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Two-body Dirac equations for meson spectroscopy

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Recently we used Dirac's constraint mechanics and supersymmetries to derive two coupled compatible 16-component Dirac equations that govern two relativistic spinning particles interacting through world scalar and vector potentials. They reduce exactly to four decoupled four-component local Schrödinger-like equations with energy-dependent quasipotentials Φ_w. Their nonperturbative covariant structure [leading to perturbative and $O(1/c^2)$ expansions that agree with field-theoretic approaches] suit these equations ideally for phenomenological applications in which the potentials have some links with relativistic field theories. (These equations are exactly solvable for singlet positronium producing a spectrum correct through order α^4.) Here we use our equations to extend the validity of various one- or two-parameter models for the heavy-quark static potential to the relativistic light-quark regime. These models include the leading-log model (for all length scales) of Adler and Piran and Richardson's potential modified by flavor-dependent vacuum corrections. They significantly improve the good results that we obtained using Richardson's potential alone. Both nonperturbative and perturbative properties of the constraint approach are responsible for the spin-dependent consequences of the potential that result in a good overall fit to the meson masses. The nonperturbative structure dictated by the compatibility of our two Dirac equations enforces an approximate chiral symmetry that may account for the goodness of our pion fit. Perturbatively, for weak potentials, the upper-upper components of our equations reduce to the appropriate Todorov equation and then for low velocities to the Breit Hamiltonian. Thus, our approach reproduces the semirelativistic spin-dependent consequences of a quantum field theory. We strengthen this connection by deriving the Todorov inhomogeneous quasipotential equation for Φ_w from the Bethe-Salpeter equation using an operator generalization of Sazdjian's quantum-mechanical transform of the Bethe-Salpeter equation. Consequently our covariant compatible coupled Dirac equations provide a nonperturbative framework for extrapolating $O(1/c^2)$ field-theoretic results into the highly relativistic regime of bound light quarks.

I. INTRODUCTION

In this paper we apply two-body Dirac equations,[1,2] to systems of spinning quarks in mesons. First we review the equations (derived in a previous paper) beginning with the two compatible 16-component Dirac equations. We introduce covariant scalar and vector interactions (timelike and spacelike) and detail the spin dependence forced by the compatibility of our equations on these different covariant structures. We set out relations (implied by compatibility) of the various covariant interactions to an underlying set of three independent invariant interaction functions. As we showed in an earlier paper, these relations are generated by nonperturbative compatibility arguments in concert with the requirement that our equations display the correct nonrelativistic and $O(1/c^2)$ or semirelativistic limits. We end that section by restating our equations as four decoupled four-component Schrödinger-type forms suitable for applications.

In Sec. II we present the potential models that we tested using our equations. They include the log-log model for all length scales derived by Adler and Piran[3] using effective-action methods as well as cruder phenomenological models such as Richardson's potential[4] (here given both with and without flavor-dependent vacuum corrections[5]). We find that, in our equations, the more physically detailed models lead to significant improvement of the good spectral results provided by the simpler relativistic Richardson model that we treated in Ref. 1. We finish this section by plotting the effective relativistic potentials seen by quarks in selected mesons so that the reader may compare them with each other and especially with the (quite different) corresponding nonrelativistic potentials.

In the remaining sections of the paper we present both nonperturbative and perturbative arguments to support the validity of our equations and to explore structures that contribute to the goodness of the resulting fits to the meson spectrum (beyond the particular choice of an input heavy-quark potential).

An important feature of our equations is that they appear to be able to realize the effects of chiral symmetry. In Sec. IV we present the results of a numerical test that indicates that in the chiral-symmetry limit (zero-mass quarks) our equations reveal zero-mass pions (Goldstone bosons). Building on recent work of Sazdjian,[6] we show

how our equations achieve this effect (at least for scalar interactions[7]).

In Sec. V we show how our equations are related to the semirelativistic (weak-potential, slow-motion) Fermi-Breit approximation to the Bethe-Salpeter equation and to the Breit equation itself. These equations possess the correct semirelativistic limits as a consequence of the structure of classical and quantum relativistic constraint mechanics detailed in a previous paper.[2] We then check that for equal-mass singlet positroniumlike systems the quasipotential reduces to a form that allows an exact solution.[8] These connections to perturbative field theory argue for the validity of the relativistic quantum wave equations derived from the constraint approach and make them attractive for phenomenological application even though their connection to field theory is not as yet completely understood.

Why do we concentrate so heavily on verifying that our equations have the correct structure for electrodynamic calculations when we intend to apply them to quark-model calculations? What we learn from such studies is that the relativistic kinematical, dynamical, and spin structures of our equations allow us to start with a static form of a potential valid in the nonrelativistic limit, and with no additional assumptions derive standard relativistic corrections to the total energy operator. This makes it plausible that our equations might accurately reflect the (hyper)fine structure of QCD with an appropriate heavy-quark static potential as input.

In the final section of this paper we derive the Todorov inhomogeneous quasipotential corresponding to the constraint potential Φ_w of our equations, thereby linking our approach to quantum field theory via the Bethe-Salpeter equation.

II. COVARIANT CONSTRAINT EQUATIONS FOR TWO SPIN-$\frac{1}{2}$ PARTICLES

We begin by setting out the covariant forms of our two-body Dirac equations containing simultaneous electromagneticlike vector, timelike vector, and scalar interactions together with their four decoupled four-component forms (in the c.m. rest frame). (The covariant spin-dependent forms appearing in them are nonperturbative features of the constraint approach enforced by the requirement that these equations be compatible.) For two relativistic spin-$\frac{1}{2}$ particles interacting through a system of world scalar and vector potentials the compatible 16-component (or 4×4 matrix) Dirac equations take the form

$$\mathcal{S}_1\psi=\gamma_{51}(\gamma_1\cdot(p_1-\tilde{A}_1)+m_1+\tilde{S}_1)\psi=0 , \tag{1a}$$

$$\mathcal{S}_2\psi=\gamma_{52}(\gamma_2\cdot(p_2-\tilde{A}_2)+m_2+\tilde{S}_2)\psi=0 . \tag{1b}$$

The particular spin dependences in the potentials are the consequences of supersymmetries of the corresponding (pseudo)classical system. In detail[2] we find, for the vector potentials,

$$\tilde{A}_1=\left[(\epsilon_1-E_1)-i\frac{G}{2}\gamma_2\cdot\left[\frac{\partial E_1}{E_2}+\partial\ln G\right]\gamma_2\cdot\hat{P}\right]\hat{P}+(1-G)p-\frac{i}{2}\partial G\cdot\gamma_2\gamma_2 , \tag{2a}$$

$$\tilde{A}_2=\left[(\epsilon_2-E_2)+i\frac{G}{2}\gamma_1\cdot\left[\frac{\partial E_2}{E_1}+\partial\ln G\right]\gamma_1\cdot\hat{P}\right]\hat{P}-(1-G)p+\frac{i}{2}\partial G\cdot\gamma_1\gamma_1 , \tag{2b}$$

and, for the scalar potentials,

$$\tilde{S}_1=M_1-m_1-\frac{i}{2}G\gamma_2\cdot\frac{\partial M_1}{M_2} , \tag{3a}$$

$$\tilde{S}_2=M_2-m_2+\frac{i}{2}G\gamma_1\cdot\frac{\partial M_2}{M_1} . \tag{3b}$$

In Eqs. (2a), (2b), (3a), and (3b) the variable $P=p_1+p_2$ is the total momentum, $-P^2=w^2$ is the c.m. energy squared, and $\hat{P}\equiv P/w$. The variables ϵ_i are the (conserved) c.m. energies of the constituent particles given by

$$\epsilon_1=(w^2+m_1^2-m_2^2)/2w , \quad \epsilon_2=(w^2+m_2^2-m_1^2)/2w . \tag{4}$$

In terms of these energies the usual relative momentum becomes $p=(\epsilon_2p_1-\epsilon_1p_2)/w$. In order that Eqs. (1a) and (1b) be compatible it is necessary that the functions E_1, E_2, G, M_1, and M_2 depend on the relative separation, $x=x_1-x_2$, only through the coordinate four-vector $x_{\perp\mu}=(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)x_\nu$ transverse to the total four-momentum P. It is also necessary that E_1, E_2, and G be related to each other as functions of two scalar functions, say $\mathcal{A}(x_\perp)$ and $\mathcal{V}(x_\perp)$ corresponding to the fact that the two vector potentials each have separate timelike and electromagneticlike parts (which contain both timelike and spacelike pieces). The relations to $\mathcal{A}$ and $\mathcal{V}$ are not unique, but the simplest choice[2] (that guarantees the correct nonrelativistic and semirelativistic limits) is

$$E_1^2(\mathcal{A},\mathcal{V})=G^2((\epsilon_1-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2) , \tag{5a}$$

$$E_2^2(\mathcal{A},\mathcal{V})=G^2((\epsilon_2-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2) , \tag{5b}$$

$$G^2=\frac{1}{1-2\mathcal{A}/w} . \tag{5c}$$

The mass potentials M_1,M_2 are parametrized not only in terms of an underlying world scalar potential S but also in terms of $\mathcal{A}$, the invariant function responsible for the electromagneticlike vector potential. The simplest choice[1,2] for these potentials consistent with the correct nonrelativistic and semirelativistic limits is

$$M_1^2(\mathcal{A},S)=m_1^2+G^2(2m_wS+S^2) , \tag{6a}$$

$$M_2^2(\mathcal{A},S)=m_2^2+G^2(2m_wS+S^2) . \tag{6b}$$

The kinematical variables $m_w=m_1m_2/w$ and $\epsilon_w=(w^2-m_1^2-m_2^2)/2w$ are the relativistic reduced mass and energy of a fictitious particle of relative motion. The corresponding value of the on-shell relative momentum squared then takes the form

$$b^2(w) \equiv [w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2]/4w^2$$
$$= \epsilon_w^2 - m_w^2$$

in terms of the c.m. energy of the two-body system w.

Equations (1a) and (1b) can be reduced with no approximation to four decoupled (with diagonal γ_i^0) four component Schrödinger-type equations. For convenience we decompose the 16-component ψ into four $\Psi_{\kappa,\lambda}$ by writing

$$\psi = \sqrt{\chi_1 \chi_2}\,\Psi \ , \tag{7}$$

where

$$\psi = \Psi_{++}\begin{bmatrix} I \\ 0 \end{bmatrix}_1 \begin{bmatrix} I \\ 0 \end{bmatrix}_2 + \Psi_{-+}\begin{bmatrix} 0 \\ I \end{bmatrix}_1 \begin{bmatrix} I \\ 0 \end{bmatrix}_2 + \Psi_{+-}\begin{bmatrix} I \\ 0 \end{bmatrix}_1 \begin{bmatrix} 0 \\ I \end{bmatrix}_2 + \Psi_{--}\begin{bmatrix} 0 \\ I \end{bmatrix}_1 \begin{bmatrix} 0 \\ I \end{bmatrix}_2 .$$

One finds[1,2]

$$(\mathbf{p}^2 + \Phi_{SI} + \Phi_{SS} + \Phi_{SO} + \Phi_T + \Phi_{DO})\Psi \equiv (\mathbf{p}^2 + \Phi_w)\Psi = b^2\Psi \ , \tag{8}$$

where

$$\Phi_{SI} = 2m_w S + S^2 + 2\epsilon_w \mathcal{V} - \mathcal{V}^2 + 2\epsilon_w \mathcal{A} - \mathcal{A}^2 \ ,$$

$$\Phi_{SS} = -\nabla^2 \ln(\chi_1\chi_2 G^{1-2\sigma_1\cdot\sigma_2/3})/2 + [\nabla \ln(\chi_1\chi_2 G^{1-2\sigma_1\cdot\sigma_2/3})]^2/4 + (\nabla \ln G)^2(3+\sigma_1\cdot\sigma_2)/18 \ ,$$

$$\Phi_{SO} = -(\partial \ln\chi_1/\partial r\, \mathbf{L}\cdot\sigma_1 + \partial \ln\chi_2/\partial r\, \mathbf{L}\cdot\sigma_2)/r \ ,$$

$$\Phi_T = S_T[-(r\partial^2 \ln G/\partial r^2 - \partial \ln G/\partial r)/r + \nabla \ln G\cdot\nabla \ln(\chi_1\chi_2)]/6 \ ,$$

$$\Phi_{DO} = (\mathcal{M} - \mathcal{E})^2/4 - \{\epsilon_2\sigma_1\cdot\nabla[\sigma_2\cdot(\mathcal{M} - \mathcal{E})]/w + \epsilon_1\sigma_2\cdot\nabla[\sigma_1\cdot(\mathcal{M} - \mathcal{E})]/w - \sigma_1\cdot\sigma_2\nabla \ln G\cdot(\mathcal{M} - \mathcal{E})$$
$$- \sigma_1\cdot\nabla \ln(\chi_1)\sigma_2\cdot(\mathcal{M} - \mathcal{E}) - \sigma_2\cdot\nabla \ln(\chi_2)\sigma_1\cdot(\mathcal{M} - \mathcal{E})\}(-)^s(\Phi_{SI} - b^2)/(2\chi_1\chi_2) \ ,$$

with

$$\chi_i = (E_i\gamma_i^0 + M_i)/G \ ,$$
$$\mathcal{M} = \nabla(M_1^2 + M_2^2)/4M_1M_2 \ ,$$
$$\mathcal{E} = \nabla(E_1^2 + E_2^2)/4E_1E_2\gamma_1^0\gamma_2^0 \ .$$

This Pauli form is the two-body analog of the standard Dirac equation reduced to two coupled two-component equations in which the characteristic form $\gamma^0(\epsilon - V) - m - S$ appears in denominators of its $\mathbf{L}\cdot\mathbf{S}$ and Darwin terms. That potential energy and ϵ-dependent denominator structure enters our equations through the χ forms. In spite of its formidable appearance, the generalized two-body Pauli equation (8) is as easy to work with as the nonrelativistic Schrödinger equation because of its simple momentum dependence and decoupling.

III. APPLICATIONS TO MESON SPECTROSCOPY

In Ref. 1 we tested the effect of the exactly covariant nonperturbative structure in our equation on relativistic quark-antiquark interactions. We found it plausible that our equation might accurately reflect the (hyper)fine structure of QCD with an appropriate heavy-quark static potential as input since the correct Breit structure for QED is generated by the static Coulomb potential alone [see Eq. (47)]. We will extend that investigation here to use these relativistic wave equations to test the range of validity of selected one- or two-parameter models for the heavy-quark static potential.

Richardson's model[4] for the static interquark potential

$$\tilde{V}(\mathbf{q}) \sim 1/\mathbf{q}^2 \ln(1 + \mathbf{q}^2/\Lambda^2)$$

interpolates in a simple way between asymptotic freedom $[\tilde{V}(\mathbf{q}^2) \sim 1/\mathbf{q}^2 \ln(\mathbf{q}^2/\Lambda^2)]$ and linear confinement $[\tilde{V}(\mathbf{q}^2) \sim \Lambda^2/\mathbf{q}^4]$. This interpolation is not tied at all in the intermediate region and only roughly tied in the large-r region to any field-theoretic data. Nevertheless it provides a convenient one-parameter form for the static quark potential:

$$V(|\mathbf{r}|) = 8\pi\Lambda^2 r/27 - 8\pi f(\Lambda r)/27r \ ,$$

where

$$f(t) = 1 - 4\int_1^\infty \frac{dx}{x}\,\frac{e^{-xt}}{[\ln(x^2-1)]^2 + \pi^2} \ . \tag{9}$$

For $r \to 0$, $f(\Lambda r) \to -1/\ln\Lambda r$, while for $r \to \infty$, $f(\Lambda r) \to 1$. In order to extend this model relativistically, we replace $\mathbf{r}$ by its covariant generalization (so that we have compatible constraints) ($r = \sqrt{x_\perp^2}$) and assign the Coulomb-type part to an electromagneticlike vector interaction while fixing the confining part to be half scalar and half timelike four-vector. That is, $S = \mathcal{V} = 8\pi\Lambda^2 r/54$ while $\mathcal{A} = -8\pi f(\Lambda r)/27r$. The separation of the confining part into one-half scalar and one-half timelike four-vector is admittedly *ad hoc*. [If one assumes however, that the confining part is pure scalar, then the $\mathbf{L}\cdot\mathbf{S}$ multiplets for the light mesons become partially inverted $M(^3P_2) < M(^3P_1)$.] This assumption, although not unique, leads to a cancellation of spin-orbit effects at long range, and consequently prevents partial multiplet inversions for the lighter mesons. As we did in Ref. 1 we perform numerical calculations of the $q\bar{q}$ bound states using (8). To solve for the eigenvalue $b^2(w)$ we use an iterative technique since the Φ's depend on w. To display the results we use a spectroscopic notation that describes the quantum numbers associated with the upper-upper decoupled four-component Schrödinger-type equation. In this investigation we use a more systematic fitting procedure than the one we employed in Ref. 1. We vary the single potential parameter Λ together with the b, c, s, and u quark masses in order to give the best χ^2 fit to the 53 mesons listed in Table I which have well-known experimental values and uncertainties. We leave out of the fit the η, η', and other mesons whose spectral properties require annihilation contributions. We also leave out the κ, δ, and S^*, ${}^3P_0$-like mesons whose quark status is ambiguous.[9] The definition of χ^2 that we use is

$$\chi^2 \equiv \sum_i \frac{(w_i^{\text{expt}} - w_i^{\text{th}})^2}{N((\Delta w_i^{\text{expt}})^2 + (\Delta w_i^{\text{th}})^2)} , \tag{10}$$

where N = number of mesons fit − number of parameters and w_i and Δw_i correspond to the meson masses and their uncertainties. For Δw_i^{th} we use the numerical uncertainties in our calculations. This best fit to the overall meson spectrum gives, in units of GeV, $\Lambda = 0.420$, $m_b = 4.912$, $m_c = 1.561$, $m_s = 0.407$, and $m_u = 0.143$. As we showed in Ref. 10, Richardson's potential, used with the nonrelativistic Schrödinger equation, is unable to give any fit at all to the light mesons. In contrast, as can be seen from Table I [the first theory column (RP) (units are GeV)], our exactly relativistic scheme gives a reasonable fit to most of the meson spectrum with just one potential parameter, its worst results being the π-ρ splitting and the masses of some of the higher l light mesons. This relativistic model yields a π-ρ splitting of 476–178 MeV below the measured value of 634 MeV. The K-K^* splitting is 75 MeV below the 397-MeV value. The D-D^* splitting is very close to the 142-MeV measured value although the absolute scale is about 100 MeV off. The F-F^* splitting agrees with experiment just as well while its absolute scale is in much better agreement with the measured value than is that of the D. The π-π' and ρ-ρ' splittings are in reasonable agreement with the observed values. (The exact covariance of our wave equations is designed to handle the highly relativistic motion exhibited by these systems.) The ψ-η_c splitting is just 3 MeV off.

The most important defect of Richardson's model is that the intermediate region of its potential is not tied to any field-theoretic data. Hence, we now apply our relativistic treatment to two potentials that correct this deficiency. The first is based on nonperturbative solutions to an effective field theory derived by Adler.[3] Adler makes a series of approximations to the field theory of QCD that lead to an effective classical Lagrange function which governs the dynamics of a number-valued quasi-Abelian field (depending only on A_3^μ and A_8^μ):

$$\mathcal{L}_{\text{eff}} = \hat{F}^{\mu\nu}\hat{F}_{\mu\nu}/[2\alpha_R(\hat{F}^2)] , \tag{11}$$

where for the leading-log model

$$\alpha_R(\hat{F}^2) = \frac{8\pi}{27}\frac{1}{\ln(\hat{F}^2/\Lambda^2)} . \tag{12}$$

For static solutions this leads to the field equation

$$\nabla\cdot\epsilon\mathbf{E} = 4\pi Q(\delta^3(\mathbf{x}-\mathbf{x}_1) - \delta^3(\mathbf{x}-\mathbf{x}_2)) \tag{13}$$

with the field-dependent permittivity

$$\epsilon = 9/16\pi\ln(\,|\,\mathbf{E}/\Lambda^2\,|\,), \quad Q = \sqrt{4/3} .$$

Adler and Piran solved this equation through combined analytic and numerical techniques and integrated the resulting energy density to obtain the static quark potential

$$V_{\text{static}}(\mathbf{x}_1 - \mathbf{x}_2) = \int d^3x\, \mathbf{E}^2(\mathbf{x},\mathbf{x}_1,\mathbf{x}_2)/8\pi . \tag{14}$$

This potential includes subdominant long-range parts in addition to the linear potential, i.e.,

$$V = \kappa Q r + 2/3 Q^{3/2}(16\pi/9)^{1/2}\kappa^{1/2}\ln\kappa^{1/2} r + O(1) , \tag{15}$$

along with a small-r behavior appropriate for the log model. Unlike Richardson's potential, its interpolation between the r extremes is not a guess but is based on the solution to an effective field theory. The Adler-Piran[3] model makes its significant approximations at the level of the field equation rather than to its solutions. The static potential that we actually use is extrapolated from their leading log-log model. Its static form is

$$V = \Lambda(U(\Lambda r) + U_0) \tag{16}$$

depending on two parameters Λ and U_0. Their nonperturbative solution is divided into four regions, giving $U(x)$ for $0 < x \le 0.0125$, $0.0125 \le x \le 0.125$, $0.125 \le x \le 2$, $2 \le x$. For $0 < x \le 0.0125$,

$$U(x) = -(16\pi/27)(1 + a_1 x^{a_2}) f(w_p)/w_p , \tag{17}$$

where $w_p = 1/(2.52x)^2$ and $f(w)$ is the function defined by $w = f(\ln f + \xi\ln\ln f)$ with $\xi = 2(51 - 19n_f/3)/(11 - 2n_f/3)^2 = \frac{64}{81}$ for $n_f = 3$. The large-w behavior of f demonstrates explicitly the asymptotic-freedom behavior. For $0.0125 \le x \le 0.125$ the fit to $U(x)$ is given by

$$U(x) = K + \alpha(x/0.125)^E , \tag{18}$$

where

$$E = \beta + \gamma\ln(1/x) + \delta(\ln(1/x))^2 + \epsilon(\ln(1/x))^3 , \tag{19}$$

and, for $0.125 \le x \le 2$,

$$U(x) = K' + \alpha'\ln x + \beta'(\ln x)^2 + \gamma'(\ln x)^3 + \delta'(\ln x)^4 + \epsilon'(\ln x)^5 . \tag{20}$$

The final part ($x \ge 2$) combines the *derived* confining and subdominant long-range parts:

$$U(x) = K'' + Qf(0)x + \gamma''\ln x + \alpha''/x^{1/2} + \beta''/x . \tag{21}$$

The values for the numerical coefficients a_1 through β'' that apply for the leading log-log model can be found in

TABLE I. The meson mass fits given here are produced by the covariant, nonperturbative generalizations via the constraint equation (8) of static potentials of Richardson (RP), Adler-Piran (APP), and vacuum-modified Richardson (RVP). The RP and RVP potentials are one-parameter potentials. The APP potential is a two-parameter potential.

Name	Expt.	RP	APP	RVP
$\Upsilon:b\bar{b}1^3S_1$	9.460	9.469	9.464	9.485
$\Upsilon:b\bar{b}2^3S_1$	10.023	10.021	10.026	10.022
$\Upsilon:b\bar{b}3^3S_1$	10.356	10.369	10.368	10.343
$\Upsilon:b\bar{b}4^3S_1$	10.577	10.647	10.634	10.596
$\Upsilon:b\bar{b}5^3S_1$	10.865	10.891	10.860	10.815
$\Upsilon:b\bar{b}6^3S_1$	11.019	11.112	11.060	11.012
$\Upsilon:b\bar{b}1^3P_0$	9.860	9.849	9.853	9.863
$\Upsilon:b\bar{b}1^3P_1$	9.892	9.887	9.887	9.903
$\Upsilon:b\bar{b}1^3P_2$	9.913	9.915	9.909	9.930
$\Upsilon:b\bar{b}2^3P_0$	10.235	10.234	10.235	10.218
$\Upsilon:b\bar{b}2^3P_1$	10.255	10.258	10.259	10.246
$\Upsilon:b\bar{b}2^3P_1$	10.271	10.278	10.275	10.265
$\psi:c\bar{c}1^3S_1$	3.097	3.114	3.116	3.135
$\psi:c\bar{c}2^3S_1$	3.686	3.685	3.684	3.670
$\psi:c\bar{c}3^3S_1$	4.030	4.126	4.102	4.071
$\chi_0:c\bar{c}1^3P_0$	3.415	3.400	3.418	3.404
$\chi_1:c\bar{c}1^3P_1$	3.511	3.490	3.492	3.494
$\chi_2:c\bar{c}1^3P_2$	3.556	3.547	3.535	3.547
$\psi:c\bar{c}1^3D_1$	3.770	3.793	3.792	3.777
$\psi:c\bar{c}2^3D_1$	4.159	4.198	4.171	4.141
$\psi:c\bar{c}3^3D_1$	4.415	4.559	4.501	4.466
$\eta_c:c\bar{c}1^1S_0$	2.981	2.996	3.023	3.000
$\eta_c:c\bar{c}2^1S_0$	3.594	3.614	3.625	3.592
$B^*:b\bar{u}1^3S_1$	5.352	5.229	5.321	5.270
$B:b\bar{u}1^1S_0$	5.271	5.169	5.287	5.211
$D^*:c\bar{u}1^3S_1$	2.007	1.918	1.996	1.962
$D^*:c\bar{u}1^3P_1$	2.420	2.322	2.377	2.332
$D:c\bar{u}1^1S_0$	1.865	1.770	1.903	1.812
$D_s^*:c\bar{s}1^3S_1$	2.110	2.119	2.071	2.122
$D_s:c\bar{s}1^1S_0$	1.971	1.981	1.975	1.975
$\phi:s\bar{s}1^3S_1$	1.020	1.063	0.975	1.055
$\phi:s\bar{s}2^3S_1$	1.685	1.831	1.793	1.782
$f_1:s\bar{s}1^3P_1$	1.422	1.499	1.430	1.468
$f_2':s\bar{s}1^3P_2$	1.525	1.630	1.554	1.596
$\phi:s\bar{s}1^3D_3$	1.853	2.069	1.981	2.003
$f_4:s\bar{s}1^3F_4$	2.026	2.436	2.334	2.341
$K^*:s\bar{u}1^3S_1$	0.892	0.839	0.911	0.885
$K_1:s\bar{u}1^3P_1$	1.406	1.336	1.387	1.341
$K_2^*:s\bar{u}1^3P_2$	1.426	1.479	1.512	1.479
$K_2:s\bar{u}1^3D_2$	1.770	1.875	1.911	1.841
$K_3^*:s\bar{u}1^3D_3$	1.780	1.948	1.947	1.907
$K_4^*:s\bar{u}1^3F_4$	2.060	2.332	2.304	2.257
$K:s\bar{u}1^1S_0$	0.494	0.485	0.504	0.480
$K_1:s\bar{u}1^1P_1$	1.270	1.347	1.371	1.353
$\rho:u\bar{u}1^3S_1$	0.770	0.629	0.889	0.745
$\rho:u\bar{u}2^3S_1$	1.590	1.606	1.746	1.606
$a_1:u\bar{u}1^3P_1$	1.275	1.222	1.372	1.256
$a_2:u\bar{u}1^3P_2$	1.318	1.367	1.495	1.394
$\rho_3:u\bar{u}1^3D_3$	1.691	1.860	1.927	1.838
$\pi:u\bar{u}1^1S_0$	0.135	0.143	0.133	0.140
$\pi:u\bar{u}2^1S_0$	1.300	1.363	1.622	1.355
$b_1:u\bar{u}1^1P_1$	1.235	1.195	1.318	1.236
$\pi_2:u\bar{u}1^1D_2$	1.680	1.774	1.867	1.761

Ref. 3. As was the case with Richardson's potential, the covariant generalization of the Adler-Piran potential beyond the replacement $|\mathbf{r}| \to \sqrt{x_\perp^2}$ is not unique. We choose, for all regions,

$$S = \mathcal{V} = \tfrac{1}{2}\Lambda(f(0)Q\Lambda r + \gamma'' \ln\Lambda r + \alpha''/\sqrt{\Lambda r} + K'' + U_0) \quad (22)$$

and

$$\mathcal{A}(r)=V(r)-S-\mathcal{V} . \quad (23)$$

Note that with this choice of $\mathcal{A}$, its large-r behavior is $\sim 1/r$ as is that of Richardson's potential. We make slight modifications in the potential in order to match $V(r)$ between various regions in such a way that its first and second derivatives are continuous so that the various relativistic spin and Darwin corrections will be well behaved. This relativistic extension yields the results that appear in the second theory column (APP) of Table I. Comparison of the results generated by this potential with those generated by Richardson's shows an improvement in most all sectors of the meson spectrum except the high-mass $u\bar{u}$ mesons, for which this potential gives mass predictions that are too large. For the heavy Υ system and ψ system this potential produces clear improvements particularly in the $\mathbf{L}\cdot\mathbf{S}$ splittings. The spin splittings produced by this potential are smaller for the ψ-η_c, D-D^*, F-F^* but larger for the K-K^* and ϕ-ρ systems than are those generated by Richardson's potential. The accompanying χ_2 value for this potential was about 980 compared to 1710 for the Richardson potential. In units of GeV, the best fit parameters were $\Lambda=0.224$, $\Lambda U_0=0.410$, $m_b=4.911$, $m_c=1.535$, $m_s=0.227$, $m_u=0.0924$. Of the 53 mesons the five that contribute the bulk of the χ^2 value are the ϕ, ψ, K^*, K, and χ_2 mesons of magnitude 434, 70, 67, 62, and 61, respectively. This compares with the top five generated by the Richardson potential of the K^*, ϕ, D, π, and ψ of 597, 387, 242, 71, and 57, respectively.

The third model that we consider is a modification of Richardson's potential based on the technique of Lichtenberg, Namgung, and Wills for including flavor-dependent vacuum effects.[5] One carries out this modification by substituting $\pi/27 \to \pi/(33-2n_f(r))$ in $\mathcal{V}$, $\mathcal{A}$, and S with $n_f(r)$ modeled as $n_f(r) = \sum \exp(-2m_f r)$. However, the connection of this potential to an effective field theory although closer than Richardson's (which assumes $n_f=3$) is not as systematic as that of Adler and Piran. Such a vacuum modification cannot be made so simply to the Adler-Piran potential. Since no extra variable parameters are introduced by the substitution, this third model still has just one potential parameter. This model produces a fit that is about as good as that produced by the Adler-Piran model (see the third theory column) being better for the lighter mesons and slightly inferior for the heavier ones. For the heavier mesons it is also slightly inferior to the fit produced by Richardson's potential without the vacuum correction. For example, the ψ-η_c splitting is 10 MeV off the measured value versus 3 MeV off for Richardson's potential. The F-F^* splitting is improved, and the D-D^* splitting about the same. The K-K^* splitting is significantly improved being only 1 MeV off. The π-ρ splitting is also significantly improved being only 38 MeV off the observed π-ρ splitting. The accompanying χ^2 value was about 1060, significantly better than that for Richardson's potential and about the same as that for the Adler-Piran potential. The best fit values for the parameters were $\Lambda=0.430$, $m_b=4.904$, $m_c=1.550$, $m_s=0.361$, and $m_u=0.134$, with units in GeV. The five mesons that contribute to the bulk of this value are ψ,ϕ,K,ψ',D of amounts of 276, 251, 114, 87, and 75, respectively. Note that in our wave equation, both the more physically detailed Adler-Piran model and the vacuum modification of Richardson's potential lead to an improved spectrum.

Based on the respective χ^2 values of 1710, 960, and 1060 of our three models there is obviously plenty of room for theoretical and phenomenological improvements in the potential. However, it would be naive to ignore the (possibly complex) effects on the decay channels on the potentials and mass eigenvalues. In lieu of a systematic treatment of this effect we can get perhaps a more realistic estimate of χ^2 by including the decay widths as the main contribution to the *theoretical* error that comes from leaving out the effects of the decay channel on the bound-state eigenvalues. When we carry this out then all three χ^2 drop in value. The Richardson potential which was the worst of the three for the light mesons involving at least one light quark improves substantially in acceptability since those mesons have the largest widths. Its χ^2 drops from 1710 to 696 in a new fit in which we choose $\Delta w_i^{\text{th}}=\max(\Gamma_i,\Delta w_i^{\text{num}})$. The five mesons contributing to the bulk of the χ^2 values are the D, ψ, ϕ, χ_2, and Υ of 218, 128, 120, 39, and 25, respectively. Note that the K^* and ϕ mesons drop from the list and the χ_2 and the Υ are added. The K^* drops from the list because its width makes the (slightly) worse fit less significant. For the same reason the χ_2 and Υ are added. The π drops out because a better fit can be obtained by varying m_u at the expense of the heavier large-width light-quark mesons. For the Adler-Piran model this new fit produces a χ^2 which drops substantially from 960 to 353 again due in greatest part to a lessening impact of the contribution of the heavier (large-width) light-quark mesons. The five mesons that contribute to the bulk of the χ^2 are the ψ χ_2, D, Υ, and ϕ of amounts 129, 59, 56, 49, and 38, respectively. Note the drastic drop of the ϕ from 434 to 38 and the dropping of the K^* and K from the list. The K^* drops from the list because its poor fit becomes less important whereas the K drops from the list because it can be fitted much better at the expense of the higher width mesons containing an s quark. Finally, the modified Richardson model has its χ^2 dropping from 1060 just to 922 with a new fit. This relatively small drop is due to the fact that its original fit to the light-quark mesons was very good and hence it did not benefit very much from inclusion of the widths as the theoretical error. With only a slight change in order the ϕ, ψ, K, ψ', and D mesons still contribute the bulk of the χ^2 with values of 262, 232, 117, 107, and 80, respectively.

This method for incorporating uncertainties due to our ignoring the decay channels is admittedly crude. However, all of the fits still have χ^2 well above 1 so that there is still substantial room for improvement in the potential (both theoretically and phenomenologically). Note that the π-ρ splitting remains unchanged for this modified χ^2 fit to the vacuum-modified Richardson potential at about 605 MeV. For the other two potentials it is not as stable. For the unmodified Richardson potential it drops 15 MeV to 471 MeV while for the Adler-Piran potential it rises from 756 to 825 MeV.

With this modified fit it is still noteworthy that in our wave equation, the physically detailed Adler-Piran model leads to an improved fit over that given by the Richardson potential. This is evidence that our relativistic wave equations are able to accurately capture the features of a physically realistic model even in nonperturbative circumstances.

If this is so, we should examine the sensitivity of our equations to the dynamical structure of the nonrelativistic heavy-quark potential that leads to our relativistic one. The potential of Eichten and Feinberg[11] (EF),

$$V=kr-\alpha_s/r \ , \tag{24}$$

has given excellent results when used nonrelativistically to fit the heavy-meson spectrum. If we split it into $S=\mathcal{V}=kr/2$ and $\mathcal{A}=-\alpha_s/r$ we too can use it to obtain an excellent fit to the heavy mesons. However, unlike the three models described above, it generates very poor fits to the light mesons with a single set of parameters for both light and heavy mesons. For example, its $K-K^*$ splitting is ~ 600 MeV instead of ~ 400 MeV and its π-ρ splitting is about 1000 MeV. The main structural difference between the EF potential on the one hand and Richardson's and the Adler-Piran potential on the other is that the EF potential does not have asymptotic freedom forms built into its short-range part. Apparently this feature plays a crucial role in generating good fits for the light mesons even though it is a very short-range effect which one might think should effect only the heavier mesons. Presumably, the improper short-range behavior of the EF potential distorts the parameter values needed for a good fit to the heavy mesons so much that they cannot yield reasonable results for the lighter ones. The more physically detailed Richardson (modified and unmodified) and Adler-Piran potentials "take the pressure off" the parameters to make the heavy-meson fits thereby enabling good fits to the light mesons.

In Table II we list the masses of meson states that are as yet unobserved that each model predicts (using the parameters for each obtained from our fits to 53 observed mesons). Note that in calculating both our meson fits and predictions we have neglected the effects of the off-diagonal tensor terms (which mix different orbital states) and the off-diagonal $L\cdot(s_1-s_2)$ terms (which mix different spin states). In a future work we will include the effects of these terms.

As a final digression on the relativistic dynamical structure contained in our equations, we display a set of plots that show how drastically different the behavior of our quasipotential Φ_w is from that of its nonrelativistic limit $(2\mu(\mathcal{A}+\mathcal{V}+S))$, particularly at short distances where the repulsive spin-spin potential present in the $^3S_1$ states dominates. Figures 1–4 show this difference for the $\Upsilon(9.460)$, $\psi(3.097)$, $\phi(1.020)$, and $\rho(0.770)$ mesons in terms of potentials generated by the Adler-Piran potential. Comparing these plots we also see the important c.m. energy (w) dependence of Φ (the nonrelativistic limit is w independent). Notice that at large separation the potential experienced by the quarks in the Υ meson virtually coincides with the nonrelativistic potential. As the

TABLE II. New meson mass predictions based on the same covariant generalizations used in producing Table I.

Name	RP	APP	RVP
$b\bar{b}7^3S_1$	11.317	11.243	11.196
$b\bar{b}1^3D_1$	10.152	10.157	10.156
$b\bar{b}1^3D_2$	10.166	10.167	10.168
$b\bar{b}1^3D_3$	10.179	10.176	10.180
$b\bar{b}1^1S_0$	9.391	9.397	9.384
$b\bar{b}2^1S_0$	9.985	9.993	9.980
$b\bar{b}1^1P_1$	9.896	9.894	9.911
$b\bar{b}2^1P_1$	10.264	10.265	10.252
$b\bar{b}1^1D_2$	10.168	10.169	10.170
$b\bar{b}2^1D_2$	10.467	10.462	10.440
$c\bar{c}4^3S_1$	4.507	4.451	4.414
$c\bar{c}1^3D_2$	3.828	3.816	3.807
$c\bar{c}2^3D_2$	4.228	4.194	4.168
$c\bar{c}1^3D_3$	3.859	3.839	3.834
$c\bar{c}2^3D_3$	4.257	4.214	4.193
$c\bar{c}1^1P_1$	3.506	3.506	3.509
$c\bar{c}2^1P_1$	3.964	3.949	3.927
$c\bar{c}1^1D_2$	3.833	3.821	3.812
$c\bar{c}2^1D_2$	4.233	4.198	4.173
$b\bar{u}2^3S_1$	5.796	5.846	5.786
$b\bar{u}1^3P_0$	5.528	5.607	5.539
$b\bar{u}1^3P_1$	5.604	5.661	5.612
$b\bar{u}1^3P_2$	5.663	5.702	5.668
$b\bar{u}1^3D_1$	5.906	5.943	5.891
$b\bar{u}1^3D_2$	5.935	5.960	5.916
$b\bar{u}1^3D_3$	5.964	5.977	5.942
$b\bar{u}2^1S_0$	5.753	5.818	5.741
$b\bar{u}1^1P_1$	5.621	5.679	5.636
$b\bar{u}2^1P_1$	6.066	6.082	6.033
$b\bar{u}1^1D_2$	5.942	5.964	5.923
$b\bar{u}2^1D_2$	6.316	6.304	6.621
$c\bar{u}2^3S_1$	2.576	2.624	2.563
$c\bar{u}1^3P_0$	2.169	2.256	2.180
$c\bar{u}1^3P_2$	2.408	2.440	2.413
$c\bar{u}1^3D_1$	2.674	2.715	2.659
$c\bar{u}1^3D_2$	2.729	2.751	2.707
$c\bar{u}1^3D_3$	2.775	2.781	2.748
$c\bar{u}2^1S_0$	2.476	2.554	2.459
$c\bar{u}1^1P_1$	2.346	2.397	2.356
$c\bar{u}2^1P_1$	2.891	2.904	2.851
$c\bar{u}1^1D_2$	2.736	2.755	2.714
$c\bar{u}2^1D_2$	3.213	3.192	3.145
$c\bar{s}2^3S_1$	2.739	2.685	2.700
$c\bar{s}1^3P_0$	2.363	2.329	2.340
$c\bar{s}1^3P_1$	2.500	2.446	2.480
$c\bar{s}1^3P_2$	2.581	2.509	2.557
$c\bar{s}1^3D_1$	2.836	2.777	2.795
$c\bar{s}1^3D_2$	2.888	2.815	2.841
$c\bar{s}1^3D_3$	2.934	2.847	2.881
$c\bar{s}2^1S_0$	2.644	2.612	2.598
$c\bar{s}1^1P_1$	2.523	2.466	2.503
$c\bar{s}2^1P_1$	3.046	2.968	2.981
$c\bar{s}1^1D_2$	2.896	2.820	2.848
$c\bar{s}2^1D_2$	3.357	3.253	3.267
$s\bar{s}1^3P_0$	1.231	1.125	1.172
$s\bar{s}1^3D_1$	1.902	1.842	1.849
$s\bar{s}1^3D_2$	1.995	1.933	1.936
$s\bar{u}2^3S_1$	1.699	1.761	1.679
$s\bar{u}1^3P_0$	0.984	1.052	0.959

TABLE II. (*Continued.*)

Name	RP	APP	RVP
$s\bar{u}\,1^3D_1$	1.768	1.809	1.742
$s\bar{u}\,2^1S_0$	1.501	1.628	1.467
$s\bar{u}\,2^1P_1$	2.061	2.049	2.006
$s\bar{u}\,1^1D_2$	1.870	1.883	1.837
$s\bar{u}\,2^1D_2$	2.641	2.424	2.378
$u\bar{u}\,3^3S_1$	2.277	2.357	2.214
$u\bar{u}\,1^3P_0$	0.731	1.055	0.748
$u\bar{u}\,1^3D_1$	1.671	1.795	1.667
$u\bar{u}\,1^3D_2$	1.800	1.915	1.783
$u\bar{u}\,2^1P_1$	1.960	1.998	1.925
$u\bar{u}\,2^1D_2$	2.387	2.400	2.317
$b\bar{s}\,1^3S_1$	5.422	5.394	5.426
$b\bar{s}\,2^3S_1$	5.987	5.915	5.931
$b\bar{s}\,1^3P_0$	5.716	5.685	5.697
$b\bar{s}\,1^3P_1$	5.784	5.736	5.764
$b\bar{s}\,1^3P_2$	5.839	5.775	5.816
$b\bar{s}\,1^3D_1$	6.078	6.016	6.037
$b\bar{s}\,1^3D_2$	6.105	6.033	6.061
$b\bar{s}\,1^3D_3$	6.132	6.051	6.085
$b\bar{s}\,1^1S_0$	5.361	5.357	5.362
$b\bar{s}\,2^1S_0$	5.926	5.885	5.886
$b\bar{s}\,1^1P_1$	5.806	5.753	5.785
$b\bar{s}\,2^1P_1$	6.234	6.156	6.176
$b\bar{s}\,1^1D_2$	6.112	6.038	6.067
$b\bar{s}\,2^1D_2$	6.478	6.377	6.400
$b\bar{c}\,1^3S_1$	6.345	6.352	6.366
$b\bar{c}\,2^3S_1$	6.886	6.892	6.877
$b\bar{c}\,1^3P_0$	6.674	6.690	6.682
$b\bar{c}\,1^3P_1$	6.728	6.733	6.736
$b\bar{c}\,1^3P_2$	6.767	6.763	6.773
$b\bar{c}\,1^3D_1$	7.005	7.009	6.996
$b\bar{c}\,1^3D_2$	7.026	7.023	7.014
$b\bar{c}\,1^3D_3$	7.046	7.037	7.031
$b\bar{c}\,1^1S_0$	6.268	6.392	6.275
$b\bar{c}\,2^1S_0$	6.844	6.856	6.829
$b\bar{c}\,1^1P_1$	6.741	6.745	6.749
$b\bar{c}\,2^1P_1$	7.141	7.135	7.114
$b\bar{c}\,1^1D_2$	7.030	7.026	7.018
$b\bar{c}\,2^1D_2$	7.367	7.347	7.322

mesons become lighter the potential develops progressively more energy dependence so that even at large separation the constraint potential differs significantly from its nonrelativistic limit. Figure 5 compares the different Φ_w's for the $\psi(3.097)$ and $\eta_c(2.980)$ demonstrating the strong spin-spin dependence of Φ_w.

In summary, using our coupled Dirac equations, we have obtained overall fits to the meson masses (including the lightest meson, the pion) that are surprisingly good. Their goodness is a consequence of both the structure of the phenomenological input potential and the way in which the covariant constraint formalism dictates the spin-dependent consequences of these potentials. In the next three sections we examine three important properties of the constraint formalism (two nonperturbative and one perturbative) that are responsible for the nature of its spin-dependent consequences.

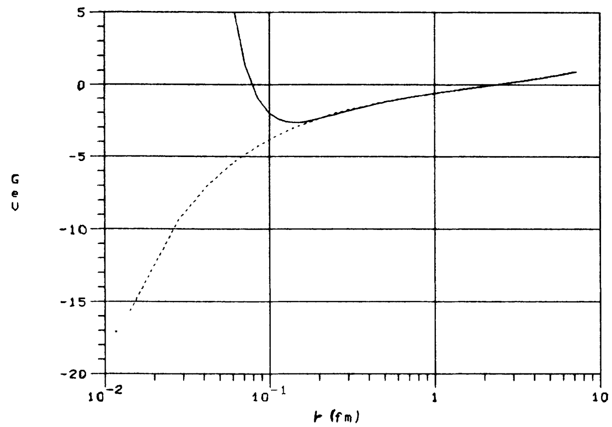

FIG. 1. The nonrelativistic potential $S+\mathcal{A}+\mathcal{V}$ (dotted curve) and the relativistic quasipotential $\Phi_w/2\mu$ (solid curve) are plotted together as functions of the invariant distance r for the $1^3S_1,\Upsilon$ meson. Notice that even though the quarks in this meson move slowly, the relativistic spin-spin corrections to the potential at short-range produce a significant repulsive barrier. The vertical axis gives the potentials in units of GeV and the horizontal axis gives r in units of fm.

IV. CHIRAL-SYMMETRY BREAKING IN THE TWO-BODY DIRAC EQUATION

The $\pi-\rho$ splitting of 605 MeV predicted by our relativistic extension of the modified Richardson model is surprisingly close to the experimental value of 634 MeV considering that we made no explicit effort to build the chiral-symmetry limit into our equations. Before investigating the presence of approximate chiral symmetry in our quantum equations, we must make clear what we mean by "chiral symmetry." We mean that the axial-vector current j_5^μ generated by our model is conserved. In field theory, this implies that the quark mass terms are

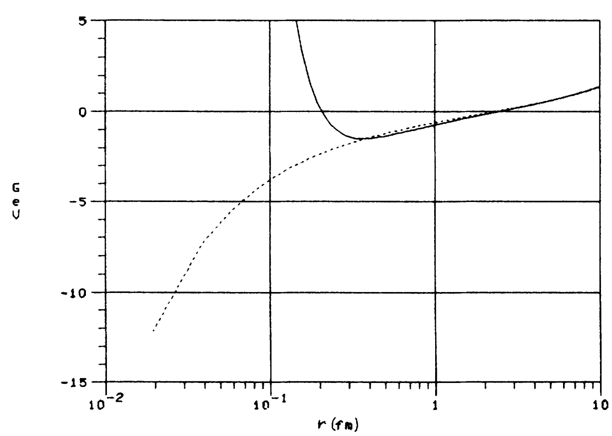

FIG. 2. The same nonrelativistic potential as in Fig. 1 is plotted together with its covariant counterpart for the $1^3S_1,\Psi$ meson. Notice that there is a slight change in the covariant potential from that of the previous figure, demonstrating the energy and mass dependence in Φ_w.

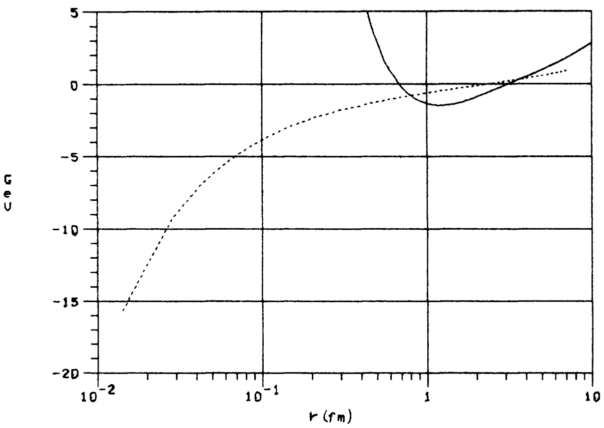

FIG. 3. The same sort of plot as that made in Fig. 2 but for the $1^3S_1, \Phi$ meson. Notice that for this case even the long-range part of the covariant potential differs significantly from that of its nonrelativistic counterpart.

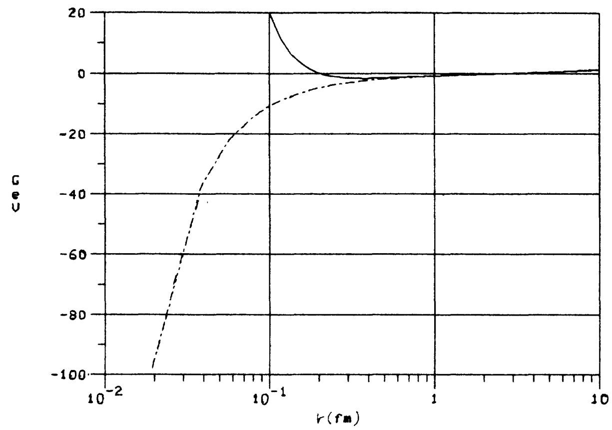

FIG. 5. This figure compares $\Phi_w/2\mu$ for the $\Psi, 1^3S_1$ (the solid curve) exhibiting a short-range repulsive behavior with that for the $\eta_c, 1^1S_0$ (the alternately dotted and dashed curve) displaying the strong spin dependence of the quasipotential.

zero in the Lagrange function and that the quarks experience no scalar or pseudoscalar interactions. Introduction of finite-quark masses or scalar or pseudoscalar interactions explicitly breaks chiral symmetry thereby producing a nonzero divergence of the axial-vector current. In the absence of such explicit interactions, spontaneous symmetry breaking may occur even when the system retains formal chiral invariance ($\partial_\mu j_5^\mu = 0$) and $m_q = 0$ and is manifested by the noninvariance of the vacuum under chiral transformations and the appearance of Goldstone bosons ($m_\pi = 0$). One demonstrates the connection with the pion as Goldstone boson as follows. The pion form factor is given by

$$\langle 0 | j_{\mu 5}(0) | \pi(P) \rangle = i P_\mu F_\pi(P) \ . \tag{25}$$

thus the on-shell matrix element of its divergence is

$$\langle 0 | \partial^\mu j_{\mu 5}(0) | \pi(P) \rangle = -P^2 F_\pi(P) = m_\pi^2 F_\pi \ . \tag{26}$$

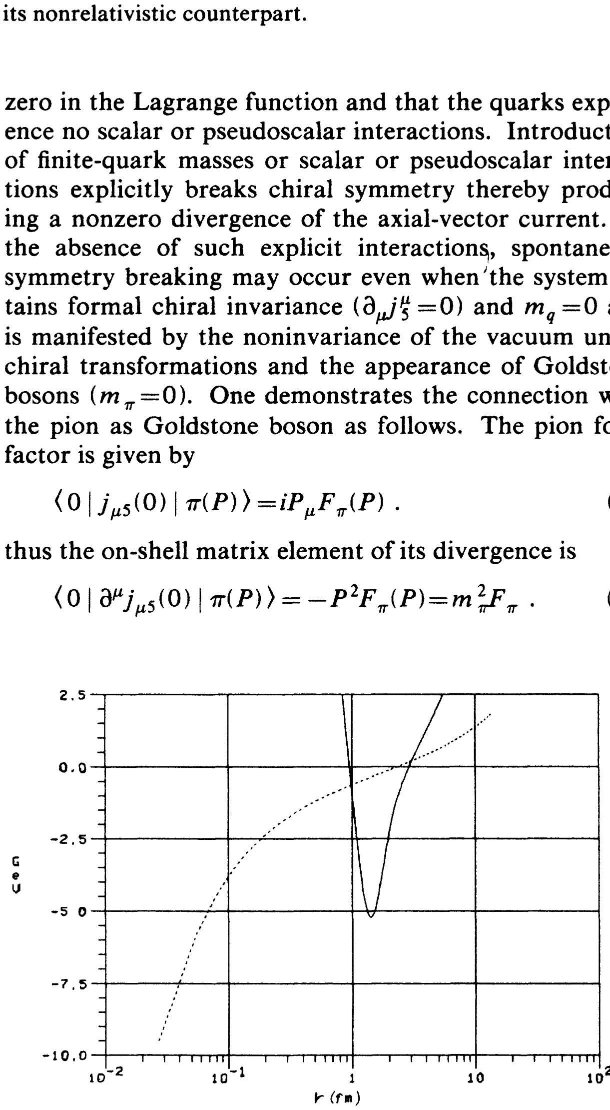

FIG. 4. The same sort of plot as in the previous figure for the $1^3S_1, \rho$ meson. The covariant potential for this case is sharply different from its nonrelativistic counterpart at both long and short ranges.

Formal chiral invariance then appears when either $m_\pi = 0$ (the spontaneous breakdown of chiral symmetry) or $F_\pi = 0$ (degenerate scalar and pseudoscalar multiplets and zero decay constant).

In order to see how our equations are structurally able to reflect the chiral-symmetry limit, we shall treat the scalar interaction using arguments similar to those presented by Sazdjian[6] in his treatment of the pseudoscalar interaction. In order to connect field-theoretic matrix elements to the constraint formalism, Sazdjian assumed that the constraint quantum-mechanical wave function ψ and the field-theoretic wave function

$$\chi_{\rm FT}(x_1, x_2)_{\alpha_1 \alpha_2} = \langle 0 | T[\psi_1(x_1)_{\alpha 1} \bar{\psi}_2(x_2)_{\alpha 2}] | \pi \rangle \tag{27}$$

are at least proportional. The explicit operator nature of this proportionality factor [see Ref. 6 and also Eqs. (61) and (65) in Sec. V of this paper] is not important for our purposes. For simplicity we shall assume that these two objects are the same ($\psi = \chi_{\rm FT}$). Reducing out the pion gives

$$\begin{aligned} \langle 0 | j_{\mu 5}(0) | \pi \rangle &\equiv \langle 0 | \bar{\psi}_2(0) \gamma_\mu \gamma_5 \psi_1(0) | \pi \rangle \\ &= -\mathrm{Tr}(\gamma_\mu \gamma_5 \psi) |_{x_1 = x_2 = 0} \end{aligned} \tag{28}$$

and

$$\langle 0 | \partial^\mu j_{\mu 5}(0) | \pi \rangle = -[(\partial_1^\mu + \partial_2^\mu) \mathrm{Tr}(\gamma_\mu \gamma_5 \psi)]_{x_1 = x_2 = 0} \ . \tag{29}$$

In this form, ψ is a 4×4-matrix wave function instead of a 16-component column vector wave function. In order to compute the trace we need the matrix forms of our constraint equations:

$$(\gamma \cdot p_1 + M_1)\psi = -\frac{i}{2} \psi \mathcal{M} \cdot \gamma \ , \tag{30}$$

$$\psi(M_2 - \gamma \cdot p_2) = -\frac{i}{2} \mathcal{M} \cdot \gamma \psi \ , \tag{31}$$

where $\mathcal{M}=\partial M_1/M_2=\partial M_2/M_1$. [Note the importance here of the existence of two (compatible) Dirac equations for computing the right-hand side of Eq. (29).] Using these equations the divergence condition becomes

$$\begin{aligned}\mathrm{Tr}(\not{p}_1\gamma_5\psi+\gamma_5\psi\not{p}_2)&=\mathrm{Tr}[\gamma_5(\psi\not{p}_2-\not{p}_1\psi)]\\&=[M_1(0)+M_2(0)]\mathrm{Tr}[\gamma_5\psi(0)]\\&=[m_1+m_2+S_1(0)\\&\quad+S_2(0)]\mathrm{Tr}[\gamma_5\psi(0)]\ .\end{aligned}\qquad(32)$$

Thus, even in the presence of scalar interactions (which formally appear to explicitly break chiral symmetry) formal chiral invariance in the sense $\langle 0\,|\,\partial^\mu j_{\mu5}(0)\,|\,\pi(P)\rangle=0$ occurs provided that $M_1(0)+M_2(0)=0$ and that $\psi(0)$ is finite. If $S_i(0)=0$ (as happens for linear confining potentials)[12] then formal chiral invariance requires $m_i=0$. Deviations from chiral symmetry occur if $m_i\neq0$ or $S_i\neq0$.

In order for our quantum equations to reflect the spontaneous breakdown of chiral symmetry that occurs in quantum field theory, they must contain two ingredients. First, the wave equation must be made noninvariant under chiral transformations through the introduction of appropriate interactions. As we have seen, however, this only leads to spontaneously broken chiral symmetry if $M_i(0)=0$. Second, the pion must appear as a Goldstone boson, namely, $m_\pi=0$. We have found suggestive numerical evidence that our full equations do, in fact, have the chiral-symmetry limit built into them for pseudoscalars. Specifically, we have found that for equal mass $q\bar{q}$, $l=0$ spin-singlet bound states, as the quark mass tends toward zero, the mass of the bound state also tends toward zero. If this were to continue to the formal chiral-symmetry limit ["zero-mass" quarks and $S_i(0)=0$] our equations would yield "zero-mass" pions (Goldstone bosons). As numerical evidence we offer the calculated sequence of masses $m_\pi=0.164$, 0.108, 0.067, 0.041, 0.023 MeV corresponding to the quark masses 0.141, 0.121, 0.100, 0.081, 0.063 MeV, respectively. This numerical evidence strongly suggests that our equations contain structures dictated by the spontaneous breakdown of chiral symmetry.

Our numerical demonstration parallels an analytic one given by Sazdjian in which he computes an analytic spectrum using two-body Dirac equations for the pion with a pseudoscalar interaction. The equations he uses are[6]

$$(\gamma_1\cdot p_1+m_1+i\gamma_2\cdot\partial V\gamma_{51}\gamma_{52})\psi=0\ ,\qquad(33)$$

$$(\gamma_2\cdot p_2+m_2-i\gamma_1\cdot\partial V\gamma_{51}\gamma_{51})\psi=0\ .\qquad(34)$$

For the potential $V=kx_\perp^2$ (a relativistic harmonic oscillator) he shows explicitly that

$$w=(m_1^2+m_2^2+4k(l+2s+2n_r)+\{[m_1^2+m_2^2+4k(l+s+2n_r)]^2-(m_1^2-m_2^2)^2\}^{1/2})^{1/2}\ ,$$

which vanishes as $m_1,m_2\to0$ for the ground state ($s=l=n_r=0$). Like his exact model, our equations for $s=1$ predict that the ρ mass does not tend to zero as $m_1,m_2\to0$ (in fact, in our model it approaches about 300 MeV).

Before extending our chiral-symmetry arguments for scalar interactions to the case of the combined scalar and vector interactions that actually appear in our quark models we compare the chiral-symmetry limit of our two-body Dirac equations with the chiral-symmetry limit of equations derived recently by Sazdjian.[13] He also uses the constraint approach but solves the problem of compatibility in an entirely different way. We restrict our comparisons to scalar interactions of spin-$\frac{1}{2}$ particles. Our approach[1,2] emphasizes the appearance of the scalar potential as a modification of each constituent mass depending on a supersymmetric position coordinate:

$$\mathcal{S}_1\psi=\gamma_{5i}(\gamma_i\cdot p_i+m_i)\psi\to\gamma_{5i}(\gamma_i\cdot p_i+m_i+\tilde{S}_i)\ .\qquad(35)$$

The supersymmetric $\tilde{x}_\perp$ dependence leads to the strong compatibility conditions $[\mathcal{S}_1,\mathcal{S}_2]=0$. The maintenance of supersymmetry forces a relativistic condition that plays the role of Newton's third law[2] ($\Phi_1=\Phi_2$) on the S_i to take the same form as that of the spinless case ($2m_1S_1+S_1^2=2m_2S_2+S_2^2$). Sazdjian starts with two modified Dirac equations

$$\mathcal{D}_1\psi=(\gamma_1\cdot p_1+m_1+U_1)\psi=0\qquad(36a)$$

and

$$\mathcal{D}_2\psi=(\gamma_2\cdot p_2+m_2+U_2)\psi=0\ .\qquad(36b)$$

First he guarantees that the squared versions of these equations are compatible. After multiplying Eq. (36a) by $(-\gamma_1p_1+m_1)$ and Eq. (36b) by $(-\gamma_2p_2+m_2)$ he finds

$$[p_1^2+m_1^2-(m_1-\gamma_1\cdot p_1)U_1]\psi=0\ ,\qquad(37a)$$

$$[p_2^2+m_2^2-(m_2-\gamma_2\cdot p_2)U_2]\psi=0\ ,\qquad(37b)$$

each of which is of the form

$$\mathcal{H}_i\psi=(p_i^2+m_i^2+\Phi_i)\psi=0\ .\qquad(38)$$

As in the spinless case compatibility follows from

$$\Phi_1=\Phi_2=\Phi(x_\perp)\ .$$

He solves the third law problem, $\Phi_1=\Phi_2$, by choosing $U_1=(m_2-\gamma_2\cdot p_2)\mathcal{U}$ and $U_2=(m_1-\gamma_1\cdot p_1)\mathcal{U}$. This leads to

$$[\gamma_1\cdot p_1+m_1+(m_2-\gamma_2\cdot p_2)\mathcal{U}]\psi=0\ ,\qquad(39a)$$

$$[\gamma_2\cdot p_2+m_2+(m_1-\gamma_1\cdot p_2)\mathcal{U}]\psi=0\ ,\qquad(39b)$$

and guarantees the strong compatibility of the squared equations $[\mathcal{H}_1,\mathcal{H}_2]=0$. His approach does not lead to $[\mathcal{D}_1,\mathcal{D}_2]=0$, the strong compatibility of the Dirac operators, but instead to the weak compatibility condition $[\mathcal{D}_1,\mathcal{D}_2]\psi=0$. That is,

$$[\mathcal{D}_1,\mathcal{D}_2]\psi=[\gamma_1 p_1,\mathcal{U}]\mathcal{D}_1\psi-(\gamma_2\cdot p_2\mathcal{U})\mathcal{D}_2\psi=0\ . \quad (40)$$

For scalar interactions

$$\mathcal{U}=\mathcal{U}(x_\perp,p_1,p_2)1_1\times 1_2\ . \quad (41)$$

At first sight there is no direct link between our $\mathcal{S}_i$ and his $\mathcal{D}_i$. That is, $\mathcal{S}_i\neq\gamma_{5i}\mathcal{D}_i$ nor is $\tilde{S}_i=(m_j-\gamma_j\cdot p_j)\mathcal{U}$. However, one can make a connection by bringing the operators $\gamma_2\cdot p_2$ of Eq. (39a) and $\gamma_1\cdot p_1$ of Eq. (39b) to the right of $\mathcal{U}$ and then using (39b) and (39a), respectively, to eliminate $\gamma_2 p_2\psi$ and $\gamma_1 p_1\psi$. This leads to

$$\mathcal{D}_1\psi=\left[\gamma_1\cdot p_1+m_1+\frac{2}{1-\mathcal{U}^2}\left[m_2\mathcal{U}+m_1\mathcal{U}^2+\frac{i}{r}\frac{\partial\mathcal{U}}{\partial r}\gamma_2\cdot x_\perp\right]\right]\psi=0\ , \quad (42a)$$

$$\mathcal{D}_2\psi=\left[\gamma_2\cdot p_1+m_2-\frac{2}{1-\mathcal{U}^2}\left[m_1\mathcal{U}+m_2\mathcal{U}^2-\frac{i}{r}\frac{\partial\mathcal{U}}{\partial r}\gamma_1\cdot x_\perp\right]\right]\psi=0\ . \quad (42b)$$

As Sazdjian points out $[\mathcal{D}_1,\mathcal{D}_2]\neq 0$ but $[\mathcal{D}_1,\mathcal{D}_2]\psi=0$. However, $[\gamma_{51}\mathcal{D}_1,\gamma_{52}\mathcal{D}_2]=0$. This indicates that there is indeed a connection between his $\mathcal{U}$ and our S_i. In fact $\mathcal{S}_i=\gamma_{5i}\mathcal{D}_i$ if

$$S_1=\frac{2m_2\mathcal{U}+m_2\mathcal{U}^2}{1-\mathcal{U}^2}\ , \quad (43a)$$

$$S_2=\frac{2m_1\mathcal{U}+m_2\mathcal{U}^2}{1-\mathcal{U}^2}\ . \quad (43b)$$

Note that these satisfy $2m_1S_1+S_1^2=2m_2S_2+S_2^2$.

Sazdjian's parametrization gives the impression that $S_i\to 0$ as $m_i\to 0$, i.e., in the chiral limit the scalar potential vanishes. However, despite the connection between our equations and his, like the field-theoretic results, our potentials S_i do not vanish in the zero-mass limit. In addition, Sazdjian's parametrization lacks a well-defined heavy-particle limit. Apparently, only through direct comparison of each potential with the corresponding field-theoretic potential could the proper mass factors be extracted which would rectify this difference (see Appendix A).

In the general case when both vector and scalar interactions are present the matrix forms of our equations are

$$[\gamma\cdot p_1+M_1/G+(E_1/G-\epsilon_1)\gamma\cdot\hat{P}]\psi+\frac{i}{2}\gamma_\perp\cdot\psi\gamma_\perp\partial\ln G\cdot\gamma+\frac{i}{2}\gamma\cdot\hat{P}\psi\gamma\cdot\hat{P}\gamma\cdot\mathcal{E}+\frac{i}{2}\psi\mathcal{M}\cdot\gamma=0\ , \quad (1a')$$

$$\psi(-\gamma\cdot p_2+M_2/G-(E_2/G-\epsilon_2)\gamma\cdot\hat{P})+\frac{i}{2}\gamma\cdot\partial\ln G\gamma_\perp\psi\cdot\gamma_\perp+\frac{i}{2}\gamma\cdot\mathcal{E}\gamma\cdot\hat{P}\psi\gamma\cdot\hat{P}+\frac{i}{2}\gamma\cdot\mathcal{M}\psi=0\ , \quad (1b')$$

where $\mathcal{M}$ is defined above and $\mathcal{E}=\partial E_1/E_2=\partial E_2/E_1$. If we use $\gamma\cdot\hat{P}=(\not{p}_1+\not{p}_2)/w$ we find that

$$\mathrm{Tr}[\gamma_5(\not{p}_1+\not{p}_2)\psi(x)]=w\frac{M_1+M_2}{E_1+E_2}\mathrm{Tr}[\gamma_5\psi(0)]\ . \quad (44)$$

We then use the expressions for M_i,E_i given in (7) and (5) and S, $\mathcal{V}\sim 0$, and $\mathcal{A}\sim 1/r\ \ln r$ to obtain

$$\mathrm{Tr}[\gamma_5(\not{p}_1+\not{p}_2)\psi]=\frac{m_1+m_2}{(1-2\mathcal{A}/w)^{1/2}}\mathrm{Tr}[\gamma_5\psi(0)]\ . \quad (45)$$

Although this result formally displays the chiral-symmetry limit as $m_1,m_2\to 0$, the uncertain behavior of the wave function at the origin makes evaluation of this limit difficult.

V. THE WEAK-POTENTIAL SLOW-MOTION APPROXIMATION AND ALTERNATIVE FORMS OF THE NONPERTURBATIVE AND COVARIANT CONSTRAINT EQUATIONS FOR TWO SPIN-$\frac{1}{2}$ PARTICLES

In this section and the next we wish to investigate two related questions about the quasipotential form given in Eq. (8). First, when one has been given a phenomenological input potential in the form of the covariant scalars $\mathcal{A}$, $\mathcal{V}$, and S does the resultant quasipotential Φ_w display relativistic spin structures that extrapolate the corresponding semirelativistic spin structures given by a related quantum field theory? Second, in cases where our equations are supposed to describe a system governed by a particular quantum field theory, is there a way of determining the invariant scalars $\mathcal{A}$, $\mathcal{V}$, and S in order to connect the relativistic quantum-mechanical description provided by our equations to quantum field theory? The answer to both questions is affirmative and is supplied by the direct connection between our equations and the Todorov equation. We find that for weak potentials the upper-upper component of Eq. (8) reduces to the Todorov equation for scalar and vector interactions. The Todorov equation itself is covariant and reproduces the correct fine-structure effects in bound-state calculations with either scalar[1] or vector potentials.[14] In this section we answer the first question by demonstrating this weak-potential connection as well as its production of slow-motion forms familiar from quantum field theory. In addition, we examine alternative forms of our unapproximated equations that are related to yet other field-theoretic results. In the next section we answer the second question by supplying the missing link between our equations and those of relativistic quantum field theory—the connection between the Todorov equation and quantum field theory. This is provided by Todorov's inhomogeneous quasipotential equation which we derive from the Bethe-Salpeter equation.

We demonstrate the connection of our equations to that of Todorov by starting from our equations given in their two-body Pauli form: Eq. (8). We remind the reader that our equations in this form (apart from the energy dependence of Φ_w) are as easy to work with as the nonrelativistic Schrödinger equation because of their simple momentum dependence and decoupling.

Once we have found the weak-potential form of our equations, we unravel their covariant structures by performing an $O(1/c^2)$ expansion. As we shall see, we then obtain a form canonically equivalent to the Fermi-Breit approximation of the Bethe-Salpeter equation. That form is more familiar than are those of our unapproximated equations but has the distinct disadvantages that it is valid only perturbatively and possesses a complicated momentum dependence.

We obtain the semirelativistic form of our equation through a two-step reduction. First we find that our equation reduces to the Todorov equation[14,1] when the potential is weak, i.e., of $O(1/c^2)$ times the total c.m. energy. In that case, all the terms in Φ_{DO} are of $O(1/c^4)$ relative to the kinetic $\mathbf{p}^2$ term. After taking the various derivatives, all potential $\mathcal{A}$, $\mathcal{V}$, and S dependences are dropped in the $\mathcal{X}_1$, $\mathcal{X}_2$, and G terms so that $M_i \rightarrow m_i$, $E_i \rightarrow \epsilon_i$, and $G \rightarrow 1$. Thus $\mathcal{X}_i \rightarrow \epsilon_i \gamma_i^0 + m_i$. Further, the gradient squared terms appearing in Φ_{SS} and Φ_T are of the same order as Φ_{DO} and can be dropped. Hence, specializing to the upper-upper component we find the following Todorov equation for simultaneous scalar and vector potentials:

$$\Bigg[\mathbf{p}^2c^2+2m_w c^2 S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\frac{1}{2w}\left[\frac{\epsilon_1}{\epsilon_2+m_2c^2}+\frac{\epsilon_2}{\epsilon_1+m_1c^2}\right]\nabla^2(\mathcal{A}+\mathcal{V}-S)+\frac{1}{2w}\nabla^2\mathcal{A}$$
$$+\frac{1}{wr}\frac{\partial}{\partial r}(\mathcal{A}+\mathcal{V}-S)\mathbf{L}\cdot\left[\frac{\epsilon_2}{\epsilon_1+m_1c^2}\boldsymbol{\sigma}_1+\frac{\epsilon_1}{\epsilon_2+m_2c^2}\boldsymbol{\sigma}_2\right]+\frac{1}{wr}\frac{\partial}{\partial r}\mathcal{A}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+\frac{1}{3w}\nabla^2\mathcal{A}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$
$$-\frac{1}{3}\left[-\frac{1}{wr}\frac{\partial\mathcal{A}}{\partial r}+\frac{\partial^2\mathcal{A}}{\partial r^2}\right]S_T\Bigg]\Psi_{++}=b^2(w)\Psi_{++}\ . \tag{46}$$

(It is important to remember that r is an invariant and equal to $|\mathbf{r}|$ only in the c.m. frame.) The Todorov form Eq. (46), although not an $O(1/c^2)$ approximation, would be inadequate for strong coupling. Its spin-dependent and Darwin interaction terms can only be treated perturbatively. As the second step, we carry out a further $O(1/c^2)$ expansion by replacing the energy variables ϵ_i by m_i and expanding the energy dependence in $2\epsilon_w\mathcal{A}$ and $2m_w c^2 S$. This results in the Fermi-Breit Hamiltonian for scalar and vector interactions when we take $\mathcal{V}=0$. Finally, we specialize to $\mathcal{A}=-\alpha/r$ and $S=0$, the form for electrodynamics to the appropriate order.[15,16] In that case, we obtain

$$\Bigg\{\Bigg[\frac{\mathbf{p}^2}{2\mu}-\frac{\alpha}{r}-\frac{1}{8}\left[\frac{1}{m_1^3c^2}+\frac{1}{m_2^3c^2}\right](\mathbf{p}^2)^2-\frac{\alpha}{2m_1m_2c^2}\mathbf{p}\cdot\left[\frac{1}{r}+\frac{\mathbf{r}\mathbf{r}}{r^3}\right]\cdot\mathbf{p}\Bigg]-\frac{1}{2c^2}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]\delta(\mathbf{r})$$
$$-\frac{1}{4c^2}\frac{\mathbf{L}}{r^3}\cdot\left[\left[\frac{1}{m_1^2}+\frac{2}{m_1m_2}\right]\boldsymbol{\sigma}_1+\left[\frac{1}{m_2^2}+\frac{2}{m_1m_2}\right]\boldsymbol{\sigma}_2\right]$$
$$+\frac{1}{4m_1m_2c^2}\left[-\frac{8\pi}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\delta(\mathbf{r})+\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3}-\frac{3\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{r}}{r^5}\right]\Bigg\}\Psi_{++}=(w-m_1c^2-m_2c^2)\Psi_{++}\ . \tag{47}$$

As part of the manipulations leading to this form, we have performed a canonical transformation[17] that yields the momentum-dependent Darwin terms (appearing between $(\mathbf{p}^2)^2$ and the δ-function piece). In summary, the relativistic kinematical, dynamical, and spin structures of our equation have allowed us to start with a static form of the potential, valid in the nonrelativistic limit, and with no additional assumptions, derive the standard relativistic corrections to the total energy operator. Note the contrast between the complicated momentum dependence of this noncovariant perturbative standard Breit form and the simple Schrödinger momentum dependence of the covariant nonperturbative form of our exact equation. Even though the quasipotentials are complicated they are local and momentum independent[18] (although dependent on the c.m. energy).

Before deriving the Todorov equation for Φ_w from field theory (so that one would know what invariant forms to use, say, for $\mathcal{A}$ and S) as a further comparison between our approach and older two-body formalisms we combine our two-body Dirac equations in the form of an effective Breit equation[15,16] for electromagneticlike interactions ($\mathcal{V}=0$, $S=0$). In that case our Dirac equations reduce to

$$\mathcal{S}_1\psi=\gamma_{51}(\gamma_1\cdot(p_1-\tilde{A}_1)+m_1)\psi=0\ , \tag{1a'}$$

$$\mathcal{S}_2\psi=\gamma_{52}(\gamma_2\cdot(p_2-\tilde{A}_2)+m_2)\psi=0\ , \tag{1b'}$$

where

$$\tilde{A}_1=[\epsilon_1(1-G)+G\mathcal{A}]\hat{P}+(1-G)p-\frac{i}{2}\partial G\cdot\gamma_2\gamma_2\ , \tag{48a}$$

$$\tilde{A}_2=[\epsilon_2(1-G)+G\mathcal{A}]\hat{P}-(1-G)p+\frac{i}{2}\partial G\cdot\gamma_1\gamma_1\ . \tag{48b}$$

In the c.m. frame these become

$$\tilde{A}\,{}^0_1=\epsilon_1(1-G)+G\mathcal{A}+\frac{i}{2}\nabla G\cdot\boldsymbol{\alpha}_2\ , \tag{49a}$$

$$\mathbf{A}_1=(1-G)\mathbf{p}+\tfrac{1}{2}\nabla G\times\boldsymbol{\sigma}_2\ , \tag{49b}$$

$$\tilde{A}\,{}^0_2=\epsilon_2(1-G)+G\mathcal{A}-\frac{i}{2}\nabla G\cdot\boldsymbol{\alpha}_1\ , \tag{50a}$$

$$\mathbf{A}_2=-(1-G)\mathbf{p}-\tfrac{1}{2}\nabla G\times\boldsymbol{\sigma}_1\ . \tag{50b}$$

Multiplying the first Dirac equation by $\beta_1\gamma_{51}$ and the second by $\beta_2\gamma_{52}$ and making the scale transformation $\psi=\Psi/G$ leads to

$$\left[\boldsymbol{\alpha}_1\cdot\left[G\mathbf{p}-\tfrac{1}{2}\nabla G\times\boldsymbol{\sigma}_2-\frac{i}{2}\nabla G\right]-G(\epsilon_1-\mathcal{A})-\frac{i}{2}\nabla G\cdot\boldsymbol{\alpha}_2+\beta_1 m_1\right]\Psi=0\ , \tag{51a}$$

$$\left[\boldsymbol{\alpha}_2\cdot\left[-G\mathbf{p}+\tfrac{1}{2}\nabla G\times\boldsymbol{\sigma}_1+\frac{i}{2}\nabla G\right]-G(e_2-\mathcal{A})+\frac{i}{2}\nabla G\cdot\boldsymbol{\alpha}_1+\beta_2 m_2\right]\Psi=0\ . \tag{51b}$$

Their sum produces the Breit-type equation

$$\left[\boldsymbol{\alpha}_1\cdot\left[G\mathbf{p}-\tfrac{1}{2}\nabla G\times\boldsymbol{\sigma}_2\right]+\boldsymbol{\alpha}_2\cdot\left[-G\mathbf{p}+\tfrac{1}{2}\nabla G\times\boldsymbol{\sigma}_1\right]-G(w-2\mathcal{A})+\beta_1 m_1+\beta_2 m_2\right]\Psi=0\ . \tag{52}$$

This is similar to the Breit equation but with a Pauli coupling. Note however that our wave function is governed by two simultaneous equations so that this Breit-type equation is accompanied by say the difference of the two Dirac equations. This difference equation is missing from those treatments that employ the single Breit equation and a similar treatment that employs a single equation recently proposed by Barut.[19] Without a difference equation of this type one cannot obtain two separate free Dirac equations from the Breit equation as the particles move out of range of their mutual interaction unless additional restrictions are imposed on the solutions. These restrictions must be compatible with the full relativistic dynamics of the interacting system. Note also that the cancellation that occurs when the two Dirac equations are added shows that is is risky to make guesses of the form of separate two-body Dirac equations based on the Breit or a Breit-type equation. They are likely to be incompatible.[20]

For the benefit of the reader who may still be hesitant to adopt a new formalism for the two-body problem we mention here an additional attractive feature. The exact equation [Eq. (8)] for $\mathcal{V}=S=0$, $\mathcal{A}=-\alpha/r$, and equal masses reduces to [using $S_T(-)^s=-S_T$, $(-)^s\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3)/2$]

$$\Bigg\{\mathbf{p}^2-\frac{2\epsilon_w\alpha}{r}-\frac{\alpha^2}{r^2}+\frac{3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{1+2\alpha/rw}\left[\frac{\pi\alpha\delta^3(\mathbf{r})}{3w}\left[\frac{5w+6mG}{w+2mG}\right]+\frac{1}{12w^2}\frac{\alpha^2}{r^4}\frac{1}{1+2\alpha/wr}\frac{21w^2+52wmG+36m^2G^2}{(w+2mG)^2}\right]$$
$$-\frac{S_T}{3(1+2\alpha/wr)}\frac{w}{w+2mG}\left[-\frac{3\alpha}{wr^3}+\frac{2}{1+2\alpha/wr}\frac{\alpha^2}{r^4}\frac{3w+4mG}{w+2mG}\right]$$
$$+\frac{4\mathbf{L}\cdot\mathbf{S}}{rw(1+2\alpha/wr)}\frac{\alpha}{r^2}\frac{w+mG}{w+2mG}\Bigg\}\Psi_{++}=b^2\Psi_{++}\ . \tag{53}$$

For singlet positronium this further reduces to

$$\left[\mathbf{p}^2-\frac{2\epsilon_w\alpha}{r}-\frac{\alpha^2}{r^2}\right]\Psi_{++}=b^2\Psi_{++}\ . \tag{54}$$

That is, residual Darwin and spin-spin terms in Φ_{SS} (of $O(1/c^4)$ since the $O(1/c^2)$ terms cancel among themselves in Φ_{SS}) cancel exactly with Φ_{DO} for singlet states. This can be shown by using $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=-3$ and the fact that since the states are spin singlets, the $\mathbf{L}\cdot\mathbf{S}$ and tensor terms vanish identically. The exact eigenvalue for the eigenvalue equation Eq. (54) is

$$w=\sqrt{2}\,m\left[1+\left[1+\frac{\alpha^2}{(n-\delta_l)^2}\right]^{-1/2}\right]^{1/2}, \tag{55}$$

where

$$\delta_l=-\tfrac{1}{2}-l+[(l+\tfrac{1}{2})^2-\alpha^2]^{1/2}\ . \tag{56}$$

We presented its derivation through an alternative reduction of the two-body Dirac equations using the chiral representation of the Dirac γ matrices in Ref. 8. The resulting spectrum is correct through order α^4 (vacuum and Lamb-shift corrections are not included). The advantage of the chiral representation is that along with the spectral results, we obtain the exact 16-component wave function as well. When one uses the chiral representation of the Dirac γ matrices, one discovers as a by-product that one of the second-order equations analogous to the Pauli forms [Eq. (8)] reduces (for weak potentials) to a relativistic equation recently postulated by Pilkuhn[21] while the other generates a new equation. This fact along with the existence of the Breit-type equation [Eq. (52)] illustrates an important but unusual property of our method. Since our wave function is governed by two simultaneous but compatible wave equations, these equations can be rearranged in large number of equivalent structures—some old, some new. The process of rearrangement supplies new relationships that exist in our method among other wise competing descriptions of the two-body system.

VI. DERIVATION OF TODOROV'S INHOMOGENEOUS QUASIPOTENTIAL EQUATION FOR THE CONSTRAINT POTENTIAL Φ_w

Beyond their incorporation of the chiral limit, why is it that our equations are able to capture light-quark and heavy-quark effects with a minimum of inserted potential structure? After all, relativistic quantum constraint equations (whether of the types used by us or Sazdjian) are the relativistic counterparts to the Schrödinger equation induced by canonical quantization of a peculiar form of relativistic classical mechanics. At the very least they incorporate correct relativistic free-particle kinematics. But more importantly Todorov has shown from a postulated inhomogeneous quasipotential equation how the dynamical potentials (Φ_w) of the constraint equation can be obtained from relativistic quantum field theory. Moreover, Sazdjian has shown how to construct constraint mechanics as the quantum-mechanical transform of the Bethe-Salpeter equation. In this section we combine and generalize these ideas to show how the constraint potential Φ_w (and thus the invariants $\mathcal{A},\mathcal{V},S$) can be obtained from the off-shell scattering amplitude T of quantum field theory through derivation of the Todorov inhomogeneous quasipotential equation from the Bethe-Salpeter equation. We limit our discussion in this section to spinless particles. (We extend the argument to include spin in the Appendix.) We use a technique based on an operator generalization of Sazdjian's quantum-mechanical transform of the Bethe-Salpeter wave function.[22,23] We finish this section by discussing some advantages of the constraint formalism over the older Bethe-Salpeter formalism.

The two defining equations of quantum constraint mechanics are

$$P\cdot p\ |\psi\rangle=0 \tag{57}$$

and

$$(p_\perp^2-b^2)|\psi\rangle=-\Phi_w(x_\perp)|\psi\rangle\ , \tag{58}$$

where

$$p=\frac{\epsilon_2 p_1-\epsilon_1 p_2}{w},\quad \epsilon_1+\epsilon_2=w,\quad \epsilon_1-\epsilon_2=\frac{m_1^2-m_2^2}{w}\ ,$$

$$b^2(w)=\frac{1}{4w^2}[w^4-2(m_1^2+m_2^2)w^2+(m_1^2-m_2^2)^2]\ ,$$

$$P=p_1+p_2, w^2=-P^2\ .$$

We will determine Φ_w from field theory using both the homogeneous Bethe-Salpeter equation written in the form

$$(G_0^{-1}+U)|\chi\rangle=0\ , \tag{59}$$

where

$$G_0=G_1G_2$$

(G_1 and G_2 are the single-particle Green's functions) and the inhomogeneous equation

$$T=U+UG_0T \tag{60}$$

to derive the system constraint equation (58). In Eq. (60), T is the off-mass-shell Feynman-scattering amplitude and U is the Bethe-Salpeter potential.

Let $|\tilde\psi\rangle=\mathcal{J}|\psi\rangle$ with $\mathcal{J}=1$ on mass shell and $[P\cdot p,\mathcal{J}]=0$ so that $P\cdot p\ |\tilde\psi\rangle=0$ just as for the constraint state vector in (57). $\mathcal{J}$ is otherwise arbitrary. We assume, as does Sazdjian, that the Bethe-Salpeter and constraint wave functions are related by a linear transformation:

$$|\tilde\psi\rangle=|\chi\rangle+BU|\chi\rangle \tag{61}$$

with B an unknown operator. Then when we use (59) in the form $|\chi\rangle=G_0U|\chi\rangle+|\chi_0\rangle$, where $|\chi_0\rangle$ is the free-particle state satisfying $(p_i^2+m_i^2)|\chi_0\rangle=0$, the first constraint condition (57) implies

$$P\cdot p\ |\tilde\psi\rangle=0=P\cdot p(G_0+B)U\ |\chi\rangle+P\cdot p\ |\chi_0\rangle\ . \tag{62}$$

But $P\cdot p\ |\chi_0\rangle=(p_1^2+m_1^2-p_2^2-m_2^2)|\chi_0\rangle=0$. Thus

$$P\cdot p(G_0+B)U\ |\chi\rangle=0\ ,$$

which implies that

$$G_0+B=\pi i\delta(P\cdot p)\mathcal{G}\ . \tag{63}$$

$\mathcal{G}$ is an operator "integration constant" and may be a function of P, $p_\perp$, and $x_\perp$ and thus commutes with $P\cdot p$. We let

$$\mathcal{G}=\mathcal{I}\frac{1}{p_\perp^2-b^2-i0} \tag{64}$$

so that the unknown operator

$$B=\pi i\mathcal{I}\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}-G_0 \tag{65}$$

is given in terms of the unknown operator $\mathcal{I}$.

We then demand that the most singular parts of B cancel in the approach to the mass shell ($U\rightarrow 0$) so that

$$|\tilde\psi\rangle\rightarrow|\chi\rangle\ .$$

Now, near the mass shell,

$$G_0\rightarrow\frac{1}{p_1^2+m_1^2-i0}\,\frac{1}{p_2^2+m_2^2-i0} \tag{66}$$

so that the most singular part of B is

$$(\pi i)^2[\mathcal{I}\delta(P\cdot p)\delta(p_\perp^2-b^2)-\delta(p_1^2+m_1^2)\delta(p_2^2+m_2^2)]\ , \tag{67}$$

which vanishes if

$$\mathcal{J}\,|_{\text{on shell}}=1 \tag{68}$$

since $\delta(P\cdot p)\delta(p_\perp^2-b^2)=\delta(p_1^2+m_1^2)\delta(p_2^2+m_2^2)$. $\mathcal{J}$ is otherwise arbitrary.

We now proceed to derive the constraint equation (58) and from that a relativistic Lippmann-Schwinger-type equation for Φ_w. Substitution of the expression for B in (65) into (61) yields

$$|\tilde{\psi}\rangle=\mathcal{I}\,|\psi\rangle=|\chi\rangle+\pi i\mathcal{J}\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}U\,|\chi\rangle-G_0U\,|\chi\rangle$$

$$=\pi i\mathcal{J}\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}U\,|\chi\rangle+|\chi_0\rangle\ . \tag{69}$$

Next we let $\mathcal{I}=\mathcal{J}$ so that $\mathcal{I}\,|\psi\rangle=\mathcal{J}\,|\psi\rangle$ and use $|\chi_0\rangle=\mathcal{J}\,|\chi_0\rangle$. Then

$$|\psi\rangle=i\pi\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}U\,|\chi\rangle+|\chi_0\rangle\ . \tag{70}$$

Using $(p_\perp^2-b^2)\,|\chi_0\rangle=(p^2-b^2)\,|\chi_0\rangle=0$ we find that

$$(p_\perp^2-b^2)\,|\psi\rangle=\pi i\delta(P\cdot p)\mathcal{J}U\,|\chi\rangle\ . \tag{71}$$

To proceed with the derivation of (58) we need to write $|\chi\rangle$ in terms of $|\psi\rangle$ by using the transform equation (61). But that transform depends on the nonunique function $\mathcal{J}$. That is,

$$\mathcal{J}\,|\psi\rangle=|\chi\rangle+\left[i\pi\mathcal{J}\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}-G_0\right]|\chi\rangle\ . \tag{72}$$

However, if we rewrite this as

$$\mathcal{J}\left[|\psi\rangle-i\pi\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}U\,|\chi\rangle\right]=|\chi\rangle-G_0\,|\chi\rangle \tag{73}$$

and use (70) we obtain

$$\mathcal{J}\left[|\psi\rangle-i\pi\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}U\,|\chi\rangle\right]=\mathcal{J}\,|\chi_0\rangle=|\chi_0\rangle=|\psi\rangle-i\pi\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}U\,|\chi\rangle=|\chi\rangle-G_0\,|\chi\rangle\ . \tag{74}$$

Thus (72) reduces to

$$|\psi\rangle=|\chi\rangle+\left[i\pi\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}-G_0\right]U\,|\chi\rangle \tag{75}$$

so that the nonunique factor $\mathcal{J}$ disappears[23] from the transform equation. Inverting this we can write (71) as

$$(p_\perp^2-b^2)\,|\psi\rangle=\pi i\delta(P\cdot p)U\left[1+\left[i\pi\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}-G_0\right]^{-1}U\right]|\psi\rangle\ . \tag{76}$$

This equation is of the form of the constraint equation (58) so that comparison leads at once to an equation for Φ_w in terms of U:

$$\left\{\Phi_w+\pi i\delta(P\cdot p)U\left[1-\left[G_0-\pi i\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}\right]U\right]^{-1}\right\}|\psi\rangle=0\ . \tag{77}$$

To obtain the Todorov inhomogeneous quasipotential equation we then solve (77) perturbatively with the use of (60). We find

$$\Phi_w^{(1)}\,|\psi\rangle=-\pi i\delta(P\cdot p)U^{(1)}\,|\psi\rangle=-\pi i\delta(P\cdot p)T^{(1)}\,|\psi\rangle\ , \tag{78}$$

$$\Phi_w^{(2)}\,|\psi\rangle=-\pi i\delta(P\cdot p)U^{(2)}\,|\psi\rangle-\pi i\delta(P\cdot p)U^{(1)}\left[G_0-\pi i\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}\right]U^{(1)}\,|\psi\rangle\ . \tag{79}$$

Note that

$$U^{(2)}=T^{(2)}-T^{(1)}G_0T^{(1)}\ . \tag{80}$$

Hence,

$$\Phi_w^{(2)}\,|\psi\rangle=\left[-\pi i\delta(P\cdot p)T^{(2)}+\pi i\delta(P\cdot p)T^{(1)}\pi i\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}T^{(1)}\right]|\psi\rangle\ . \tag{81}$$

Continuing with this perturbative solution we see that (77) is formally solved by

$$\left[\Phi_w+\pi i\delta(P\cdot p)T+\Phi_w\pi i\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}T\right]|\psi\rangle=0\ . \quad (82)$$

This equation which we have just derived is a form of the Todorov equation more restrictive than the requirement of an operator equation of the form

$$\Phi+\pi i\delta(P\cdot p)T+\Phi\pi i\frac{\delta(P\cdot p)}{p_\perp^2-b^2-i0}T=0 \quad (83)$$

with no further restrictions on T. However, since T operates on a $|\psi\rangle$ which satisfies $P\cdot p\,|\psi\rangle=0$ while $\delta(P\cdot p)$ appears on the left, we see that (83) is an operator identity only if the scattering matrix T is evaluated off shell such that in momentum space its matrix elements satisfy

$$T(q_1',q_2';p_1',p_2')=T_{P'}(p',q')\ , \quad (84)$$

where the relative momenta p' and q' satisfy $P'\cdot p'=0$ and $P'\cdot q'=0$. But this is just the off-mass-shell extrapolation of T that Todorov originally postulated[14] for the quasipotential equation (83) for Φ_w. In our applications we use (82) or its perturbative forms (78) and (81).

In summary, if we start with just the constraint condition $P\cdot p\,|\psi\rangle=0$ (a covariant control of the relative energy), then the Bethe-Salpeter equation (in both its homogeneous and inhomogeneous forms) leads to the constraint eigenvalue equation for $|\psi\rangle$ (58) with Φ_w given by the Todorov quasipotential equation (82).

The lowest-order potential identification (78) leads to a local potential. As an example, consider a simple scalar Yukawa field theory. From field theory we know that, in momentum space,

$$T_{P'}^{(1)}(p',q')=\frac{\delta(p_1'+p_2'-q_1'-q_2')}{(p'-q')^2+\mu^2-i0}g_1g_2\ . \quad (85)$$

If we let $\langle x_1x_2|\psi\rangle=e^{iP\cdot X}\psi(x_\perp)$ where $X=(\epsilon_1x_1+\epsilon_2x_2)/w$, then the bound-state equation

$$(p_\perp^2+\Phi_w^{(1)})\,|\psi\rangle=b^2(w)\,|\psi\rangle \quad (86)$$

becomes

$$(-\partial_\perp^2-b^2)\psi(x_\perp)+\pi g_1g_2\int\frac{d^4p'd^4z}{(2\pi)^4}e^{ip'\cdot(x-z)}\delta(P\cdot p')\Delta_F(z,\mu)\psi(z_\perp)=0\ . \quad (87)$$

After we perform the $z_\parallel$ and $p_\parallel$ integrations we find that

$$(-\partial_\perp^2-b^2)\psi(x_\perp)+\frac{\pi g_1g_2}{w}\int\frac{d^3z_\perp d^3p_\perp'}{(2\pi)^4}e^{ip'\cdot(x-z)_\perp}\frac{e^{-\mu z_\perp}}{4\pi z_\perp}\psi(z_\perp)=0\ , \quad (88)$$

which in turn becomes the local covariant Schrödinger-type equation:

$$\left[-\partial_\perp^2-2m_w\alpha\frac{e^{-\mu|x_\perp|}}{|x_\perp|}\right]\psi(x_\perp)=b^2(w)\psi(x_\perp)\ , \quad (89)$$

where

$$m_w=\frac{m_1m_2}{w},\quad g_1g_2=16\pi m_1m_2\alpha\ . \quad (90)$$

Equation (89) is a local homogeneous quasipotential equation of the type given by Todorov in 1971 (Ref. 14). Comparison of the form of (89) with that of (8) then determines the invariant S, the scalar interaction corresponding to a field theory for spinless particles as $-\alpha e^{-\mu|x_\perp|}/|x_\perp|$. Notice that the relativistic reduced mass m_w appears as a natural outgrowth of the field-theoretic connection. (Of course this S is not the S we used in the quark model calculations. No one has yet as far as we know developed a method for extracting the scalar portion of the nonperturbative QCD potential.)

Even though we have been able to connect the constraint approach to quantum field theory and we have been able in previous sections to give a successful phenomenological test of the constraint approach, why should we adopt this approach rather than the time honored Bethe-Salpeter equation? There are two sets of reasons why we choose the constraint equation. The first set consists of its advantages over the four-dimensional form of the Bethe-Salpeter equation, in which there is no separation of an instantaneous part of the interaction (but which like the constraint approach is manifestly covariant) while the second set consists of its advantages over the instantaneous (three-dimensional) forms of the Bethe-Salpeter equation.

The original four-dimensional form of the Bethe-Salpeter equation has a set of extra unphysical solutions that arise from the presence of an uncontrolled relative time (or the conjugate variable, the relative energy).[24] These solutions are absent from the constraint approach because the difference of the defining constraints on the wave function yields $P\cdot p\,|\psi\rangle=0$. In addition, workers often apply the Bethe-Salpeter equation to the two-particle bound-state problem with only the ladder approximation. This approximation does not yield (in the four-dimensional form) the correct limit of a relativistic (Klein-Gordon or Dirac) equation in the limit in which one particle becomes infinitely heavy. In fact, the ladder approximation leads, in calculations of relativistic corrections to incorrect $O(1/c)$ as opposed to $O(1/c^2)$ modifications.[24–27] In the context of the Balmer formula, this would give incorrect contributions of order $\alpha^3\ln\alpha$ to the total c.m. energy.[24–27] In contrast the constraint equations automatically give the correct heavy-particle limits in the ladder approximation and do not lead to spurious $O(1/c)$ or $\alpha^3\ln\alpha$ energy terms. This makes them well suited for phenomenological applications. These problems with the Bethe-Salpeter equation make its use in phenomenological calculations suspect. While the erroneous contributions in the Bethe-Salpeter equation can be eliminated by including the irreducible cross ladder diagram in the Bethe-Salpeter kernel U, one need only use the Born term in the constraint equation to obtain the same accuracy. In fact with the Born diagram alone as input, the quasipotential equation [or constraint equation (58)] sums up all cross ladder and ladder diagrams in the limit of small exchanged mass and momentum transfer.[14] The Bethe-Salpeter equation[26,27] does not give this correct result in the ladder approximation. Tiktopoulos and Treiman[28] found that the reason for this is that the leading asymptotic term in each ordinary ladder diagram is exactly canceled by the leading terms of the corresponding set of cross ladder diagrams so that one has to take into account an infinite number of irreducible diagrams in the kernel of the Bethe-Salpeter equation in order to obtain the result (the eikonal approximation[29]) which is produced by taking the Born term alone in the constraint or quasipotential approach.

These problems with the Bethe-Salpeter equation in the ladder approximation occur when one uses the four-dimensional kernel $1/k^2$ to compute the unperturbed wave function. However, most of these problems do not appear in the various equal time reductions of the Bethe-Salpeter equation, in which one uses the three-dimensional kernel $1/k_\perp^2$ ($1/\mathbf{k}^2$ in the c.m. system) to compute the unperturbed wave function.

In scalar field theories, one separates the four-dimensional kernel $1/k^2$ as

$$\frac{1}{k^2}=\frac{1}{k_\perp^2}+\frac{1}{k^2}\frac{(k\cdot\hat{n})^2}{k_\perp^2},$$

where $k_\perp = k + k\cdot\hat{n}\hat{n}$ (satisfying $k_\perp\cdot\hat{n}=0$). $\hat{n}$ is an arbitrary timelike four-vector (usually chosen to be $\hat{P}$) that defines the equal-time frame. In vector field theories in the Feynman gauge,

$$\frac{\gamma_1\cdot\gamma_2}{k^2}=-\frac{\gamma_1\cdot\hat{n}\gamma_2\cdot\hat{n}}{k_\perp^2}+\frac{1}{k^2}\left[\gamma_{1\perp}\cdot\gamma_{2\perp}-\frac{(k\cdot\hat{n})^2\gamma_1\cdot\hat{n}\gamma_2\cdot\hat{n}}{k_\perp^2}\right],$$

while in the Coulomb gauge,

$$\frac{\gamma_1\cdot\gamma_2}{k^2}\to-\frac{\gamma_1\cdot\hat{n}\gamma_2\cdot\hat{n}}{k_\perp^2}+\frac{1}{k^2}\left[\gamma_\perp\cdot\gamma_{2\perp}-\frac{\gamma_1\cdot k_\perp\gamma_2\cdot k_\perp}{k_\perp^2}\right].$$

These three-dimensional rearrangements into instantaneous (in the c.m. system) Coulomb plus perturbation do lead to the correct α^4 energy corrections if the first part is used to compute the unperturbed wave functions.[30] Furthermore these divisions do not produce any spurious $O(\alpha^3\ln\alpha)$ terms in addition to the correct $O(\alpha^2)$ terms in the unperturbed part of the spectrum.[31] Note that the choice of the Coulomb gauge in effect forces a three-dimensional ($1/k_\perp^2$) rather than four-dimensional ($1/k^2$) propagator in the kernel so that like the constraint approach the Bethe-Salpeter equation in the Coulomb gauge provides a legitimate tool for phenomenological quark model studies. (It is in standard use in e^+e^- calculations.)

However, in order to maintain the manifest covariance of these three-dimensional rearrangements of the Bethe-Salpeter equation one must introduce by hand certain invariants (e.g., $k\cdot\hat{P}$, $k_\perp$) that do not arise naturally in the theory. One must decide in which frame the instantaneous part of the interaction is to be instantaneous. In contrast, in the constraint approach, the variable $x_\perp$, unlike its conjugate counterpart, $k_\perp$ in the Bethe-Salpeter equation, arises as a mathematical necessity for the consistency of the theory and is not put in by hand. In addition, unlike the "instantaneous" Bethe-Salpeter equation in the ladder approximation, the constraint equation is local because of the simple momentum dependence of its Schrödinger-type form. Finally, one need not divide the single-photon-exchange diagram into instantaneous plus perturbative parts. This fact is most strikingly demonstrated by the existence of an exact solution of the constraint equation for singlet positronium with correct spectrum through order α^4. The Bethe-Salpeter equation has no known exact solution for singlet positronium with single-photon exchange.

VII. CONCLUSION

In this paper we have used two coupled compatible Dirac equations to calculate the mass spectrum of mesonic states composed of light or heavy quarks. The quality of the resulting fits and predictions is a consequence of the accuracy of the original nonrelativistic potentials that our method uses as its starting point, the physical effects of choices we must make in order to identify a family of relativistic potentials that together extrapolate the nonrelativistic behavior of heavy-quark potentials to the relativistic regime, and (of greatest concern here) the spin-dependent interaction structures produced by our extension of Dirac's single-particle wave equation to two coupled Dirac equations.

The various connections of this work to relativistic quantum field theory argue that as more becomes known experimentally and theoretically about the relativistic potential structure generated by quantum chromodynamics, more detailed equations of this form may lead to still better spectral results. Conversely, the relativistic spin structures of our two-body Dirac equations already distinguish among extrapolations of heavy-quark potentials of various types (e.g., suggesting that the real chromodynamic structure is badly modeled by the EF potential at short distance or that schemes that possess only relativistic scalar potentials at long distance may invert spin-orbit splittings). As we have seen, these equations add their own relativistic treatment of mass parameters to their handling of relativistic potentials to generate automatically the symptoms of approximate chiral symmetry. Their spin structures, which are fixed by the unfamiliar algebraic principle of compatibility of two independent constituent Dirac equations, already contain not only the usual semirelativistic spin structure of atomic physics (where such expansions are valid) but also the relativistic weak potential structure of Todorov's quasipotential equation. Beyond any particular structures that these equations assume in the presence of particular potentials, their general significance lies in their connection to quantum field theory as a quantum-mechanical transform of the Bethe-Salpeter equation. In this light, our investigations here assume the same character as those of other workers who attempt to flesh out the full nonperturbative field-theoretic two-body interaction structure generated by a strong field by studying the effects of candidate interactions or approximations to the full kernel in the Bethe-Salpeter equation.

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APPENDIX: DERIVATION OF THE QUASIPOTENTIAL Φ_w AND INVARIANTS $\mathcal{A}$ and S FOR TWO SPIN-$\frac{1}{2}$ PARTICLES

To include spin we need only make a few modifications in the analysis given in Sec. VI. First, the spinless Green's functions G_1 and G_2 of that section are replaced by the corresponding Dirac forms. One then multiplies both sides of (59) by $(\gamma_1\cdot p_1+m_1)(\gamma_2\cdot p_2+m_2)/m_1m_2$. This has the effect of replacing the U in the spinless form of that equation by U multiplied by this same factor. After performing a similar operation on the spin-$\frac{1}{2}$ version of (60) one can proceed with the proof as in the spinless case. The question that remains is how, say, $\mathcal{A}$ and S are determined from T? Let us recall that our Pauli equation [Eq. (8)] separates into four decoupled four-component equations. Now the T matrix is a 16×16 matrix of the form $e_1^*e_2^*f(s,t,u)1^{(1)}1^{(2)}$ for scalar interactions and $e_1e_2f(t,s,u)\gamma_1^\mu\gamma_{2\mu}$ for vector interactions. [For timelike four-vector interactions the matrix amplitude would be of the form $e_1e_2f(t,s,u)\gamma_1\cdot\hat{P}\gamma_2\cdot\hat{P}$.] Following the work of Aneva, Karpchev, and Rizov in Ref. 14 (for similar constructions see Ref. 32), we construct the corresponding upper-upper 4×4 amplitude from the 16×16 amplitude in the following way. Define first the invariant scattering amplitude

$$\mathcal{T}(p_1,\lambda_1,p_2,\lambda_2;q_1,\kappa_1,q_2,\kappa_2)=\sum\sum\bar{u}_{\lambda_1\alpha_1}(p_1)\bar{u}^{\,c}_{\lambda_2\alpha_2}(p_2)\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}(p_1,p_2;q_1,q_2)u_{\kappa_1\beta_1}(q_1)u^{c}_{\kappa_2\beta_2}(q_2)\;. \tag{A1}$$

$\mathcal{T}$ is a 16×16 amplitude such as given above. We are interested only in a lowest-order (nonrelativistic) determination of $\mathcal{A}$ and S, so we need only use

$$\Phi^{(1)}|\psi\rangle=-\pi i\delta(P\cdot p)T^{(1)}|\psi\rangle\;. \tag{A2}$$

We specialize this equation to its upper-upper components since this is sufficient to determine $\mathcal{A}$ and S to lowest order. The $4\times 4T$ matrix needed for computing Φ_w is defined by

$$\mathcal{T}(p_1,\lambda_1,p_2,\lambda_2;q_1,\kappa_1,q_2,\kappa_2)=\sum\sum\bar{e}^{(1)a_1}_{\lambda_1}\bar{e}^{(2)a_2}_{\lambda^2}T_{a_1a_2b_1b_2}(p_1,p_2;q_1,q_2)e^{(1)b_1}_{\kappa_1}e^{(2)b_2}_{\kappa_2}\;. \tag{A3}$$

Using

$$u_{\kappa_1}(q_1)=\frac{\epsilon_1+m_1+\boldsymbol{\alpha}_1\cdot\mathbf{q}_1}{\sqrt{\epsilon_1+m_1}}\begin{bmatrix}e^{(1)}_{\kappa_1}\\0\end{bmatrix},\quad \bar{u}_{\lambda_1}(p_1)=(\bar{e}^{(1)}_{\lambda_1}0)\frac{\epsilon_1+m_1-\bar{\boldsymbol{\alpha}}_1\cdot\mathbf{p}_1}{\sqrt{\epsilon_1+m_1}}\;,$$

we find

$$\bar{u}_{\lambda_1}(p_1)1^{(1)}u_{\kappa_1}(q_1)=\frac{\bar{e}^{(1)}_{\lambda_1}[(\epsilon_1+m_1)^2-b^2+\frac{1}{2}(\mathbf{p}_1-\mathbf{q}_1)^2-i\boldsymbol{\sigma}_1\cdot\mathbf{q}_1\times\mathbf{p}_1]e^{(1)}_{\kappa_1}}{\epsilon_1+m_1}\;,$$

$$\bar{u}^{\,c}_{\lambda_2}(p_2)1^{(2)}u^{c}_{\kappa_2}(q_2)=\frac{\bar{e}^{(2)}_{\lambda_2}[(\epsilon_2+m_2)^2-b^2+\frac{1}{2}(\mathbf{p}_2-\mathbf{q}_2)^2-i\boldsymbol{\sigma}_2\cdot\mathbf{q}_2\times\mathbf{p}_2]e^{(2)}_{\kappa_2}}{\epsilon_2+m_2}\;,$$

$$\bar{u}_{\lambda_1}(p_1)\gamma_1^0u_{\kappa_1}(q_1)=\frac{\bar{e}^{(1)}_{\lambda_1}[(\epsilon_1+m_1)^2-b^2-\frac{1}{2}(\mathbf{p}_1-\mathbf{q}_1)^2+i\boldsymbol{\sigma}_1\cdot\mathbf{q}_1\times\mathbf{p}_1]e^{(1)}_{\kappa_1}}{\epsilon_1+m_1}\;,$$

$$\bar{u}^{\,c}_{\lambda_2}(p_2)\gamma_2^0u^{c}_{\kappa_2}(q_2)=\frac{\bar{e}^{(2)}_{\lambda_2}[(\epsilon_2+m_2)^2-b^2-\frac{1}{2}(\mathbf{p}_2-\mathbf{q}_2)^2+i\boldsymbol{\sigma}_2\cdot\mathbf{q}_2\times\mathbf{p}_2]e^{(2)}_{\kappa_2}}{\epsilon_2+m_2}\;,$$

$$\bar{u}_{\lambda_1}(p_1)\gamma_1^iu_{\kappa_1}(q_1)=\bar{e}^{(1)}_{\lambda_1}[(q_1+p_1)_i+i\epsilon_{ijk}(q_1-p_1)_j\sigma_{1k}]e^{(1)}_{\kappa_1}\;,$$

$$\bar{u}^{\,c}_{\lambda_2}(p_2)\gamma_2^iu_{\kappa_2}(q_2)=\bar{e}^{(2)}_{\lambda_2}[(q_2+p_2)_i+i\epsilon_{ijk}(q_2-p_2)_j\sigma_{2k}]e^{(2)}_{\kappa_2}\;.$$

These can be simplified using

$$(\mathbf{p}_1+\mathbf{q}_1)\cdot(\mathbf{p}_2+\mathbf{q}_2)=-(\mathbf{p}_1+\mathbf{q}_1)^2$$
$$=-4b^2+(\mathbf{p}_1-\mathbf{q}_1)^2 ,$$

$$\frac{(\epsilon_i+m_i)^2-b^2}{\epsilon_i+m_i}=2m_i, \quad \frac{(\epsilon_i+m_i)^2+b^2}{\epsilon_1+m_1}=2\epsilon_i ,$$

and

$$4\epsilon_1\epsilon_2-4b^2=4w\epsilon_w .$$

Then we find that for combined scalar and vector interactions to lowest order that the 4×4 amplitude analogous to (85) is

$$T_P^{(1)}(p,q)=e_1^* e_2^* \frac{\delta(p_1+q_1-p_2-q_2)4m_1m_2}{(p-q)^2+\mu_s^2-i0} + e_1e_2\frac{(\delta(p_1+q_1-p_2-q_2)4w\epsilon_w}{(p-q)^2+\mu_A^2-i0} . \quad (A4)$$

Following steps analogous to (67)–(69) we find

$$\left[-\partial_\perp^2-2m_w\alpha_s\frac{e^{-\mu_s|x_\perp|}}{|x_\perp|}-2\epsilon_w\alpha_A\frac{e^{-\mu_A|x_\perp|}}{|x_\perp|}\right]\psi(x_\perp) = b^2(w)\psi(x_\perp) \quad (A5)$$

which through comparison with (8) would allow us to identify $\mathcal{A}$ and S. One could further show that if the $O(1/c^2)$ terms are included then with the same identifications of $\mathcal{A}$ and S we would reproduce (to that order) the corresponding Darwin, spin-orbit, tensor, and spin-spin terms appearing in (8). What is noteworthy is that we only need to identify the nonrelativistic parts, i.e., the lowest-order forms of $\mathcal{A}$ and S. The spin-dependent and covariant structure of the two-body Dirac formalism then automatically stamps out the correct semirelativistic corrections.

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[31]If one did not perform either of these three-dimensional rearrangements but instead made the naive division

$$\frac{\gamma_1\cdot\gamma_2}{k^2}=-\frac{\gamma_1\cdot\hat{n}\gamma_2\cdot\hat{n}}{k^2}+\frac{\gamma_{1\perp}\cdot\gamma_{2\perp}}{k^2}$$

in the Feynman gauge into unperturbed and perturbed parts then the first part would lead to an unperturbed wave function with spurious $\alpha^3\ln\alpha$ terms in the energy (Ref. 27). In the Feynman gauge one can make the "wrong choice." This is very difficult to do in the Coulomb gauge.

[32]Note that similar quasipotential-like formalisms have been used in recent works by S. Gupta, S. F. Radford, and W. W. Repko, Phys. Rev. D **28**, 1132 (1983); and Godfrey and Isgur (Ref. 9).