

## Colliding electromagnetic shock waves in general relativity

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We derive a new, exact solution for the Einstein-Maxwell equations that describes the collision (interaction) of two arbitrarily polarized electromagnetic shock waves. In the limit that the polarization angle vanishes, our solution reduces to the Bell-Szekeres solution.

### I. INTRODUCTION

Plane waves in general relativity, whether pure gravitational, scalar, electromagnetic (em), neutrino, or any combination of these are known to exhibit nonlinear features, attributed to the gravitational interaction of their general-relativistic energy-momenta. The problem of collision, in particular, between such waves has been considerably important in moving toward a better understanding of the gravitational interaction at a classical (e.g., nonquantum) level. A number of exact solutions available on this subject have been considered; the main guidelines shed further light on the deeper understanding of a number of unresolved questions. The physical results to be drawn from many publications on the topic of colliding waves in general relativity do not extend beyond a handful of significant ones. We have learned, for instance, that pure plane gravitational waves scatter each other to yield a space-time singularity,<sup>1,2</sup> whereas for cylindrical gravitational waves<sup>3,4</sup> the emergence of a singularity is not imperative. By the same token, two linearly polarized plane em waves, in contrast with their gravitational counterparts, interact in such a way that the resulting space-time happens to be nonsingular.

In this paper we present the solution of an open problem related to colliding em waves (cemw's). This problem was formulated first by Bell and Szekeres<sup>5</sup> (BS) who gave an exact solution to satisfy the appropriate boundary conditions. In the solution given by BS the plane em waves were both linearly polarized. The principal task in this paper is to remove this restriction and solve the Einstein-Maxwell (EM) equations, which are more suitable for the more general boundary conditions than those imposed by BS. The second polarization of the em waves in collision serves to bring a nontrivial cross term in the metric. This extension of the BS solution is similar to the Nutku-Halil<sup>6</sup> extension of the Khan-Penrose<sup>7</sup> solution. We have already considered various generalizations of the BS solution from different viewpoints. These include the interaction of superposed em shocks<sup>8</sup> and the interaction between shocks with nonconstant profiles.<sup>9</sup>

In Secs. II and III we reformulate the problem of cemw's and present the new solution. In Sec. IV we study some of its physical properties and in Sec. V we provide a conclusion.

### II. COLLIDING em WAVES

Following BS we assume a space-time metric that is  $C^0$  and piecewise  $C^1$  as the requirements of the shock em

conditions. From the  $+z$  and  $-z$  directions  $\phi_0(v) = F_{\mu\nu} l^\mu m^\nu$  and  $\phi_2(u) = F_{\mu\nu} \bar{m}^\mu n^\nu$ , respectively, are moving toward each other until they make a head on collision at the origin  $u = v = 0$ . The space-time line element describing the cemw's for all the regions is given by<sup>5</sup>

$$ds^2 = 2e^{-M} du dv - e^{-U} (e^V \cosh W dx^2 + e^{-V} \cosh W dy^2 - 2 \sinh W dx dy), \quad (1)$$

where all the metric functions depend on the null coordinates  $u$  and  $v$  alone. The nontrivial Maxwell Equations are

$$\phi_{2,v} = -\frac{1}{2}(V_u \cosh W + iW_u)\phi_0 + \frac{1}{2}(U_v + iV_v \sinh W)\phi_2, \quad (2)$$

$$\phi_{0,u} = -\frac{1}{2}(V_v \cosh W - iW_v)\phi_2 + \frac{1}{2}(U_u - iV_u \sinh W)\phi_0, \quad (3)$$

whereas

$$\phi_1 = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + m^\mu \bar{m}^\nu) = 0$$

throughout the space-time regions. The Einstein-Maxwell field equations can be quoted directly from BS:

$$U_{uv} = U_u U_v, \quad (4)$$

$$2U_{uu} - V_u^2 + 2U_u M_u = W_u^2 + V_u^2 \cosh^2 W + 4k |\phi_2|^2, \quad (5)$$

$$2U_{vv} - V_v^2 + 2U_v M_v = W_v^2 + V_v^2 \cosh^2 W + 4k |\phi_0|^2, \quad (6)$$

$$2M_{uv} + U_u U_v - W_u W_v = V_u V_v \cosh^2 W, \quad (7)$$

$$2W_{uv} - W_u U_v - W_v U_u = 2V_u V_v \sinh W \cosh W - 2ik(\phi_2 \bar{\phi}_0 - \bar{\phi}_2 \phi_0), \quad (8)$$

$$2V_{uv} - V_u U_v - V_v U_u = -2(V_u W_v + V_v W_u) \tanh W + 2k(\phi_2 \bar{\phi}_0 + \bar{\phi}_2 \phi_0) \frac{1}{\cosh W}, \quad (9)$$

where, as in the BS, solution, the constant  $k$  has the value  $k = G/8c^4$ . [I would like to thank Dr. J. B. Griffiths for drawing my attention to a misprint in Eq. (13) of the article by BS.]

The problem of cemw can now be summarized as follows. Given the initial data  $\phi_2(u)$  on  $v=0$ , and  $\phi_0(v)$  on  $u=0$ , determine all the functions  $\phi_0$ ,  $\phi_2$ ,  $U$ ,  $V$ ,  $M$ , and  $W$  in the interaction region ( $u > 0$ ,  $v > 0$ ). Our main objective is to solve the foregoing Eqs. (2)–(9) for the case

$W \neq 0$ , since the special case,  $W=0$ , was already considered by BS.

EM equations are known to be cast into the pair of complex Ernst equations given by<sup>10</sup>

$$(\xi\bar{\xi} + \eta\bar{\eta} - 1)\nabla^2\xi = 2\nabla\xi \cdot (\bar{\xi}\nabla\xi + \bar{\eta}\nabla\eta), \quad (10)$$

$$(\xi\bar{\xi} + \eta\bar{\eta} - 1)\nabla^2\eta = 2\nabla\eta \cdot (\bar{\xi}\nabla\xi + \bar{\eta}\nabla\eta), \quad (11)$$

where  $\xi$  and  $\eta$  represent the gravitational and em complex potentials, respectively. The gradient and Laplacian operators depend in general on the geometry of the base manifold, i.e., whether it is stationary axially symmetrical, cylindrical, or planar. Usually, once a pure gravitational solution ( $\xi$ ) is known, there are well-established methods, initiated first by Ernst<sup>10</sup> to obtain a corresponding EM solution with  $(\xi, \eta)$ . However, in this paper since we are interested in pure em solutions, this accustomed trend does not help our objective, simply because we make the choice  $\xi=0$ , and the metric functions with em field strengths must be constructed from  $\eta$  alone. Under this assumption Eqs. (10) and (11) reduce to the single equation

$$(\eta\bar{\eta} - 1)\nabla^2\eta = 2\bar{\eta}(\nabla\eta)^2, \quad (12)$$

where the operators are to be defined on the geometry

$$ds^2 = 2du dv + e^{-2U}d\phi^2, \quad (13)$$

suitable for the cemw. Here  $\phi$  is a Killing coordinate and  $U$  is fixed by the coordinate condition. The Ernst equation (12) is given under these conditions by<sup>2</sup>

$$2\eta_{uv} - U_u\eta_v - U_v\eta_u = 4\bar{\eta}\eta_u\eta_v(\eta\bar{\eta} - 1)^{-1}. \quad (14)$$

We parametrize  $\eta$  now in accordance with

$$\eta = Ye^{i\delta}, \quad (15)$$

where  $Y$  and  $\delta$  are both real functions of a single function  $X$ , which satisfies the Euler-Darboux equation

$$2X_{uv} - U_uX_v - U_vX_u = 0. \quad (16)$$

After substituting (15) into (14) and imposing (16) we obtain the system of equations

$$\frac{d\delta}{dX} = b_0 \frac{(Y^2 - 1)^2}{Y^2}, \quad (17)$$

$$\frac{d^2Y}{dX^2} + \frac{2Y}{1 - Y^2} \left[ \frac{dY}{dX} \right]^2 = -b_0^2 \frac{(Y^2 + 1)(Y^2 - 1)^3}{Y^3}, \quad (18)$$

in which  $b_0$  is a constant of integration. A particular solution of this pair of equations is given by<sup>11</sup>

$$Y^2 = \frac{\cosh 2X - \cos\theta}{\cosh 2X + \cos\theta}, \quad (19)$$

$$\tan\delta = -(\tan\theta)\coth 2X, \quad (20)$$

where we have used the reparametrization,  $2b_0 = \tan\theta$ . Essentially, this is the solution that we shall adopt in solving the cemw problem with second polarization. For  $\theta=0$  we have  $Y = \tanh X$  and  $\delta=0$ , which yields the BS solution provided the metric function  $e^{-U}$  is chosen as

$$e^{-U} = \cos(au + bv)\cos(au - bv), \quad (21)$$

in which  $a$  and  $b$  are constants, as defined in BS.

As a matter of fact,  $e^{-U}$  corresponds to the coordinate  $\rho$  in the cylindrical and axially symmetrical fields. The only field equation that determines  $U$  is (4) and the choice (21) provides the proper choice for our purpose. For the BS solution we have to make the choice for  $X$ ,

$$\cosh X = \frac{1}{\cos(au - bv)}, \quad (22)$$

which correctly solves the Euler-Darboux equation (16). As we have already stated elsewhere,<sup>9</sup> there is much benefit in employing new, prolate- (oblate-)type coordinates for the problem of cemw. For this purpose we introduce new coordinates by

$$\begin{aligned} \tau &= \sin(au + bv), \\ \sigma &= \sin(au - bv) \quad (a, b = \text{const}), \end{aligned} \quad (23)$$

such that the metric function  $U$  is expressed by

$$e^{-U} = (1 - \tau^2)^{1/2}(1 - \sigma^2)^{1/2}. \quad (24)$$

Let us note that, since we are seeking the solution in the interaction region ( $u > 0, v > 0$ ), we have dropped the Heaviside unit step function in the arguments. In the final solution we will have to make the substitutions  $u \rightarrow u\theta(u)$  and  $v \rightarrow v\theta(v)$ , where the Heaviside unit step function  $\theta(x)$  satisfies (this is not to be confused with the polarization angle  $\theta$ )

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \quad (25)$$

Furthermore, in the new coordinates the wave equation (16) is given by

$$[(1 - \tau^2)X_\tau]_\tau - [(1 - \sigma^2)X_\sigma]_\sigma = 0, \quad (26)$$

and the BS solution takes the form

$$\begin{aligned} ds^2 &= \frac{1}{2ab} \left[ \frac{d\tau^2}{1 - \tau^2} - \frac{d\sigma^2}{1 - \sigma^2} \right] \\ &\quad - (1 - \tau^2)dx^2 - (1 - \sigma^2)dy^2. \end{aligned} \quad (27)$$

### III. THE NEW SOLUTION

The next, and crucial, stage is to consider the case  $\theta \neq 0$  in the Ernst solution (19) and (20), and to determine the remaining metric functions while  $U$  is kept unchanged. Another invariant expression is the form of the solution of the Euler-Darboux equation that we shall consider: namely, (22). The next step, in principle, is to transform all field equations into  $(\tau, \sigma)$  coordinates and integrate them; however, this route is far from being practical and therefore we shall follow a different method. We recall the cylindrically symmetrical geometry that describes cross-polarized cylindrical waves,

$$ds^2 = e^{2(\gamma - \psi)}(dt^2 - d\rho^2) - e^{2\psi}(dz + wd\phi)^2 - \rho^2 e^{-2\psi}d\phi^2, \quad (28)$$

where all metric functions depend on  $\rho$  and  $t$  alone. This metric can locally be identified as the metric we have adopted for cemw. The integrability equation for  $w$  in this line element is given by (i.e., special form of those given by Chandrasekhar in Ref. 3)

$$\begin{aligned} w_t &= 2\rho e^{-4\psi} \text{Im}(\eta\bar{\eta}_\rho), \\ w_\rho &= 2\rho e^{-4\psi} \text{Im}(\eta\bar{\eta}_t). \end{aligned} \quad (29)$$

The corresponding integrability equations for cemw can be obtained by making the identifications

$$\begin{aligned} \eta &= Y e^{i\delta}, \quad e^{2\psi} = 1 - Y^2, \\ \rho &= e^{-U}, \quad w = \tanh W e^{-V}, \\ w e^{2\psi} &= e^{-U} \sinh W. \end{aligned} \quad (30)$$

The results are

$$w_u = (\tan\theta) e^{-U} X_u, \quad w_v = -(\tan\theta) e^{-U} X_v, \quad (31)$$

in which  $X$  is given by (22). These equations are integrated to yield  $w = \tan\theta \sin(au + bv)$ , and the metric functions  $V$  and  $W$  are given by

$$\begin{aligned} e^{-V} \tanh W &= \tan\theta \sin(au + bv), \\ \sinh W &= \frac{\cos(au - bv)}{\cos(au + bv)} \\ &\times \frac{\sin(au + bv) \sin\theta}{\cos^2 \frac{\theta}{2} + \sin^2(au - bv) \sin^2 \frac{\theta}{2}}. \end{aligned} \quad (32)$$

What remains now is to determine  $M$  from quadratures and  $\phi_0$  and  $\phi_2$  from the Maxwell equations. In obtaining  $M$  we have been guided by an interesting principle,<sup>4</sup> as follows. In cylindrical gravitational waves (28), the metric function  $\gamma$  is known to represent the energy content<sup>3</sup> of the waves, which has the same value for both linearly and cross polarized waves. From the local equivalence of the metrics (1) and (28), the metric function  $M$  of cemw is related to  $\gamma$  and  $\psi$  of cylindrical waves by  $M = 2(\psi - \gamma)$ . For the BS equivalent solution we have  $M_0 = 2(\psi_0 - \gamma_0) = 0$ , which means that  $\psi_0 = \gamma_0$ . For the double polarized case  $M = 2(\psi - \gamma)$ , and since  $\gamma = \gamma_0 = \psi_0$  we obtain  $M = 2(\psi - \psi_0)$ . As a result we find

$$e^{-M} = \frac{1 - Y_0^2}{1 - Y^2} = \frac{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}}{\cos\theta}, \quad (33)$$

where  $Y_0$  corresponds to the  $\theta = 0$  (BS) case, while  $Y$  corresponds to the  $\theta \neq 0$  case. Direct substitution of (33) into field equations proves that the metric function  $M$  obtained as above provides the correct value.

Finally,  $\phi_0$  and  $\phi_2$  are calculated from the Maxwell and EM equations. In the null coordinates the calculation is

rather tedious, but in the  $\tau, \sigma$  coordinates it becomes relatively simpler. We summarize our solution:

$$\begin{aligned} e^{-U} &= (1 - \tau^2)^{1/2} (1 - \sigma^2)^{1/2}, \\ e^{-M} &= \frac{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}}{\cos\theta}, \\ \sinh W &= \left[ \frac{1 - \sigma^2}{1 - \tau^2} \right]^{1/2} \frac{\tau \sin\theta}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}}, \\ e^{-V} \tanh W &= \tau \tan\theta, \end{aligned} \quad (34)$$

$$\begin{aligned} \phi_2 &= \frac{a\theta(u)}{\sqrt{k}} \left[ \frac{\cos\theta}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}} \right]^{1/2} e^{i\alpha}, \\ \phi_0 &= \frac{b\theta(v)}{\sqrt{k}} \left[ \frac{\cos\theta}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}} \right]^{1/2} e^{i\beta}, \end{aligned}$$

where the phase functions are determined by the expressions

$$\sin(\alpha - \beta) = \tanh W, \quad \tan \left[ \frac{\alpha + \beta}{4} \right] = \sigma \tan \frac{\theta}{2}, \quad (35)$$

and the coordinates  $\tau, \sigma$  are to be chosen with the step functions, i.e.,

$$\begin{aligned} \tau &= \sin[au\theta(u) + bv\theta(v)], \\ \sigma &= \sin[au\theta(u) - bv\theta(v)]. \end{aligned}$$

#### IV. PROPERTIES OF THE SOLUTION

It is readily observed that for  $\theta = \alpha = \beta = 0$  our solution reduces to the BS solution. In order to eliminate the apparent difficulty for the particular value  $\theta = \pi/2$ , as it occurs in the metric function  $M$ , we can reparametrize the second polarization in accordance with  $\tan\theta \rightarrow \sinh\theta$ , which takes care for all values of  $\theta$ . In order to see the form that the second polarization couples to the field strengths we would like to give the exact initial data for the cemw. In the  $+z$  direction the incoming em field strength is given by

$$\begin{aligned} \phi_2(u) &= \frac{a\theta(u)}{\sqrt{k}} \left[ \frac{\cos\theta}{\cos^2 \frac{\theta}{2} + \sin^2 au \sin^2 \frac{\theta}{2}} \right]^{1/2} e^{i\alpha(u)}, \\ \phi_0 &= 0, \end{aligned} \quad (36)$$

where

$$\alpha(u) = \frac{1}{2} \arcsin \left[ \frac{\sin au \sin\theta}{\left[ \left[ \cos^2 \frac{\theta}{2} + \sin^2 au \sin^2 \frac{\theta}{2} \right]^2 + \sin^2 au \sin^2 \frac{\theta}{2} \right]^{1/2}} \right] + 2 \arctan \left[ \sin au \tan \frac{\theta}{2} \right].$$

The incoming em data from the  $-z$  direction is given similarly by

$$\phi_2 = 0, \quad (37)$$

$$\phi_0 = \frac{b\theta(v)}{\sqrt{k}} \left[ \frac{\cos\theta}{\cos^2\frac{\theta}{2} + \sin^2bv \sin^2\frac{\theta}{2}} \right]^{1/2} e^{i\beta(v)},$$

where

$$\beta(v) = \alpha(u \rightarrow v, \theta \rightarrow -\theta, a \rightarrow b)$$

so that the initial waves are out of phase by  $2\theta$ .

For  $(u < 0, v < 0)$  the space-time line element reduces to

$$ds^2 = \frac{2du dv}{1 - \tan^2\frac{\theta}{2}} - dx^2 - dy^2, \quad (38)$$

which is the flat metric in a scaled coordinate system. The unusual factor of  $1/[1 - \tan^2(\theta/2)]$  does not pose any difficulty since it can be absorbed by a redefinition of the coordinates  $x$  and  $y$ . (This factor can best be handled by adding a constant term of  $[1 - \tan^2(\theta/2)]$  into  $e^{-M}$ , which does not change any feature of the problem at hand.)

Another property of the solution is that in the incoming regions the phase factors cannot be assigned with arbitrary values simultaneously. Starting from the flat metric we apply the coordinate transformation (this is equivalent to a duality rotation on the em fields)

$$\begin{aligned} x &\rightarrow x \cosh \frac{a}{2} + y \sinh \frac{a}{2}, \\ y &\rightarrow x \sinh \frac{a}{2} + y \cosh \frac{a}{2} \quad (a = \text{const}), \end{aligned} \quad (39)$$

which transforms the flat metric into

$$ds^2 = 2du dv - \cosh a (dx^2 + dy^2) + 2 \sinh a dx dy. \quad (40)$$

Such an incoming state, however (i.e., with constant phases in the em fields), does not exist in our general solution.

Also we would like to remark that since we have introduced  $\theta$  as a measure of second polarization, the limit of single polarization (i.e.,  $W=0$ ) should require also that  $\theta=0$ . Otherwise, from the general solution (34) the particular choices  $\alpha=\beta=W=0, \theta \neq 0$ , naturally raises ambiguity and should be discarded.

In order to calculate the scalar curvature components, we make use of the Newman-Penrose formalism in which our choice of null tetrads are given by<sup>2</sup>

$$\begin{aligned} l_\mu &= e^{-M/2} \delta_\mu^0, \quad n_\mu = e^{-M/2} \delta_\mu^1, \\ m_\mu &= \frac{1}{\sqrt{2}} e^{-U/2} \left[ e^{V/2} \left[ i \sinh \frac{W}{2} - \cosh \frac{W}{2} \right] \delta_\mu^2 \right. \\ &\quad \left. + e^{-V/2} \left[ \sinh \frac{W}{2} - i \cosh \frac{W}{2} \right] \delta_\mu^3 \right]. \end{aligned} \quad (41)$$

Following Szekeres<sup>1,5</sup> we delete a common scale factor in the Weyl components and define the scale-invariant components. By virtue of the  $(u, v)$  symmetry the  $\psi_4$  and  $\psi_0$  components differ only by the replacements  $au \rightleftharpoons \pm bv$  and  $\theta \rightarrow -\theta$ ; therefore it suffices to calculate  $\psi_2$  and  $\psi_4$  alone. The results are

$$\psi_2 = 2ab\theta(u)\theta(v) \left[ \sin \frac{\theta}{2} \right] \frac{(1-\sigma^2)\sin\frac{\theta}{2} + \left[ i\sigma \cos\frac{\theta}{2} - \frac{1}{2}\sin\frac{\theta}{2} \right] \left[ 1 + (1-\sigma^2)\sin^2\frac{\theta}{2} \right]}{\left[ \cos^2\frac{\theta}{2} + \sigma^2\sin^2\frac{\theta}{2} \right]^2}, \quad (42)$$

$$\begin{aligned} -\frac{2}{a^2} (\cosh W) (\text{Re}\psi_4) &= \frac{1}{a} \delta(u) \left[ \frac{\tau}{\sqrt{1-\tau^2}} - \frac{\sigma \left[ 1 + (1-\sigma^2)\sin^2\frac{\theta}{2} \right]}{\sqrt{1-\sigma^2} \left[ \cos^2\frac{\theta}{2} + \sigma^2\sin^2\frac{\theta}{2} \right]} - \frac{\tau(1-\sigma^2)\sin^2\theta}{\sqrt{1-\tau^2} \left[ \cos^2\frac{\theta}{2} + \sigma^2\sin^2\frac{\theta}{2} \right]^2} \right] \\ &\quad + 1 + \frac{\sin^4\frac{\theta}{2}(1-\sigma^4) + 6\sigma^2\sin^2\frac{\theta}{2} - 1}{\left[ \cos^2\frac{\theta}{2} + \sigma^2\sin^2\frac{\theta}{2} \right]^2} - \left[ \frac{1-\sigma^2}{1-\tau^2} \right] \frac{\sin^2\theta}{\left[ \cos^2\frac{\theta}{2} + \sigma^2\sin^2\frac{\theta}{2} \right]^2} \\ &\quad + \frac{3\tau\sigma\sin^2\theta}{\left[ \cos^2\frac{\theta}{2} + \sigma^2\sin^2\frac{\theta}{2} \right]^3} \left[ \frac{1-\sigma^2}{1-\tau^2} \right]^{1/2} \left[ 1 + (1-\sigma^2)\sin^2\frac{\theta}{2} \right], \end{aligned} \quad (43)$$

$$\begin{aligned}
-\frac{2}{a^2}(\cosh W)(\text{Im}\psi_4) &= \frac{1}{a} \frac{\delta(u) \sin\theta}{\sqrt{1-\tau^2} \left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]} \left[ \left( \frac{1-\sigma^2}{1-\tau^2} \right)^{1/2} - \frac{\tau\sigma \left[ 1 + (1-\sigma^2) \sin^2 \frac{\theta}{2} \right]}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}} \right] \\
&- \frac{\theta(u)\tau\sigma \sin\theta \left[ 1 + (1-\sigma^2) \sin^2 \frac{\theta}{2} \right]}{\left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]^2 \cosh^2 W} \left( \frac{1-\sigma^2}{1-\tau^2} \right)^{1/2} \\
&\times \left[ \frac{\sigma \left[ 1 + (1-\sigma^2) \sin^2 \frac{\theta}{2} \right]}{(1-\sigma^2) \left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]} - \frac{2}{\tau\sqrt{1-\tau^2}\sqrt{1-\sigma^2}} \right] \\
&+ \frac{\theta(u)\tau \sin^3\theta}{\cosh^2 W (1-\tau^2)^2 \left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]^3} \left( \frac{1-\sigma^2}{1-\tau^2} \right)^{1/2} \\
&\times \left[ \frac{2\sqrt{1-\tau^2}\sqrt{1-\sigma^2} \left[ 1 + (1-\sigma^2) \sin^2 \frac{\theta}{2} \right]}{2\tau^2 - \tau^4 - \sigma^2 + \frac{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}}} \right. \\
&\quad \left. - \frac{\tau^2\sigma^2(1-\tau^2)}{\left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]^2} \left[ 1 + (1-\sigma^2) \sin^2 \frac{\theta}{2} \right]^2 \right] \\
&- \frac{\theta(u)\tau \sin^3\theta(1-\sigma^2)}{\left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]^3} \left( \frac{1-\sigma^2}{1-\tau^2} \right)^{1/2} + \frac{\sin\theta\theta(u)}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}} \\
&\times \left[ \frac{\tau(2-\tau^2)}{\cosh^2 W (1-\tau^2)} \left( \frac{1-\sigma^2}{1-\tau^2} \right)^{1/2} - \frac{2\sigma \left[ 1 + (1-\sigma^2) \sin^2 \frac{\theta}{2} \right]}{\sqrt{1-\tau^2} \left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]} \right. \\
&\quad \left. - \frac{\tau \left[ 1 + (1-\sigma^2) \sin^2 \frac{\theta}{2} \right]}{\sqrt{1-\tau^2}\sqrt{1-\sigma^2} \left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]} + \frac{2\tau\sigma^2 \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}} \left( \frac{1-\sigma^2}{1-\tau^2} \right)^{1/2} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{4\tau\sigma^2 \sin^2 \frac{\theta}{2} \left[ 1 + (1 - \sigma^2) \sin^2 \frac{\theta}{2} \right]}{\left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]^2} \left[ \frac{1 - \sigma^2}{1 - \tau^2} \right]^{1/2} - \frac{\tau \sqrt{1 - \sigma^2}}{(1 - \tau^2)^2} \\
& + \frac{\tau\sigma^2 \left[ 1 + (1 - \sigma^2) \sin^2 \frac{\theta}{2} \right]}{\sqrt{1 - \tau^2} (1 - \sigma^2)} - \frac{3\sigma \left[ 1 + (1 - \sigma^2) \sin^2 \frac{\theta}{2} \right]}{\cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2}} \\
& + \frac{\tau\sigma^2 \left[ 1 + (1 - \sigma^2) \sin^2 \frac{\theta}{2} \right]^2}{\sqrt{1 - \tau^2} \sqrt{1 - \sigma^2} \left[ \cos^2 \frac{\theta}{2} + \sigma^2 \sin^2 \frac{\theta}{2} \right]^2} \Bigg\}. \tag{44}
\end{aligned}$$

## V. CONCLUSIONS

By studying the scalar curvatures  $\psi_2$  and  $\psi_4(\psi_0)$  we observe that the only possible singularities occur at  $\tau=1$  and  $\sigma=1$ , which correspond to the values  $au \pm bv = \pi/2$ . These points arise also in the collision of linearly polarized waves; however, as it was shown in BS, these are not genuine singularities since they can be removed by an appropriate coordinate transformation. Across the incoming-interaction regions, the curvatures  $\psi_4$  and  $\psi_0$  suffer from  $\delta$ -function discontinuities. Furthermore, in the presence of second polarization the em waves cease to interact minimally, i.e., there are other terms beside the terms containing  $\delta$  functions. It was observed that for the linearly polarized em waves the incoming fields retain the same form in the interaction region. We observe now that for a more general solution with cross polarization, this feature does not hold true any more. Rather, the

cross polarization manifests itself in a highly nonlinear form that reminds us of the inherent nonlinearity occurring in the pure gravitational waves.

We would also like to add that it is possible to derive more general solutions for colliding waves when gravitational waves are coupled with em waves. Although this can be done in principle, it is our belief that collision of pure gravitational or pure em waves are more important than the collisions of mixtures of such waves. The latter cases may be interesting in cases that the resultant solution admits both gravitational and em limits independently.

Finally we remark that our method of adding cross polarization described in Sec. II applies in particular to the problem of pure gravitational waves. By choosing our function  $X$  as the metric function  $V$  of Szekeres,<sup>1</sup> it enables us to obtain an infinite family of colliding gravitational waves with cross polarization.

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