+\frac{1}{32\pi^2}\frac{1}{4}\lambda^2\chi^4\tilde{a}^{-1}=0\,,$$
$$\chi'+\frac{1}{3!}\lambda\chi^3-\frac{1}{32\pi^2}\lambda^2\chi^3\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]=0\,. \tag{3.10}$$

This has a first integration given by

$$-\tfrac{1}{2}\tilde{a}'^2+\rho_r+\tfrac{1}{2}\chi'^2+\frac{1}{4!}\lambda\chi^4-\frac{1}{32\pi^2}\frac{1}{4}\lambda^2\chi^4\left[\ln\tilde{\mu}\,\tilde{a}+i\frac{\pi}{2}\right]=E\,. \tag{3.11}$$

At early times $\eta\to 0$ the only asymptotic solution for $\tilde{a}\to 0$ is

$$\tilde{a}\sim A\eta+a'\eta^5+\cdots\,,\quad \tilde{\chi}\sim B\eta+B'\eta^5\ln\eta+\cdots\,, \tag{3.12}$$

where A,B are constants satisfying $A^2-B^2=2\tilde{\rho}_r$ and

$$A'=-\frac{\lambda^2}{2560\pi^2}\frac{B^4}{A}\,,\quad B'=-\frac{\lambda}{640}B^3\,.$$

At late times $\eta\to\infty$, several characteristic solutions are allowed.

(1) One class of solutions is

$$\tilde{a}\sim C\eta+\frac{C'}{\eta^3(\ln\eta)^4}+\cdots\,,\quad \chi\sim\frac{D}{\eta\ln\eta}+\cdots\,, \tag{3.13}$$

where

$$C=\sqrt{2\tilde{\rho}_r}\,,\quad C'=-\frac{8\pi^2}{3\lambda^2 C}\,,\quad D=8\pi/\lambda\,.$$

Here $\tilde{a}\to\infty$ while $\chi\to 0$ at $\eta\to\infty$.

(2) A second class of solutions is

$$\tilde{a}\sim F+\frac{F'}{\eta^2}\,,\quad \chi\sim\frac{H}{\eta^2}\,, \tag{3.14}$$

where F is arbitrary and

$$F'=-\frac{1}{128\pi^2}\lambda^2\frac{H^4}{F}\,,\quad H=\frac{8\pi}{\lambda(\ln\mu A+i\pi/2)^{1/2}}\,.$$

In this case, $\tilde{a}\to F$ and $\chi\to 0$ as $\eta\to\infty$. This solution however does not satisfy the first integral unless $\tilde{\rho}_r=0$. It is an "empty" (matter-free) RW solution with scalar field as source.

(3) Another class of solution is incomplete in conformal time. As $\eta\to\eta_0$, $\mathrm{Im}\tilde{a}\to\infty$ and $\mathrm{Re}\tilde{a}\to 0$:

$$\tilde{a}=\pm\frac{i4\pi}{\lambda}\frac{1}{\eta-\eta_0}\frac{1}{\left|\ln\left[\frac{1}{\eta-\eta_0}\right]\right|}\,,$$
$$\chi\to\pm\frac{8\pi}{\lambda}\frac{1}{\eta-\eta_0}\frac{1}{\left|\ln\left[\frac{1}{\eta-\eta_0}\right]\right|^{1/2}}\,. \tag{3.15}$$

This case suggests a closed universe with lifetime $\sim\eta_0$.

Out of these possibilities, the first case is a physically acceptable solution. Note that a $(\ln\eta)^{-1}$ dependence always enters for massless fields. The magnitude of the correction term is proportional to C', which contains the scale $(\lambda^2\tilde{\rho}_r)^{-1/4}$. Here in contradistinction to the massive zero-field case, it is the coupling constant λ which enters in determining the scale where deviation from the Friedmann-Robertson-Walker (FRW) behavior appears.

The particle production probability is given by $P=2\,\mathrm{Im}\Gamma$, from which we obtain the rate of particle production as

$$\mathcal{P}=\begin{cases}(256\pi^2)^{-1}\lambda^2B^4\eta^4 & \text{at early time}\,,\\ (256\pi^2)\lambda^2H^4\dfrac{1}{\eta^4(\ln\eta)^4} & \text{at late time}\,.\end{cases} \tag{3.16}$$

The period of dominant production occurs at the same scale which causes departure from the classical FRW behavior.

B. Massive, conformal fields ($m\neq 0,\xi=0$)

As remarked before, since quantum effects calculated by the perturbative method can become unreliable for $ma\gtrsim 1$, one should use it only to derive the early-time behavior. However, since quantum effects associated with massive particle production in an expanding universe is usually most pronounced at a time T from the Planck time t_P to the Compton time $t_C\sim m^{-1}$ (see, e.g., Parker and Fulling in Ref. 3) much earlier than the time t_r when the Universe assumes the classical radiation-dominated FRW solution $a\sim\eta$. For the interim period ($T<t<t_r$) we can assume that the massive particles produced give rise to a matter energy density ρ_m, which together with the radiation density ρ_r determine the late-time behavior of the Universe and the background field χ. We shall first discuss the early-time behavior and then the late-time behavior for each subcase.

1. Early-time (quantum) behavior

a. Zero background field ($\chi=0$). This yields the same result as a free field ($\lambda=0$). The back-reaction problem for this subcase has also been considered earlier by Anderson (his paper III in Ref. 4) using a different method (canonical quantization and adiabatic regularization).

The effective Lagrangian is [setting $\chi=\xi=0$ in (3.7)]

$$\mathcal{L}=-\tfrac{1}{2}\tilde{a}'^2-\tilde{\rho}_r+(\tilde{m}^2\tilde{a}^2)^2\frac{1}{32\pi^2}\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right] . \quad (3.17)$$

The equation of motion for the effective geometry is

$$\tilde{a}''+\frac{\tilde{m}^4}{8\pi^2}\tilde{a}^3\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}+\frac{1}{4}\right]=0 . \quad (3.18)$$

This has a first integral given by

$$-\tfrac{1}{2}\tilde{a}'^2-\frac{\tilde{m}^4}{32\pi^2}\tilde{a}^4\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]+\tilde{\rho}_r=E . \quad (3.19)$$

Setting $E=0$ as initial condition [cf. paper I (Ref. 4)], we get a general solution

$$\int\frac{d\tilde{a}}{\left[1-\frac{\tilde{m}^4}{32\pi^2\tilde{\rho}_r}\tilde{a}^4\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]\right]^{1/2}}=\pm\int d\eta\sqrt{2\tilde{\rho}_r} . \quad (3.20)$$

The asymptotic solution for early time $\eta\to 0$ is given by

$$\tilde{a}\approx\sqrt{2\tilde{\rho}_r}\eta\left[1-\frac{\tilde{m}^4\tilde{\rho}_r^2}{40\pi^2}\eta^4\times\left[\ln\tilde{\mu}\sqrt{2\tilde{\rho}_r}\eta-\tfrac{1}{5}+\frac{i\pi}{2}\right]+\cdots\right] , \quad (\eta\to 0) \quad (3.21)$$

assuming the initial condition $\tilde{a}=0$ when $\eta=0$. Notice the η^5 correction to the FRW solution occurs at a time scale $(\tilde{m}\rho_r^{1/4})^{-1}$. This is expected, as the only scale in the system is the mass of the field enveloped by the dimensionless number $\tilde{\rho}_r$ which determines the "size" of the Universe, or how large a Universe a given amount of radiation quanta can sustain. At this scale, particle production and its back reaction is expected to be dominant. The probability for the production of a particle pair is given by

$$P=2\,\mathrm{Im}\Gamma=\frac{\tilde{m}^4V}{8\pi^2}\mathrm{Im}\int_0^T d\eta\,\tilde{a}^4\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right] . \quad (3.22)$$

Using the solution (3.21) for $\tilde{a}$, we find the rate of particle production to be

$$\mathcal{P}=\frac{\tilde{m}^4}{16\pi}V(2\rho_r)^2\eta^4 \text{ for early time} . \quad (3.23)$$

From our asymptotic analysis on massive fields based on the use of simple analytic functions, we cannot find any singular solution ($\tilde{a}\to 0$ at $\eta\to 0$) which is asymptotically Friedmann at late times. We do not, however, rule out the possibility that asymptotically Friedmannian solutions with bounce ($\tilde{a}\to\tilde{a}_0$ at $\eta\to 0$) may exist from numerical investigations.

b. Nonzero, free background field ($\chi\neq 0,\lambda=0$). This case is different from that treated by Hartle[4] in paper V in that the background scalar field is nonvanishing. It yields a coupled set of equations

$$\tilde{a}''-\tilde{m}^2\tilde{a}^2\chi^2+\frac{\tilde{m}^4}{8\pi^2}\tilde{a}^3\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}+\frac{1}{4}\right]=0 ,$$
$$\chi''+\tilde{m}^2\tilde{a}^2\chi=0 . \quad (3.24)$$

This system has a first integral

$$-\tfrac{1}{2}\tilde{a}'^2+\tilde{\rho}_r+\tfrac{1}{2}\chi'^2+\tfrac{1}{2}\tilde{m}^2\tilde{a}^2\chi^2-\frac{1}{32\pi^2}\tilde{m}^4\tilde{a}^4\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]=E . \quad (3.25)$$

We find an asymptotic solution at early times $\eta\to 0$

$$\tilde{a}\sim A\eta+A'\eta^5\ln\eta+\cdots,\quad \chi\sim B\eta+B'\eta^5 , \quad (3.26)$$

where $A^2-B^2=2\tilde{\rho}_r$, $A'=-(1/160\pi^2)\tilde{m}^4A^3$, $B'=-\frac{1}{20}\tilde{m}^2A^2B$. At late times, a class of solution behaves differently from the matter- or radiation-dominated classical solution (discussed in case 2 below):

$$\tilde{a}\sim\pm\sqrt{2/\tilde{m}}\frac{1}{\eta},\quad \chi\sim\pm i\frac{1}{2\pi}\frac{1}{\eta}(\ln\eta)^{1/2} . \quad (3.27)$$

This solution does not satisfy the first integral unless $\tilde{\rho}_r=0$. Thus it is a vacuum RW solution sustained by a scalar field. It begins with $\tilde{a}=\chi=0$ but evolves away from the solution obtained by the precise balance of a positive kinetic energy in χ'^2 and a negative kinetic energy in a'^2.

c. Nonzero, interacting background field ($\chi\neq 0,\lambda\neq 0$). The dynamical equations are

$$\tilde{a}''-\tilde{m}^2\tilde{a}\chi^2+\frac{1}{8\pi^2}\tilde{m}^2\tilde{a}(\tilde{m}^2\tilde{a}^2+\tfrac{1}{2}\lambda\chi^2)\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]+\frac{1}{32\pi^2}(\tilde{m}^2\tilde{a}^2+\tfrac{1}{2}\lambda\chi^2)^2\frac{1}{\tilde{a}}=0 , \quad (3.28)$$

$$\chi''+\tilde{m}^2\tilde{a}^2\chi+\frac{\lambda}{6}\chi^3-\frac{1}{16\pi^2}(\tilde{m}^2\tilde{a}^2+\tfrac{1}{2}\lambda\chi^2)\times\lambda\chi\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]=0 . \quad (3.29)$$

They have a first integral given by

$$-\tfrac{1}{2}\tilde{a}'^2+\tilde{\rho}_r+\tfrac{1}{2}\chi'^2+\tfrac{1}{2}\tilde{m}^2\tilde{a}^2\chi^2+\frac{1}{4!}\lambda\chi^4-\frac{1}{32\pi^2}(\tilde{m}^2\tilde{a}^2+\tfrac{1}{2}\lambda\chi^2)^2\left[\ln\tilde{\mu}\,\tilde{a}+\frac{i\pi}{2}\right]=E . \quad (3.30)$$

We find three classes of asymptotic solutions for early time $\eta\to 0$.

(1) One class of solutions is

$$\tilde{a}\sim A\eta,\quad \chi\sim B\eta,\quad \text{where } A^2-B^2=2\tilde{\rho}_r . \quad (3.31)$$

(2) A second class of solutions is

$$\tilde{a}\sim\frac{\lambda}{16\pi^2}C^2\eta\sqrt{-\ln\eta} ,$$
$$\chi\sim C+\frac{1}{64\pi^2}\lambda^2B^3\eta^2\ln\eta,\quad \text{where } \tfrac{1}{4}\lambda C^4+\tilde{\rho}_r=0 . \quad (3.32)$$

(3) A third class of solutions is

$$\tilde{a} \sim \frac{\pm i}{16\sqrt{3}\pi} \lambda D^2 \eta^3 ,$$
$$\chi \sim \pm D\eta, \quad \text{where } \tfrac{1}{2}D^2 + \tilde{\rho}_r = 0 . \tag{3.33}$$

A special case occurs when $\tilde{m}^2 = \frac{1}{3}\lambda$, or $\lambda = m^2 l_P^2/4$, where one can find an exact solution to (3.28) and (3.29):

$$\chi = \pm i (2\tilde{m}^2/\lambda)^{1/2} \tilde{a}^2 ,$$

and

$$\tilde{a}'^2 + \tfrac{1}{3}\tilde{m}^2 \tilde{a}^4 = \tfrac{6}{5}\tilde{\rho}_r . \tag{3.34}$$

$\tilde{a}(\eta)$ is a periodic function of η but the oscillation is anharmonic. The Universe expands and recontracts (in conformal time). By using the first integral one sees that there is no particle production.

In carrying out a perturbation expansion in orders of λ or m^2 one gets, for early times,

$$\tilde{a} = A\eta + \left[\frac{AB^2}{20}\tilde{m}^2 - \frac{1}{160\pi^2}\tilde{m}^2 A(\tilde{m}^2 A^2 + \tfrac{1}{2}\lambda B^2) \left[\ln \tilde{\mu} A\eta + \frac{i\pi}{2} + \frac{1}{20} \right] - \frac{1}{640\pi^2 A}(\tilde{m}^2 A^2 + \tfrac{1}{2}\lambda B^2)^2 \right] \eta^5$$
$$+ \tfrac{1}{1440}[\tilde{m}^4 AB^2(B^2 - 2A^2) - 2\tilde{m}^2 \lambda AB^4]\eta^9 + \cdots , \tag{3.35}$$

$$\chi = B\eta + \left[\frac{-1}{20}(\tilde{m}^2 A^2 B + \tfrac{1}{6}\lambda B^3) + \frac{\lambda B}{320\pi^2}(\tilde{m}^2 A^2 + \tfrac{1}{2}\lambda B^2) \left[\ln \tilde{\mu} B\eta + \frac{i\pi}{2} + \frac{1}{20} \right] \right] \eta^5$$
$$+ \frac{1}{1440} \left[\tilde{m}^4 A^2 B(A^2 - 2AB^2) + \tfrac{2}{3}\tilde{m}^2 \lambda \left[A^2 B^3 + \frac{\lambda^2 B^5}{12} \right] \right] \eta^9 + \cdots , \tag{3.36}$$

where $A^2 - B^2 = 2\tilde{\rho}_r$.

For this solution the probability to produce a particle pair for early time is given by

$$P = 2\,\mathrm{Im}\Gamma = \frac{1}{32\pi} \int d\eta (\tilde{m}^2 A^2 + \tfrac{1}{2}\lambda B^2)^2 \eta^4$$
$$\approx \frac{(\tilde{m}^2 A^2 + \frac{1}{2}\lambda B^2)^2}{160\pi} T^5 , \tag{3.37}$$

where again $A^2 - B^2 = 2\tilde{\rho}_r$.

We now discuss the late-time behavior.

2. *Late-time (classical) behavior*

If there were no massive particle production, the problem would be that of a classical massive scalar field in a radiation-filled universe. To examine the classical effect of nonlinear fields, it is perhaps instructive to begin our discussion with this simple case. The classical equations of motion for $\tilde{a}$ and χ are [from (3.28)]

$$\tilde{a}'' = \tilde{m}^2 a\chi^2, \quad \chi'' = -\tilde{m}^2 \tilde{a}^2 \chi - \frac{\lambda}{6}\chi^3 . \tag{3.38}$$

They depict two coupled oscillators with amplitudes $\tilde{a}$ and χ. The field $\chi(\eta)$ being subjected to a restoring anharmonic force can be approximated by a circular function with decreasing envelope and periodicity, like a damped harmonic wave. The geometry is subjected to a time-dependent force with a negative string constant. This results in the growth of the scale factor $a(\eta)$ faster than the FRW solution ($a \sim \eta$). Thus nonlinear fields can introduce deviations from the Friedmann behavior already on the classical level. Note that quantum correction due to particle production enters at early time as higher-power terms in η [η^9 in the solution (3.36)]. We now consider the effect of particle production in affecting the late-time behavior.

The baryon density from the production of massive particles is

$$\rho_m = 2mP = \frac{1}{80\pi}(\tilde{m}^2 A^2 + \tfrac{1}{2}\lambda B^2)^2 T^5 m , \tag{3.39}$$

where P is given in (3.37) and T is the time when significant amounts of particles are produced. After this time quantum field effects are assumed to be insignificant. To determine the late-time behavior, we assume that the total particles produced at early time give rise to a background baryon density which can be treated as an additional classical matter source. This system is described by the action

$$S[a,\chi] = V \int d\eta \left[-\frac{6}{l_P^2}a'^2 - \tilde{\rho}_r - \rho_m a + \tfrac{1}{2}\chi'^2 - \tfrac{1}{2}m^2 a^2 \chi^2 - \frac{1}{4!}\lambda\chi^4 \right] \tag{3.40}$$

yielding the equations of motion

$$\tilde{a}'' = \tilde{\rho}_m + \tilde{m}^2 \tilde{a}\chi^2, \quad \chi'' = -\tilde{m}^2 \tilde{a}^2 \chi - \frac{1}{3!}\lambda\chi^3 , \tag{3.41}$$

with a first integral

$$-\tfrac{1}{2}\tilde{a}'^2 + \tfrac{1}{2}\chi'^2 + \tilde{\rho}_m \tilde{a} + \tilde{\rho}_r + \tfrac{1}{2}\tilde{m}^2 \tilde{a}^2 \chi^2 + \frac{1}{4!}\lambda\chi^4 = 0 . \tag{3.42}$$

Here $\tilde{a} = (\sqrt{12}/l_P)a$ and $\tilde{\rho}_m = (l_P/\sqrt{12})\rho_m$. The resulted parameters $\tilde{\rho}_r = \rho_r a^4$ and $\tilde{m}^2\lambda$ are as before assumed to be small.

a. Zero background field ($\chi = 0$). Here $\chi = 0$ has the same effect as a massless ($m=0$) noninteracting ($\lambda=0$) field. The late-time behavior is the combination of matter- and radiation-dominated solutions:

$$\tilde{a} \to \tfrac{1}{2}\tilde{\rho}_m \eta^2 + \sqrt{2\tilde{\rho}_r}\,\eta \ . \tag{3.43}$$

The demarcation of the two classical eras depends on the parameter

$$\zeta = \frac{2\sqrt{2\tilde{\rho}_r}}{\tilde{\rho}_m} \ . \tag{3.44}$$

Typically when

(i) $\eta \gg \zeta$, $\tilde{a} \to \frac{1}{2}\tilde{\rho}_m \eta^2$ matter-dominated solution ,

(ii) $\eta \ll \zeta$ (but $\eta > t_P$) , $\tilde{a} \to \sqrt{2\tilde{\rho}_r}\,\eta$ radiation-dominated solution ,

(iii) $\eta \sim \zeta$ matter and radiation coexist .

Using (3.39) for ρ_m ($A^2 = 4\tilde{\rho}_r$) we get

$$\zeta = 11\,520\pi\sqrt{6}\left[\frac{t_C}{t_P}\right]^5 \frac{1}{\tilde{\rho}_r^{3/2} T^5} \ , \tag{3.45}$$

where $t_C = m^{-1}$ is the Compton time and t_P is the Planck time (since $[\rho_r^{-1/4}] \sim [\eta]$ and $[T] \sim [\eta]$, ζ has the dimension of η). One can assume an upper bound for T to be t_C (Compton time). In general if one can calculate the total amount of particle production in the whole history of the Universe, then one does not need an explicit expansion for T, as T is implicitly related to t_C and t_P. To transcribe our notation to that of Hartle's paper V in Ref. 4 note that our $\rho_m = \tilde{\rho}_b$, and $\zeta = (4\sqrt{3}/\tilde{\rho}^{1/4})\xi^{-1}$.

b. Free background field ($\lambda=0$):

$$\tilde{a}'' = \tilde{\rho}_m + \tilde{m}^2 \tilde{a}\chi^2, \quad \chi'' = -\tilde{m}^2 \tilde{a}^2 \chi \ , \tag{3.46}$$

and

$$-\tfrac{1}{2}\tilde{a}'^2 + \tfrac{1}{2}\chi'^2 + \tilde{\rho}_m \tilde{a} + \tilde{\rho}_r + \tfrac{1}{2}\tilde{m}^2 \tilde{a}^2 \chi^2 = 0 \ .$$

If we assume $\tilde{m}^2 \tilde{a}^2 \chi^2 \ll \tilde{\rho}_m \tilde{a}$ or $\tilde{\rho}_r$, then at large η, to leading order

$$\tilde{a} = \tfrac{1}{2}\tilde{\rho}_m \eta^2 + \sqrt{2\tilde{\rho}_r}\,\eta \tag{3.47}$$

and

$$\chi = B\eta^{-n} \sin\!\left(\tfrac{1}{6}\tilde{m}\,\tilde{\rho}_m \eta^3 + \tfrac{1}{2}\tilde{m}\sqrt{2\tilde{\rho}_r}\,\eta^2\right) ,$$

where $n > 1$ in order to have $\tilde{m}^2 \tilde{a}\chi^2 \ll \tilde{\rho}_m$ and B is an arbitrary constant. The power-law dependence of $\chi \sim \eta^{-n}$ is such that it satisfies the convergence condition. (The η^{-n} factor cancels identically on both sides of the differential equation.) To the next order

$$\tilde{a} = \tfrac{1}{2}\tilde{\rho}_m \eta^2 + \sqrt{2\tilde{\rho}_r}\,\eta + \tilde{a}_1 \ , \tag{3.48}$$

we obtain three sets of solutions for $\tilde{a}_1$ at large η:

$$\text{(i)}\ \tilde{a}_1 = \tfrac{1}{2}\tilde{m}^2 B^2 \left[\tfrac{1}{2}\tilde{\rho}_m \frac{1}{(4-2n)(3-2n)}\eta^{4-2n} + \sqrt{2\tilde{\rho}_r}\frac{1}{(3-2n)(2-2n)}\eta^{3-2n}\right] \quad \text{for } n>1 \text{ and } n \neq \tfrac{3}{2},\ n \neq 2 \ . \tag{3.49a}$$

$$\text{(ii)}\ \tilde{a}_1 = \tfrac{1}{2}\tilde{m}^2 B^2 \left[\tfrac{1}{2}\tilde{\rho}_m \eta(\ln\eta - 1) - \sqrt{2\tilde{\rho}_r}\ln\eta\right] \quad \text{for } n = \tfrac{3}{2} \ . \tag{3.49b}$$

$$\text{(iii)}\ \tilde{a}_1 = \tfrac{1}{2}\tilde{m}^2 B^2 \left[-\tfrac{1}{2}\tilde{\rho}_m \ln\eta + \sqrt{2\tilde{\rho}_r}\frac{1}{2\eta}\right] \quad \text{for } n = 2 \ . \tag{3.49c}$$

Up to second order as we have considered here, n is a free parameter. It may however be dependent on parameters in the third- and higher-order solutions. Finally we have the most general case.

c. Nonzero, interacting background field ($\chi \neq 0, \lambda \neq 0$):

$$\tilde{a}'' = \tilde{\rho}_m + \tilde{m}^2 \tilde{a}\chi^2, \quad \chi'' = -\tilde{m}^2 \tilde{a}^2 \chi - \tfrac{1}{3}\lambda\chi^3 \ . \tag{3.50}$$

We see that since $(1/3!)\lambda\chi^3$ is a higher-order correction compared to $\tilde{m}^2 a^2 \chi$, the solutions obtained for *case b* remain valid in this case. If the condition $\tilde{m}^2 \tilde{a}^2 \chi^2 \ll \tilde{\rho}_m \tilde{a}$ or $\tilde{\rho}_r$ is not satisfied, there may not be analytic forms of asymptotic solutions at large η. From the equations of motions we know qualitatively that $\chi \to 0$ and $\tilde{a} \to \infty$ much faster than $\tilde{a} \sim \eta$ at large η. The nonlinear coupling between the two oscillators with amplitudes a and χ (with negative and positive "spring constant") could show chaotic behavior. We do not intend to pursue this case further here.

C. Massless, nonconformal fields

By comparing this with case A one can discern the difference in the types of field coupling with spacetime curvature. Note that nonconformal coupling introduces higher-derivative terms (in $\tilde{a}$) coming from the curvature. Thus the dynamic equations for $\tilde{a}$ are more complicated. The action is given by

$$S = V\int d\eta \left[-\frac{1}{2}\left[1 - \xi\frac{\chi^2}{\tilde{a}^2}\right]\tilde{a}'^2 + \xi\chi\frac{\tilde{a}'}{\tilde{a}}\chi' + \frac{1}{2}\chi'^2 - \frac{1}{4!}\lambda\chi^4 - \tilde{\rho}_r + \frac{1}{32\pi^2}\left[\xi\frac{\tilde{a}''}{\tilde{a}} + \frac{1}{2}\lambda\chi^2\right]^2 \times \left[\ln\tilde{\mu}\,\tilde{a} + \frac{i\pi}{2}\right]\right] . \tag{3.51}$$

This leads to the dynamical equations

$$\left[-\left[1-\xi\frac{\chi^2}{\tilde a^2}\right]\tilde a''-\xi\frac{\chi}{\tilde a}\chi''-\xi\frac{\chi^2}{\tilde a^3}\tilde a'+2\xi\frac{\chi}{\tilde a}\tilde a'\chi'-\frac{\xi}{\tilde a}\chi'^2\right.$$

$$-\frac{1}{16\pi^2}\left[\xi^2\left[\frac{\tilde a'''\tilde a'}{\tilde a^3}+\frac{3}{2}\frac{\tilde a''^2}{\tilde a^3}-5\frac{\tilde a''\tilde a'^2}{\tilde a^4}\right]-\lambda\xi\left[\frac{\chi^2}{\tilde a^2}\tilde a''-\frac{3}{2}\frac{\chi^2}{\tilde a^3}\tilde a'^2+2\frac{\chi}{\tilde a}\tilde a'\chi'\right]+\frac{1}{8}\lambda^2\frac{\chi^4}{\tilde a}\right]$$

$$-\frac{1}{16\pi^2}\left[\ln\tilde\mu\,\tilde a+\frac{i\pi}{2}\right]\left[\xi^2\left[\frac{\tilde a''''}{\tilde a^2}-4\frac{\tilde a'''\tilde a'}{\tilde a^3}-3\frac{\tilde a''}{\tilde a^3}+6\frac{\tilde a''\tilde a'^2}{\tilde a^4}\right]\right.$$

$$\left.\left.-\xi\lambda\left[-\frac{\chi}{\tilde a^2}\tilde a''+\frac{\chi}{\tilde a}\chi''+\frac{\chi^2}{\tilde a^3}\tilde a'^2-2\frac{\chi}{\tilde a^2}\tilde a'\chi'+\frac{\chi'^2}{\tilde a}\right]\right]\right]=0 \quad (3.52a)$$

and

$$\left[\chi''+\xi\frac{\chi}{\tilde a}\tilde a''+\frac{1}{6}\lambda\chi^3\right]-\frac{1}{16\pi^2}\lambda\chi\left[\xi\frac{\tilde a''}{\tilde a}+\frac{1}{2}\lambda\chi^2\right]\left[\ln\tilde\mu\,\tilde a+\frac{i\pi}{2}\right]=0\,. \quad (3.52b)$$

This system has a first integral

$$\left[-\frac{1}{2}\left[1+\xi\frac{\chi^2}{\tilde a^2}\right]\tilde a'^2+\xi\frac{\chi}{\tilde a}\tilde a'\chi'+\frac{1}{2}\chi'^2+\frac{1}{4!}\lambda\chi^4+\tilde\rho_r\right]-\frac{1}{16\pi^2}\xi\frac{\tilde a'^2}{\tilde a^2}\left[\xi\frac{\tilde a''}{\tilde a}+\frac{1}{2}\lambda\chi^2\right]$$

$$-\frac{1}{32\pi^2}\left[\ln\tilde\mu\,\tilde a+\frac{i\pi}{2}\right]\left[4\xi^2\left[\frac{1}{2}\frac{\tilde a'''\tilde a'}{\tilde a^2}-\frac{\tilde a'^2\tilde a''}{\tilde a^3}-\frac{1}{4}\frac{\tilde a''^2}{\tilde a^2}\right]-2\xi\lambda\left[\frac{\chi}{\tilde a}\tilde a'\chi'-\frac{1}{2}\frac{\chi^2}{\tilde a^2}\tilde a'^2\right]+\tfrac{1}{4}\lambda^2\chi^4\right]=E\,. \quad (3.53)$$

It is instructive to separate the classical solution so as to compare the effect of quantum corrections. The classical equations for $\tilde a$ and χ consist of terms in the first set of square brackets in (3.52) and (3.53). The early time $\eta\to 0$ classical solutions have asymptotic solutions

$$\tilde a\sim A\eta+\tfrac{1}{2}\xi\frac{B^2}{A}\eta-\frac{1}{5!}\lambda\xi\frac{B^4}{A}\eta^5,\quad \chi\sim B\eta-\frac{1}{5!}\lambda B^3\eta^5\,, \quad (3.54)$$

where $A^2-B^2=2\rho_r$. At late time $\eta\to\infty$ the classical solutions have asymptotic behavior

$$\tilde a\sim A'\eta+\frac{\xi}{2A'}\frac{1}{\eta}g_0^2(\eta),\quad \chi\sim g_0(\eta)+\frac{g_1(\eta)}{\eta^2}\,, \quad (3.55)$$

where $\frac{1}{2}\dot g_0^2+(1/4!)\lambda g_0^4=\frac{1}{2}A'^2-\tilde\rho_r$. $\tilde a$ has a harmonically modulated Friedmann ($a\sim\eta$) behavior, while χ is an oscillating function. Adding the quantum corrections, we find that the small-η behavior is the same as the classical solutions. This is because the dominant $\xi R/6$ term already appears in the classical effective mass for nonconformal field. We see also that owing to the coupling with spacetime curvature where higher-derivative terms enter, the early-time behavior is significantly different from the conformal case. The late-time $\eta\to\infty$ asymptotic solution is

$$\tilde a\sim C\eta-\frac{\xi}{C\lambda}\frac{1}{\eta^3},\quad \chi\sim\frac{D}{\eta\sqrt{\ln\eta}}+2D\frac{1}{\eta(\ln\eta)^{3/2}}\,, \quad (3.56)$$

where $C=\sqrt{2\tilde\rho_r}$, $D=\pm 8\pi/\lambda$. All solutions are conformally complete. No solutions of other kinds are found. Note that the late-time quantum solutions are different (although $\tilde a$ has the same leading-η behavior). This is because the quantum terms introduce additional forces which act opposite to the classical force terms. Thus they tend to smooth out the runaway behavior of $a(\eta)$ and add an exponential component to an otherwise classical oscillatory solution (for $\chi\sim\eta$). The scale which the quantum effects of nonconformal fields become important is tied in with the scalar curvature R, and thus can be significant even at late times until the scale factor assumes the Friedmann behavior ($a\sim\eta$ or $R=0$). A measure for the magnitude and the duration of significant quantum contribution is given by the particle production rate which has the asymptotic behavior

$$\mathcal{P}\sim\begin{cases}\dfrac{V}{32\pi^2}(\tfrac{1}{2}\lambda B^2)^2\eta^4 & \text{for early time}\,,\\[2ex] \dfrac{V}{32\pi^2}(\tfrac{1}{2}\lambda D^2)^2\dfrac{1}{\eta^4(\ln\eta)^2} & \text{for late time}\,.\end{cases} \quad (3.57)$$

D. Effect of trace anomaly

We know from experience with previous problems[4] that the effect of trace anomaly is only significant near or before the Planck time. The effect of massive interacting fields are important at other scales, usually lower than the Planck energy. The equation of motion for nonconformal fields contains terms coupled with R^2, which is on the same footing as the trace-anomaly terms. To separate the sole effect of the trace anomaly we shall consider only massive, conformal self-coupled fields. The massless conformal case is not much different from the corresponding free-field case, which has been studied in detail in paper I (Ref. 4).

The additional term $\Gamma^{(1)}_{\rm TA}$ in the effective action responsible for the trace anomaly is given by (2.31) explicitly for the Robertson-Walker universe. This introduces an additional term in the dynamic equation for the background spacetime given by

$$T_A=\frac{\delta\Gamma^{(1)}_{\rm TA}}{\delta\tilde a}$$
$$=6\alpha\left[-\frac{\tilde a''''}{\tilde a}+4\frac{\tilde a'''\tilde a'}{\tilde a^3}+3\frac{\tilde a''^2}{\tilde a^3}-6\frac{\tilde a''\tilde a'^2}{\tilde a^4}\right]$$
$$+12\beta\left[-\frac{\tilde a''\tilde a'^2}{\tilde a}+\frac{\tilde a'^4}{\tilde a^5}\right], \tag{3.58}$$

where, for scalar fields,

$$\alpha=\beta=(2880\pi^2)^{-1} .$$

By inspection, it is easy to see that if the background geometry has a FRW behavior $\tilde a(\eta)\sim A\eta$, then

$$T_A=12\beta(A\eta)^{-5}\to 0 \quad \text{at } \eta\to\infty ,$$
$$\to\infty \quad \text{at } \eta\to 0 . \tag{3.59}$$

The high-power-η dependence shows that the trace anomaly is negligible at scales greater than the Planck scale, but is dominant at scales smaller than the Planck scale. Indeed this dominance over the other scales (mass and self-coupling parameters) at the Planck time renders the problem almost identical to that of the free-field cases studied previously. For a conformally coupled, massive $\lambda\phi^4$ field with trace anomaly, we find that the leading behavior of $\tilde a$ and χ at early time is given by

$$\tilde a\sim A_0\eta^{\sqrt{3/2}}+A_1\eta^{2+3\sqrt{3/2}}$$
$$+(\text{terms with } \lambda \text{ and } m^2 \text{ dependence}) ,$$
$$\chi\sim B_0\eta+B_1\eta^{2+2\sqrt{3/2}}\ln\eta \tag{3.60}$$
$$+(\text{terms with } \lambda \text{ and } m^2 \text{ dependence}) .$$

At late time ($t>t_P$), the asymptotic solutions are the same as the corresponding cases without the trace anomaly.

IV. DISCUSSION

We have presented the major analytic results on the back reaction of interacting quantum fields in the last section. We will now try to give a qualitative explanation of these results and put them in perspective with related works. Our discussion contains a dimensional study of the relative importance of the field parameters, the effect of interactions and quantum corrections.

Although the main aim of this work is to distinguish the quantum effects of interacting fields, the generality of this problem also enabled us to cover some of the simpler free-field cases, specifically the massive, conformal fields (cf. paper III of Anderson in Ref. 4) and the massless, nonconformal fields (cf. paper V of Hartle in Ref. 4). By the nature of the problem and the approach we have taken, this work is most closely related to that of paper V in Ref. 4, where the perturbative effective action method was applied to massless, nonconformally coupled free fields.

As we remarked in Sec. II our one-loop effective action for $\lambda\phi^4$ fields [Eq. (2.33)] is identical in form to the free-field case except for the important difference in the replacement of the mass term m^2 by the effective mass $m^2_{\rm eff}=m^2-\xi R/6+\lambda\hat\phi^2/2$. Effect of self-interaction thus enters both through the classical potential term $V(\hat\phi)=\lambda\phi^4/4!$ and the one-loop quantum correction term [the $\frac{1}{2}\lambda\chi^2$ term from $V''(\hat\phi)$] in the conformally related effective mass $m^2_{\rm eff}a^2$. One may find that physics is more easily interpretable if one examines the coupled equations of motion. Each of the individual terms in $M^2_{\rm eff}=m^2-\xi R/6+\lambda\hat\phi^2/2$ carries a scale: the Compton wavelength $l_m\sim m^{-1}$ of the particle, the "radius" of curvature $l_r\sim R^{-1/2}$, which can include intrinsic $(1/a^2)$ and extrinsic curvature $(\sim\ddot a/a)$ terms, and the interaction scale $l_i\sim(\sqrt{\lambda}\hat\phi)^{-1}$, which depends on the interaction strength measured by λ and the extent of interaction measured by the background field $\hat\phi$. Their effects can become important at these respective scales. The interesting feature is that the curvature and interaction scales vary with space and time according to a coupled set of equations of motion. They are determined by, and in turn determine, the geometry (a) and the field ($\hat\phi$). Both effects are significant on the classical level. The difference between the curvature ($\xi R/6$) and the interaction term ($\lambda\hat\phi^2/2$) is that the former depends on higher derivatives of the scale factor $\sim a''/a$ (it vanishes for conformal fields or classical Friedmann solutions) whereas the latter depends on the square of the field itself (acting like a variable-mass term).

In addition to the three field parameters l_m, l_r, and l_i, there are two additional scales which together determine the cosmological importance of any particular process. One is the intrinsic scale, the Planck length $l_P\sim\sqrt{G}$, which measures when quantum gravitational effects become important. Any particular quantum field effect will become important at the specific scale but always in relation to l_P. For example, production of particles with mass m due to strong gravitational fields will be most dominant at the Compton scale l_m, with production amplitude scaled by powers of l_m/l_P, etc. The other scale is a cosmological one. For the radiation-filled RW universe, the dimensionless number $\tilde\rho_r=\rho_r a^4$ which is related to the number of radiation quanta, determines roughly the maximal "size" of the Universe (for a closed Universe, it is the maximum radius $a_{\max}$). Our Universe containing $\sim 10^{80}$ radiation quanta presently has a size of 10^{28} cm, whereas a Universe with unit quanta can expand only to the Planck length $\sim 10^{-33}$ cm. Thus a cosmologically meaningful quantity which characterizes any field-theoretical process is given by that particular microphysical scale weighted with $\tilde\rho_r$ (to some power). For this reason the characteristic scale of cosmologically induced massive particle production is proportional to $l_m\tilde\rho_r$ and that of interaction $\sim l_i\tilde\rho_r$.

The effect of these individual terms m^2, $\xi R/6$, and $\lambda\hat\phi^2/2$ are better understood by studying the subcases in Sec. III where the zero field ($\hat\phi=0$), free field ($\lambda=0$), and massless field ($m=0$) limits are assumed. Hartle treated the free, zero-background field case. Our results on nonconformal coupling to scalar curvature is consistent with his qualitative analysis of the equations for the effective geometry. For massive conformal fields the present perturbative method can only yield quantum corrections at early but not late times. Anderson treated this case via canonical quantization and adiabatic regularization. The non-Friedmannian late-time behavior for solutions evolving from singularity he found may be a special effect due to the trace anomaly rather than particle production. Asymptotically Friedmannian solutions may exist for massive nonconformal fields with quantum-induced bounce or runaway solutions at early times.

The usefulness of comparison of these simple subcases with previous work on free, zero-background field notwithstanding, the problem in its full generality (with nonzero interacting background fields) is a lot more involved. The major difference from the free, zero-background-field cases is that the equations of motion for geometry and fields are nonlinearly coupled. One can get an idea of the overall picture by first looking at the simpler classical solutions, and then examining how quantum effects change their behavior. This was discussed in Secs. III B and III C. Classically the dynamics of a self-interacting scalar field in an FRW universe is similar to the system of two coupled oscillators (with amplitudes $\tilde{a}$ and χ), with a restoring force on χ (positive spring constant) and an accentuating force on $\tilde{a}$ (negative spring constant) [see Eq. (30) or (53), with a saddlelike potential]. This causes the field to oscillate and the scale factor to grow without bound. Therefore an interacting classical field tends to drive the Universe away from the Friedmann ($\tilde{a}\sim\eta$) behavior at late times. For certain ranges of initial conditions the system may even possess chaotic behavior. For massive fields the first departure occurs between the Compton scale l_m and the interaction scale l_i (case B), while for nonconformal fields it should occur between the curvature scale l_r and the interaction scale l_i (case C). The effective forces corresponding to quantum corrections [Eqs. (3.29) or (3.52)] act opposite to the classical forces, and thus tend to "soften" the classical runaway behavior for the scale factor and to "harden" the classical oscillatory behavior of the scalar field. However, for nonconformal fields quantum effects are less pronounced compared to classical effects. This is because the $\xi R/6$ term already dominates in the classical effective mass. One can see this by comparing results in cases B and C. Thus the curvature effect can be significant extending to classical periods, as long as the Universe does not approach an exact radiation-dominated Friedmann behavior (whereby $R=0$). Finally, since the effect of the trace anomaly occurs at times earlier than the Planck time [due to the $(2880\pi^2)^{-1}$ factor], the result is similar to the free-field case, which has been studied in detail before. (For unusually strong self-interactions our result based on the perturbative approach would not apply anyway.) By examining the interplay of the basic field and geometric processes as characterized by their respective physical scales one can gain a qualitative understanding of the combined action of classical and quantum effects of the cosmological and field-theoretical processes in the manner we have presented.

As mentioned in the Introduction, the problem studied here bears not only on quantum gravitational processes involving quantum fields near the Planck time, but its classical and quantum attributes could also be relevant to understanding the evolution of the Higgs field in the GUT epoch useful for a better description of the inflation and reheating processes in the curved-spacetime context. In turn, the present problem is itself the semiclassical limit of corresponding problems in quantum cosmology, that involving solutions of Wheeler-DeWitt equations[16] for a quantum scalar field and quantum geometry, in the spirit of Hartle, Hawking, and Misner.[17] The results obtained here can also be useful for analyzing nonperturbative processes involving dynamical fields in curved space such as particle production in tunneling processes and critical dynamics in the early Universe, problems we hope to address in future communications.

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