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From QCD to the low-energy effective action through composite fields: Goldstone's theorem and f_π

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We elaborate upon a formal derivation from QCD of a composite-field (bilocal) effective chiral Lagrangian which we presented in a previous work. We give here a detailed treatment of the transition to the local form of the chiral Lagrangian in the low-energy limit, when chiral symmetry is spontaneously broken. We find an explicit form of the Goldstone-boson mode wave function and use it to derive expressions for the pion decay constant in terms of the quark dynamical mass Σ by two different methods. Using several models for Σ appropriate to QCD and one appropriate to technicolor, we compare the values given by our two different f_π expressions with each other and with the values for f_π given by an expression due to Pagels and Stokar. We find good agreement among the various methods of calculating f_π, and the agreement with the experimental value is satisfactory for a condensate mass scale in the range 0.5–1.0 GeV.

I. INTRODUCTION

A complete derivation of the low-energy properties of asymptotically free, non-Abelian gauge theories is still beyond present-day theoretical capabilities, requiring as it does the solution to a strongly coupled, nonlinear system. However, interest in achieving such a derivation runs very deep. At stake is the confirmation that the tremendous amount of phenomenological success of chiral dynamics in accounting for low-energy hadron phenomenology is indeed explained by quantum chromodynamics (QCD). An important, related issue is that much of the speculation of new dynamics at high-energy frontiers, such as technicolor at 1 TeV, is based on the pattern laid down by QCD with spontaneous chiral-symmetry breaking, would-be Goldstone-boson pions and corresponding low-energy effective Lagrangians.

In the past several years, a number of papers[1–6] have presented lines of argument, with partial derivations, which lead to chiral effective Lagrangians[7] of composite Goldstone bosons starting from a non-Abelian gauge theory of quarks and gluons (QCD, for our purpose of discussion).[8] After integrating over gluon degrees of freedom, the resulting action functional is naturally expressed in terms of bilocal quark pairings,[2–4] which lend themselves to replacement by auxiliary, bilocal, bosonic degrees of freedom—i.e., composite bosons.[9] It is on this bilocal effective-field-theory approach and on the steps by which a local, chiral effective action in terms of Goldstone-boson fields is achieved that we concentrate in the present work. Related issues of obtaining the effective, local bosonic chiral Lagrangian, including the Wess-Zumino term, for a local "quark" plus composite boson Lagrangian[10–16] are not addressed here. An alternative and complementary approach to the effective action formulation which we adopt here is that of introducing the bilocal fields in the QCD functional integral in order to derive the usual Schwinger-Dyson and Bethe-Salpeter equations used to study the properties of the bosonic spectrum and couplings.[17,18] The results of previous works[19] using those equations to generalize the findings of Nambu and Jona-Lasinio[20] on chiral-symmetry breaking provide guidance in our considerations in Sec. III on the conditions under which a local, Goldstone mode Lagrangian is achieved when a spontaneously broken chiral-symmetry-breaking solution is assumed in the bilocal fields situation.

Our aim in this work is to clarify and make more explicit the steps by which a *local* pseudoscalar field, representing pion (or pionlike) degrees of freedom, emerges from the *bilocal* low-energy effective field theory which results from integrating over the degrees of freedom of a fundamental Lagrangian. We develop the bilocal field formalism in Sec. II, where we specify the approximations to be used in Sec. III, in which an explicit solution to the low-energy limit of the composite theory is displayed, and the local (free) field theory of the pion is discussed (only the kinetic energy term is developed in detail). By way of illustration, the manner in which the Nambu and Jona-Lasinio[20] model emerges is discussed. In Sec. IV we derive alternate, approximate expressions for the pion decay constant f_π, by considering W-π mixing on the one hand and $\pi^0 \to \gamma\gamma$ on the other. The numerical values of these expressions are compared with each other and with the formula of Pagels and Stokar[21] in several different models for the quark dynamical mass,[4,22,23] and in a model for the techniquark dynamical mass where the coupling is slowly running.[24] Within the errors inherent in the soft-pion approximation used, the results of the different f_π expressions are consistent with each other and in agreement with the experimental value[25] of about 90 MeV, given a reasonable condensation scale of 0.5–1.0 GeV. Section IV summarizes our results and discusses them further. Details of the derivative expansion of the effective action arising from the fermion determinant, the specifics of the kinetic energy diagonalization and normalization integral, and some details of the calculations of f_π are presented in the Appendix.

II. BILOCAL FIELD FORMALISM AND EFFECTIVE CHIRAL LAGRANGIAN

In approaching QCD-type theories at low energy, we consider first the result of integrating over gauge and ghost fields in the generating functional integral for QCD. (Our considerations apply to any non-Abelian gauge theory of fermions and gauge fields, but we will simply refer to QCD for brevity.) Suppressing the explicit dependence on the source terms, we write the generating functional Z schematically as

$$Z=\int D\psi D\bar{\psi} DG\, Dg\, e^{i(S_\psi+S_G+S_g+S_{\rm gf})} , \tag{2.1}$$

and the result of integrating over gauge G and ghost g fields is

$$Z=\int D\psi D\bar{\psi} \exp\left[i\left[\int \bar{\psi} i\partial\!\!\!/ \psi\, dx - \mathrm{Tr}\sum_{n=2}^{\infty}\frac{1}{n!}G_c^{(n)}(x_1,x_2,\ldots,x_n)\Gamma\psi(x_1)\bar{\psi}(x_2)\cdots\Gamma\psi(x_n)\bar{\psi}(x_1)\right]\right] . \tag{2.2}$$

Here S_ψ, S_G, S_g, and $S_{\rm gf}$ refer to the fermionic, gauge, ghost, and gauge-fixing actions, respectively. The (gauge-dependent) connected n-point function for the pure Yang-Mills gauge action is denoted $G_c^{(n)}$, while Γ's refer to the product of Dirac and internal-symmetry matrices, all of whose indices and labels are suppressed. Tr indicates integration over x's and sum over discrete indices.

The bilocal, auxiliary fields $\eta(x,y)$ and $B(x,y)$ can now be introduced in the functional integral in such a way that the fermions appear only quadratically in the exponential.[2,18] We insert the functional δ:

$$1=\int DB\,\delta(B-\psi\bar{\psi})\sim\int D\eta\, DB\,\exp\{-i\,\mathrm{Tr}\eta(x,y)[B(x,y)-\psi(y)\bar{\psi}(x)]\} , \tag{2.3}$$

and rewrite Z in the form

$$Z=\int D\psi D\bar{\psi} D\eta\, DB\,\exp\left[i\,\mathrm{Tr}\left[-i\delta(x-y)\partial\!\!\!/_y\psi(y)\bar{\psi}(x)+\eta(x,y)\psi(y)\bar{\psi}(x) -\eta(x,y)B(y,x)-\sum_{n=2}^{\infty}\frac{1}{n!}G_c^{(n)}\Gamma B(x_1,x_2)\Gamma B(x_2,x_3)\cdots\Gamma B(x_n,x_1)\right]\right] . \tag{2.4}$$

Now the general form of the effective action one obtains after integrating over the B degrees of freedom is

$$W[\psi,\bar{\psi},\eta]=\int dx\,\bar{\psi}(x)i\partial\!\!\!/\psi(x)-\int dx\int dy\,\bar{\psi}(x)\eta(x,y)\psi(y)+W[\eta] , \tag{2.5}$$

where

$$Z=\int D\psi D\bar{\psi} D\eta\, e^{iW[\psi,\bar{\psi},\eta]} \tag{2.6}$$

and

$$e^{iW[\eta]}\equiv\int DB\,\exp\left[-i\,\mathrm{Tr}\left[\eta(x,y)B(y,x)+\sum_n\frac{1}{n!}G_c^{(n)}\Gamma B\cdots\Gamma B\right]\right] . \tag{2.7}$$

The original massless QCD Lagrangian is globally chiral invariant, so the action functional $W[\psi,\bar{\psi},\eta]$ is also chiral symmetric, where one defines the transformation properties of η in accordance with those of ψ so that $\bar{\psi}\eta\psi$ remains chiral invariant. Namely, we have $\psi'=e^{-i\theta_V}\psi$ and $\eta'=e^{-i\theta_V}\eta e^{i\theta_V}$ under vector flavor transformations, and we have $\psi'=e^{-i\theta_A\gamma_5}\psi$ with $\eta'=e^{i\theta_A\gamma_5}\eta e^{i\theta_A\gamma_5}$ under axial-vector flavor transformations. The θ_V and θ_A are $n\times n$ matrices in SU(n)-flavor space, assuming that the fermions are in the fundamental representation.

To be more explicit about the transition to an effective action of composite, bosonic degrees of freedom represented by the bilocal field η, we integrate over the B coordinates as indicated in (2.7). Formally, one can treat this problem similarly to a perturbative expansion by rewriting (2.7) as

$$\exp\left[-i\,\mathrm{Tr}\sum_{n=3}^{\infty}\frac{1}{n!}G_c^{(n)}\Gamma\left[i\frac{\delta}{\delta\eta(x_1,x_2)}\right]\Gamma\left[i\frac{\delta}{\delta\eta(x_2,x_3)}\right]\cdots\Gamma\left[i\frac{\delta}{\delta\eta(x_n,x_1)}\right]\right] \times\int DB\,\exp\{-i\,\mathrm{Tr}[\eta(x,y)B(y,x)+\tfrac{1}{2}G_c^2\Gamma B(x,y)\Gamma B(y,x)]\} . \tag{2.8}$$

The Gaussian integral by itself provides the leading term in the expansion and serves as a first approximation to QCD and is the exact result for QED. The integration over B in Eq. (2.8) can be performed by replacing $\eta(x,y)$ by a new bilocal field $\omega(x,y)$ with

$$\eta(x,y)=G_c^{(2)}(x-y)\Gamma\omega(x,y)\Gamma\equiv\overline{\omega}(x,y)\ ,$$

and by a shift of integration variable. We have then

$$\int DB\exp\{-i\,\mathrm{Tr}[\eta(x,y)B(y,x)+\tfrac{1}{2}\overline{B}(x,y)B(x,y)]\}=\exp\left[\frac{i}{2}\mathrm{Tr}\omega(x,y)\overline{\omega}(y,x)\right]N\ , \qquad (2.9)$$

where N is a number independent of $\omega(x,y)$. Keeping only linear and quadratic terms in $\omega(x,y)$, the generating functional (2.6) has the form, up to a normalization factor and source terms,

$$Z=\int D\psi D\overline{\psi}D\omega\exp\{i\,\mathrm{Tr}[\overline{\psi}(i\partial\!\!\!/-\overline{\omega})\psi+\tfrac{1}{2}\omega\overline{\omega}]\}\ . \qquad (2.10)$$

In the next section we will examine in detail the expression (2.10), which one expects to be dominant in the low-energy and large-N_c limits,[8] and consider the bilocal field version of the Goldstone theorem.

Regarding the SU(N_c)-color content of the bilinear form $\mathrm{Tr}\overline{\omega}\omega$, we note that the color-singlet part dominates for large N_c, providing some justification for the color-singlet confinement assumption which we adopt. In detail, we have

$$\begin{aligned}\mathrm{Tr}_{\mathrm{color}}(\overline{\omega}\omega)&\sim\lambda^a_{\alpha\alpha'}\overline{\omega}_{\alpha'\beta'}\lambda^a_{\beta'\beta}\omega_{\beta\alpha}\\&=\overline{\omega}_{\alpha'\beta'}\omega_{\beta\alpha}2\left[\delta_{\alpha\beta}\delta_{\alpha'\beta'}-\frac{1}{N_c}\delta_{\alpha\alpha'}\delta_{\beta\beta'}\right]\\&=2\left[\overline{\omega}_{\alpha\alpha}\omega_{\beta\beta}-\frac{1}{N_c}\overline{\omega}_{\alpha\beta}\omega_{\beta\alpha}\right]\\&=2\left[(N_c^2-1)\overline{\omega}_1\omega_1-\frac{2}{N_c}\overline{\omega}\,^a_8\omega^a_8\right]\ ,\end{aligned} \qquad (2.11)$$

where $\omega\equiv 1\omega_1+\lambda^a\omega^a_8$.

III. LOW-ENERGY EFFECTIVE ACTION AND THE GOLDSTONE THEOREM FROM THE BILOCAL FIELD POINT OF VIEW

Projecting out gauge-singlet terms and keeping only the leading term in the expansion (2.8), one is left with an effective Abelian model for the dynamical fermion system. We write therefore the generating functional (2.6) as

$$Z=\int D\psi D\overline{\psi}D\omega\exp[i(\mathrm{Tr}\{-[\delta(x-y)i\partial\!\!\!/_y-\overline{\omega}(x,y)]\psi(y)\overline{\psi}(x)+\tfrac{1}{2}\omega(x,y)\overline{\omega}(y,x)\})]\ . \qquad (3.1)$$

Flavor indices are suppressed in Eq. (3.1) [e.g., $\psi_i(x)\overline{\psi}_j(y)\to\psi(x)\overline{\psi}(y)$] and, adopting the Feynman gauge for the following discussion, we have

$$\begin{aligned}\overline{\omega}(x,y)&\equiv\gamma_\mu\omega(x,y)\gamma_\mu G(x-y)\equiv G(x-y)[4\omega_S(x,y)-4\gamma_5\omega_P(x,y)-2\omega\!\!\!/_V(x,y)+2\omega\!\!\!/_A(x,y)\gamma_5]\\&\equiv\overline{\omega}_s-\overline{\omega}_P\gamma_5-\overline{\omega\!\!\!/}_V+\overline{\omega\!\!\!/}_A\gamma_5\ ,\end{aligned} \qquad (3.2)$$

where

$$\omega(x,y)\equiv\omega_S(x,y)+\gamma_5\omega_P(x,y)+\omega\!\!\!/_V+\omega\!\!\!/_A\gamma_5$$

and

$$D\omega=D\omega_S D\omega_P D(\omega_V)_\mu D(\omega_A)_\lambda$$

in the Feynman gauge.

We introduce an effective action $W_{\mathrm{eff}}[\omega]$ according to

$$Z=\int D\omega\exp(iW_{\mathrm{eff}}[\omega])\ , \qquad (3.3)$$

after having integrated over fermion fields which gives

$$W_{\mathrm{eff}}[\omega]=-i\,\mathrm{Tr}\ln(i\partial\!\!\!/-\overline{\omega})+\tfrac{1}{2}\mathrm{Tr}\omega\overline{\omega}\ . \qquad (3.4)$$

The translationally invariant part of $\omega(x,y)$ is related to the fermion propagator as can be seen from the following argument. First we define

$$\omega(x,y)\equiv\sigma(x-y)+\phi(x,y)\ , \qquad (3.5)$$

and introduce $S=(i\partial\!\!\!/-\overline{\sigma})^{-1}$ in order to rewrite Z as

$$Z=\int D\phi\exp\{i[-i\,\mathrm{Tr}\ln(1-S\overline{\phi})+\tfrac{1}{2}\mathrm{Tr}\sigma\overline{\sigma}+\mathrm{Tr}\sigma\overline{\phi}+\tfrac{1}{2}\mathrm{Tr}\phi\overline{\phi}+i\,\mathrm{Tr}\ln S]\}\ . \qquad (3.6)$$

Minimizing $W_{\mathrm{eff}}[\sigma]$ with respect to $\overline{\sigma}$ produces the Schwinger-Dyson, or gap, equation

$$iS(x-y)+\sigma(x-y)=0\ ,$$

i.e.,

$$i(i\partial\!\!\!/-\overline{\sigma})^{-1}=-\sigma\ . \qquad (3.7)$$

Defining

$$\overline{\sigma}(t)\equiv\frac{1}{(2\pi)^4}\int d^4k\,e^{-ik\cdot t}\Sigma(k)$$

and

$$S(t)\equiv\frac{1}{(2\pi)^4}\int d^4k\,e^{-ik\cdot t}S(k)\ ,$$

we have

$$S(k)=[k\!\!\!/-\Sigma(k)]^{-1} \qquad (3.8)$$

and the Schwinger-Dyson equation reads

$$\Sigma(k)=-\frac{i}{(2\pi)^4}\int d^4q\,\gamma_\mu[\not q-\Sigma(q)]^{-1}\gamma_\mu G(k-q) \quad (3.9)$$

in the Feynman gauge. The specific form of $\Sigma(k)$ will depend on the choice of gauge since Green's functions and bilocal fields are by construction not gauge-invariant objects. However, the physical results of the theory should, of course, be gauge independent. In particular, the signal for dynamical chiral-symmetry breaking is that $\Sigma(k)$ has a nonvanishing "mass term": namely,

$$\Sigma(k)=\not k\Sigma_1(k)+\Sigma_2(k)\ ,$$

with $\Sigma_2(k)\neq 0$. In other words, the dynamics is such that the solution of the equation

$$\Sigma_2(k)+\frac{4i}{(2\pi)^4}\int d^4q\frac{\Sigma_2(q)G(k-q)}{D(q)}=0 \quad (3.10)$$

which minimizes the effective potential is one where $\Sigma_2(k)\neq 0$. In this equation, $D(q)\equiv q^2[1-\Sigma_1(q)]^2-\Sigma_2^2(q)$.

For completeness, we note that $\Sigma_1(k)$ satisfies the equation

$$\not k\Sigma_1(k)-\frac{2i}{(2\pi)^4}\int d^4q\frac{\not q[1-\Sigma_1(q)]G(k-q)}{D(q)}=0\ . \quad (3.11)$$

We wish to determine the implication of the $\Sigma_2\neq 0$ solution of the Schwinger-Dyson equation, namely, the implication of spontaneous chiral-symmetry breaking, when it is used in the part of the bilocal effective action bilinear in ϕ. This part has the form

$$W_{\rm eff}^{(2)}=\frac{i}{2}{\rm Tr}S\bar\phi S\bar\phi+\tfrac{1}{2}{\rm Tr}\phi\bar\phi\ . \quad (3.12)$$

We expect that an extension of the Goldstone theorem holds in the bilocal case, so we examine the mass term for the pseudoscalar piece in the decomposition of $\phi(x,y)$. Choosing variables $z\equiv(x+y)/2$ and $t\equiv x-y$ and defining a double Fourier transform, we identify the mass term for the momentum-space pseudoscalar bilocal field $\phi_P(q,p)$, where

$$\phi_P(z;t)=\frac{1}{(2\pi)^8}\int d^4p\int d^4q\,\phi_P(q,p)\times\exp[i(-z\cdot p-t\cdot q)]\ . \quad (3.13)$$

The general form of the bilinear term in momentum space is given by

$$W_{\rm eff}^{(2)}=\frac{1}{2}\frac{1}{(2\pi)^8}{\rm tr}\left[i\int d^4p\int d^4q[S(q-\tfrac{1}{2}p)\bar\phi(p,q)S(q+\tfrac{1}{2}p)\bar\phi(-p,q)]+\int d^4p\int d^4q\,\phi(p,q)\bar\phi(-p,q)\right] \quad (3.14)$$

and the mass term for pseudoscalar fields is obtained by singling out $\gamma_5\phi_P(p,q)$ and by setting $p=0$ in the fermion propagators $S(q\mp\frac{1}{2}p)$. In this fashion we obtain the expression

$$W_{\rm mass}^{(2)}[\phi_P]=-\frac{1}{2}\frac{1}{(2\pi)^8}\int d^4p\int d^4q\int\left[\frac{4i}{(2\pi)^4}\frac{1}{D(q)}\int\phi_P(p,k)G(q-k)d^4k+\phi_P(q,p)\right]\bar\phi_P(-p,q)\ . \quad (3.15)$$

It is expected that the Bethe-Salpeter equation which can be obtained from the bilinear action of Eq. (2.13) for the pseudoscalar state wave function admits a $p^2=0$, massless Goldstone-boson solution when chiral symmetry is spontaneously broken. Using this fact as a guide, we expand the effective action about $p^2=0$, assuming that the physical ground state is such that $\Sigma_2(q)\neq 0$, signifying that chiral symmetry is spontaneously broken. In this expansion, the mass term, identified in Eq. (3.15) should be equal to zero, which happens if

$$\phi_P(p,q)=\psi(p)\frac{\Sigma_2(q)}{D(q)}\ . \quad (3.16)$$

Indeed, this wave function is the solution to the pseudoscalar Bethe-Salpeter equation which signals the presence of the Goldstone mode.[19] In our treatment it is the logical wave function to choose as part of the basis for the general expansion of the bilocal field.

Since we have not gone into any specific calculation of the solutions of the Schwinger-Dyson equation or the Bethe-Salpeter equation, the details related to their renormalization are not discussed. It is assumed that the original local field Lagrangian contains the necessary counterterms and, or, renormalization constants. Once the renormalized solutions are found the calculation of the bilocal action should not encounter further divergences.

These are the principal results of this section. Namely, that in the bilocal case the $\Sigma_2\neq 0$ solution of the gap equation does not by itself ensure that the necessary condition for a massless pseudoscalar mode is satisfied. However, an expansion of the action in powers of momentum p or, equivalently, center-of-mass coordinate derivatives, requires the vanishing of the mass term in order to achieve consistency with the Bethe-Salpeter equation, and that leads to the choice of Eq. (3.16).

In order to contrast the bilocal field situation to that in the local field case, recall that in the Nambu and Jona-Lasinio model,[20] which one can obtain[18] by the choice $G(x-y)=G\delta(x-y)$, the vanishing of the mass term is automatic when the gap equation is satisfied.

In that model,

$$\Sigma_2(k)=-4i\frac{G}{(2\pi)^4}\frac{1}{(2\pi)^4}\int\frac{\Sigma_2(q)d^4q}{D(q)}=\text{const} \qquad (3.17\text{a})$$

and

$$k\!\!\!/\Sigma_1(k)=2i\frac{G}{(2\pi)^4}\frac{1}{(2\pi)^4}\int\frac{d^4q\, q\!\!\!/[1-\Sigma_1(q)]}{D(q)}=0\ , \qquad (3.17\text{b})$$

and so

$$\Sigma_2\left[1+4i\frac{G}{(2\pi)^4}\frac{1}{(2\pi)^4}\int\frac{d^4q}{D(q)}\right]=0 \qquad (3.17\text{c})$$

by Eq. (3.17a), where now $D(q)=q^2-\Sigma_2^2$ according to (3.17b) and the definition following Eq. (3.10). Then

$$W^2_{\text{mass}}[\phi_P]=\frac{1}{2}\frac{1}{(2\pi)^8}\left[4i\frac{G}{(2\pi)^4}\frac{1}{(2\pi)^4}\int\frac{d^4k}{D(k)}+1\right]\int d^4p\, d^4q\, \phi_P(p,q)\bar\phi_P(-p,q)=0 \qquad (3.18)$$

automatically because of the gap equation (3.17c).

The kinetic energy term for the field $\psi(p)$ can be developed by expanding all terms up to second order in p and normalizing the result canonically. The interesting question of mixing between the pseudoscalar and axial bilocal field components arises here, and we sketch this development next.

The relevant terms come from the bilinear form in the fermion loop expansion, which we write as

$$W^{(2)}_{\text{KE}}=\frac{i}{2}\frac{1}{(2\pi)^8}\int d^4p\int d^4q\ \text{tr}\{S^-[\gamma_5(\bar\phi_1^{(+)}+q\!\!\!/(p\cdot q)\bar\phi_2^{(+)}+p\!\!\!/\bar\phi_3^{(+)}+\bar A\!\!\!/^{(+)})]$$
$$\times S^+[\gamma_5(\bar\phi_1^{(-)}-q\!\!\!/(p\cdot q)\bar\phi_2^{(-)}-p\!\!\!/\bar\phi_3^{(-)}+\bar A\!\!\!/^{(-)})]\}\ . \qquad (3.19)$$

We have kept only the terms proportional to γ_5 since we are interested in the pseudoscalar mode and its relation to the dynamical breaking of chiral symmetry. The notation is

$$S^\pm=S(q\pm\tfrac{1}{2}p)=(q\!\!\!/\pm\tfrac{1}{2}p\!\!\!/)S_1^\pm+S_2^\pm\ ,$$

$$S_i^+=S_i(q\pm\tfrac{1}{2}p),\quad \bar\phi_i^{\,+}=\bar\phi_i(\pm p,q)\ .$$

In order to define diagonal fields, one removes the mixing between ϕ and A_μ by setting to zero the coefficients of the terms proportional to $q\cdot A$ and to $p\cdot A$ independently. One obtains

$$0=\bar\phi_1(S_1^+S_2^- -S_2^+S_1^-)$$
$$+p\cdot q\bar\phi_2\left[S_1^+S_1^-\left[q^2+\frac{p^2}{4}\right]-S_2^+S_2^-\right]$$
$$+2p\cdot q\bar\phi_3S_1^+S_1^- \qquad (3.20)$$

and

$$0=\tfrac{1}{2}\bar\phi_1(S_1^+S_2^- +S_2^+S_1^-)$$
$$-\tfrac{1}{2}q\cdot p\bar\phi_2S_1^+S_1^- -\bar\phi_3\left[S_1^+S_1^-\left[q^2+\frac{p^2}{4}\right]+S_2^+S_2^-\right]\ , \qquad (3.21)$$

where $\bar\phi_i=\bar\phi_i(+p,q)$ in Eqs. (3.20) and (3.21). These two equations allow one to eliminate $\bar\phi_2$ and $\bar\phi_3$ in favor of $\bar\phi_1$.

To identify the kinetic energy, one expands $S_1^\pm$ and $S_2^\pm$ around $p=0$, writes $\bar\phi_2$ and $\bar\phi_3$ in terms of $\bar\phi_1\equiv\bar\phi_P$, performs the Dirac traces and keeps the terms quadratic in p. Using the factorization hypothesis to represent the Goldstone mode as described in our discussion of the mass term (zeroth order in p), one has the simple relationship, from (3.2) and (3.5),

$$\bar\phi_P(p,q)=\frac{4}{(2\pi)^4}\int d^4k\ \phi_P(p,k)G(q-k)$$
$$=\frac{4}{(2\pi)^4}\psi(p)\int\frac{d^4k\,\Sigma_2(k)G(q-k)}{D(k)}$$

or

$$\bar\phi_P(p,q)=i\Sigma_2(q)\psi(p)\ , \qquad (3.22)$$

using (3.10). The kinetic energy is then

$$W_{\text{KE}}[\phi_P]=-\frac{i}{2}\frac{1}{(2\pi)^8}\int d^4p\ \text{tr}\psi(p)\psi(-p)p^2$$
$$\times\int d^4qF(q^2)\ ,$$

where the complicated dependence of $F(q)$ on $G,\Sigma_1(q)$, $\Sigma_2(q)$, and their derivatives is presented in the Appendix, since the explicit form is not of interest to us here. The kinetic energy normalization requires that $\pi(p)\equiv\psi(p)\sqrt{N}$ where $N=-i\int d^4q\,F(q)$, which is in principle needed in the solution to the problem of identifying the pseudoscalar decay constant. Indeed, as we discuss in more detail in the Appendix, this normalization is consistent with the normalization of the Bethe-Salpeter pion wave function. In the next section we consider the question of applying our results to the calculation of the pion decay constant.

IV. CONSIDERATIONS ON f_π

With the kinetic energy of the Goldstone-pion field properly normalized, as was done in the preceding section, we can identify the pion decay constant f_π. We do this directly by introducing the weak interaction W bosons as external potentials in the effective action of fermions and composite bosons, performing the one-fermion-loop expansion, and identifying the hadronic current coupled to the W bosons. In the following discussion we shall compare the result of this method of computing f_π to the result of defining f_π from the coefficient of the $\pi^0\to\gamma\gamma$ interaction term in the loop expansion. The values of f_π computed in these ways in several different models will then be compared to each other and to the values obtained by using Pagels and Stokar's[21] formula for f_π.

The relevant weak interaction will include a quark Lagrangian term

$$\bar\psi(x)\frac{g}{\sqrt{2}}\tau\cdot\mathbf{W}(x)\left[\frac{1-\gamma_5}{2}\right]\psi(x)\equiv\bar\psi(x)\not{W}(x)L\psi(x)\ , \tag{4.1}$$

where $L\equiv\frac{1}{2}(1-\gamma_5)$.

In Eq. (3.6) we will have now, instead of

$$W_{\ln}^{(2)}\equiv -i\,\mathrm{Tr}\ln(1-S\bar\phi)\ ,$$

the form

$$W_{\ln}^{(2)}=-i\,\mathrm{Tr}\ln[1-S(\not{W}L+\bar\phi)]\ . \tag{4.2}$$

The W-ϕ mixing term involving just the pseudoscalar component of $\bar\phi$ reads, in momentum space,

$$W_{\ln}^{(2)}=\frac{-i}{(2\pi)^8}\mathrm{tr}\int dp\int dq\,S(q-p/2)\not{W}(p)LS(q+p/2)\gamma_5\bar\phi_P(-p,q)+\cdots\ , \tag{4.3}$$

where γ_5 in $\bar\phi_P$ has been explicitly displayed $[\bar\phi=\gamma_\mu(\gamma_5\bar\phi_P+\cdots)\gamma_\nu G_{\mu\nu}=-\gamma_5\bar\phi_P+\cdots]$.

Now we expand the fermion propagators $S(q\mp p/2)$ in powers of p, the "center-of-mass" momentum of the bilocal system, to obtain

$$W_{\ln}^{(2)}=\frac{gN_c}{16\pi^2}\int dp\,(ip\cdot\mathbf{W})\cdot\psi(-p)\int_0^\infty dl^2 l^2\Sigma_2[S_1S_2-(l^2/2)(S_1S_2'-S_2S_1')]\ , \tag{4.4a}$$

$$W_{\ln}^{(2)}\equiv\pi^2 g\int dx\,\mathbf{W}_\mu\cdot\partial_\mu\psi(x)I_{\mathrm{mix}}\ , \tag{4.4b}$$

where $\bar\phi_P(p,q)=i\psi(p)\Sigma_2(q)$, as obtained from our identification of the unrenormalized Goldstone-boson field to lowest orders in p, Eq. (3.22). N_c is the number of colors and I_{mix} is the integral over l^2 which appears in Eq. (4.4a). In obtaining this equation, we have expanded

$$S(k)=\not{k}S_1(k^2)+S_2(k^2)$$

and

$$S_i(p^2\pm q\cdot p+p^2/4)=S_i(q^2)\pm(q\cdot p)S_i'(q^2)+\cdots\ .$$

We are retaining only the part of the pseudoscalar wave function which is proportional to the Dirac unit matrix, ϕ_1 [see (3.19)]. From our work on the kinetic energy normalization of the last section we have $\psi(p)=\pi(p)/\sqrt{N}$, where $\pi(p)$ is the Fourier transform of the properly normalized local pseudoscalar Goldstone field. The normalization integral N is given by

$$N=\frac{N_c}{16\pi^2}\int dl^2 l^2\Sigma_2^2(S_1^2+l^2A_1+A_2)\equiv\frac{1}{16\pi^2}I_N\ , \tag{4.5}$$

where

$$A_i=-2S_iS_i'-S_iS_i''l^2+l^2(S_i')^2\ .$$

Identifying f_π by the charged-current effective interaction

$$L_{\mathrm{cc}}=\frac{g}{2}W_\mu^+\partial_\mu\pi^- f_\pi+\mathrm{H.c.}\ ,$$

we find finally

$$f_\pi=\frac{\sqrt{N_c}}{\sqrt{2}\pi}\frac{I_{\mathrm{mix}}}{\sqrt{I_N}}\ . \tag{4.6}$$

We turn now to the problem of calculating $\pi^0\to\gamma\gamma$ in the framework of our model in order to identify an alternative expression for f_π. The relevant interaction for $\pi^0\to\gamma\gamma$ is obtained from the expansion of the effective action in Eq. (4.2), where $\not{W}L$ is replaced by $\not{A}Q$ with A_μ the electromagnetic field and Q the quark charge matrix. We find, in momentum space, the expression

$$W_{\mathrm{eff}}(\phi_P,AA)=\frac{iN_cQ^2}{16}\frac{\mathrm{tr}}{(2\pi)^{12}}\int dp\,dk\,dq\left[\bar\phi_P(p,q)\gamma_5 S\left[q-\frac{p}{2}\right]\not{A}\left[\frac{p+k}{2}\right]S\left[q-\frac{k}{2}\right]\not{A}\left[\frac{p-k}{2}\right]S\left[q+\frac{p}{2}\right]\right]\ . \tag{4.7}$$

Expanding to lowest order in momenta and using our solution to the gap equation, Eq. (3.16), and the normalization condition $\pi=\psi\sqrt{N}$, we obtain the result

$$W_{\mathrm{eff}}(\pi,A,A)=\frac{N_ce^2(\mathrm{tr}Q^2)}{(2\pi)^4\sqrt{N}}\pi^2\int_0^\infty dl^2\frac{l^2}{2}S_1(S_1S_2-S_1l^2S_2'+l^2S_1'S_2)\epsilon_{\mu\nu\lambda\sigma}\int dx\,\pi(x)F_{\mu\nu}(x)F_{\lambda\sigma}(x)\ . \tag{4.8}$$

Calling I_{π^0} the integral appearing in Eq. (4.8), we compare to the standard result[26] of gauged Wess-Zumino chiral Lagrangian[27,28] calculations to obtain

$$f_\pi = \frac{\sqrt{N_c}\sqrt{N}}{\sqrt{2}I_{\pi^0}} = \frac{1}{4\pi}\frac{\sqrt{N_c}}{\sqrt{2}}\frac{\sqrt{I_N}}{I_{\pi^0}} . \quad (4.9)$$

As we remarked during the course of our foregoing developments, in expanding the bilocal field we have not included the explicitly momentum-dependent terms arising from pseudoscalar-axial-vector mixing in evaluating the effective action. In order to gain an idea of the consistency between the different methods of identifying f_π within this lowest order in momentum development, we compare f_π in two models where $\Sigma_1=0$, which is a generally assumed case. In the Nambu and Jona-Lasinio model, $\Sigma_1=0$ and $\Sigma_2=m$, a constant. In the model where the gluon propagator is Coulombic, the fermion propagator has $\Sigma_1=0$ in Landau gauge. Examples of papers which give explicit solutions for Σ_2 in this latter case are Chang and Chang[22] and Munczek,[23] and we compute the value for f_π in this model. We also compute f_π by using a $\Sigma_2(q^2)\equiv\Sigma(q^2)$ obtained by Cahill and Roberts[4] by joining smoothly confining[18] low-q^2, and perturbative[21] high-q^2, forms of Σ. We add also the calculation of f_π when, instead, the confining form joins the $1/q$ "semiasymptotic" falloff of Σ advocated by Appelquist and Wijewardhana.[24] In these latter models and in order to mimic the Landau gauge we have made the further approximation of neglecting $\Sigma_1(q^2)$, which is not zero although it vanishes faster than $\Sigma_2(q^2)$ at larger q^2. Finally, we include the calculations of f_π from the formula of Pagels and Stokar,[21] who worked in Landau gauge, for each different functional form of Σ. The results are shown in Table I. The integrals in terms of fermion mass function, the only relevant function in this class of models, are listed in the Appendix.

V. DISCUSSION

Based on our general bilocal field formalism developed in Sec. II and summarized in Eqs. (2.5), (2.7), and (2.8) our analysis in Sec. III of the first term in the expansion (2.8) produced the factorized wave function (3.16) for the bilocal, pseudoscalar Goldstone-boson field $\phi_P(p,q)$. Namely, $\phi_P(p,q)=\psi(p)\Sigma_2(q)/D(q)$, where p is the center-of-mass momentum and q the relative momentum.

This field, for which there is no mass term in the effective Lagrangian, can obviously be obtained only if the solution to the Schwinger-Dyson or "gap" equation (3.10) spontaneously breaks chiral symmetry (i.e., if $\Sigma_2\neq0$). Its emergence is to some degree heuristic since we have only shown that it can be consistently introduced. In a strictly local theory, instead, the masslessness follows from the gap equation automatically, and we illustrated this by reviewing the (local) Nambu–Jona-Lasinio model[20] in our formalism in Eqs. (3.17) and (3.18).

Closing Sec. III, we obtained the factor N needed to normalize the kinetic energy term of the local field $\psi(p)$ canonically, where general mixing with the axial field is admitted in the analysis.

In Sec. IV, as an application of the use of the wave function of the pseudoscalar field derived in Sec. III, we computed the pion decay constant in several different models for $\Sigma(q)$, in the approximation of keeping the part of the pion wave function which is proportional to the Dirac unit matrix. We used two different identifications of the pion decay constant. In the first, we introduced an external weak boson potential into the fermion plus pseudoscalar-boson effective action and computed the effective π-W interaction from which f_π is defined. In the second method, we computed the effective (anomalous) $\pi^0\to\gamma\gamma$ vertex which relates to f_π through partial conservation of axial-vector current, the

TABLE I. Values of f_π in terms of each model's characteristic mass scale μ for three different derivations of f_π—$L_{\pi^0\to\gamma\gamma}$, $L_{W\text{-}\pi}$, and Ref. 21—in four different models as listed across the top of the table. In each case $N_c=3$ is used for uniformity of comparison ($N_c=4$ is actually appropriate to the slow α_s model).

Source of f_π formula \ Model	Nambu and Jona-Lasinio (Ref. 20) (cutoff $=\Lambda$)	Chang and Chang (Ref. 22), Munczek (Ref. 23)	Cahill and Roberts (Ref. 4)	Slow α_s (Ref. 24) + (Refs. 4 and 18)
$L_{\pi^0\to\gamma\gamma}$	$\frac{\mu N_c^{1/2}}{2\pi}\left[\ln\left[\frac{\Lambda^2+\mu^2}{\mu^2}\right]-\frac{3}{4}\right]^{1/2}$	0.150μ	0.146μ	0.182μ
$L_{W\text{-}\pi}$	$\frac{\frac{\mu N_c^{1/2}}{2\pi}\left[\ln\left[\frac{\Lambda^2+\mu^2}{\mu^2}\right]-1\right]}{\left[\ln\left[\frac{\Lambda^2+\mu^2}{\mu^2}\right]-\frac{3}{4}\right]^{1/2}}$	0.111μ	0.099μ	0.137μ
Pagels and Stokar (Ref. 20)	$\frac{\mu N_c^{1/2}}{2\pi}\left[\ln\left[\frac{\Lambda^2+\mu^2}{\mu^2}\right]-1\right]^{1/2}$	0.111μ	0.099μ	0.137μ

soft-pion approximation and the axial anomaly.

The f_π values were compared with each other and with the formula of Pagels and Stokar[21] in the Nambu and Jona-Lasinio model,[20] the $\Sigma(q)$ model of Chang and Chang,[22] and Munczek,[23] the $\Sigma(q)$ model of Cahill and Roberts,[4] and a slowly running coupling model of $\Sigma(q)$ which patches a long-distance model[4,18] onto a "semiasymptotic" $1/q$ falloff.[24]

In the Nambu and Jona-Lasinio model the three differently obtained f_π values agree with each other to leading order in the cutoff. The π-W mixing value of f_π and that using the formula of Ref. 19 agree with each other within the accuracy of the numerical integrations in both of the QCD-inspired models,[22,23,4] and the f_π expression for $\pi^0\to\gamma\gamma$ is about 30% higher than the other f_π values in these models, a discrepancy which one might expect from comparing different chiral perturbation calculations. As summarized in the table, the various values of f_π range from 60–150 MeV as the scale factor in the models ranges from 0.5–1.0 GeV, typical QCD condensation scale values. In the technicolor case, of course, GeV is replaced by TeV.

In conclusion, we believe we have explicated the steps by which a local effective Lagrangian for Goldstone pions emerges from the bilocal effective action which results from integrating out gluons and quarks in the generating functional, replacing them with composite, bosonic degrees of freedom at low energy. An explicit solution for the Goldstone mode was displayed, and the computation of f_π based on the wave function from this solution was performed by two complementary methods. These were compared with each other and with the dynamical chiral perturbation theory result of Pagels and Stokar, and overall numerical consistency demonstrated for several different models of the dynamical fermion mass $\Sigma(q)$.

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APPENDIX

In this appendix we collect a number of results which are not given in detail in the text.

(i) The kinetic energy term normalization coefficient. We had $N=\int d^4q\,F(q)$ in Sec. III of the text. Referring to (3.21) and (3.22), we define

$$\phi_2=A\phi_1,\quad \phi_3=B\phi_1\ ,$$
$$A=-2(S_1^2B+S_1'S_2-S_2'S_1)/(S_1q^2-S_2^2)\ ,$$
$$B=-S_1S_2/(S_1q^2+S_2^2)\ ,$$

where S_i are defined in Sec. III.

The integrand of the normalization integral is given by

$$F(q)=(-q^2A_1+S_1^2+A_2)-2Aq^2(-2\Delta_{12}+S_1S_2)-8B(-\tfrac{1}{2}q^2\Delta_{12}+S_1S_2)$$
$$-2Aq^2(Aq^2+B)(S_1^2q^2-S_2^2)+2B^2(q^2S_1^2-2S_2^2)\ , \quad \text{(A1)}$$

where A_1 and A_2 are defined in Eq. (4.5) and below it, and where $\Delta_{12}\equiv S_1S_2'-S_1'S_2$. When $\phi_2=\phi_3=0$, as in our approximation in calculating f_π in Sec. IV, then only the leading term contributes, and we have $F(q)=(-q^2A_1+S_1^2+A_2)$, the Euclidean space version of which appears in Eq. (4.5).

(ii) f_π formulas. Summarizing the formulas in Sec. IV which are used to compute f_π, we have the integrals

$$I_N=\int_0^\infty dl^2\frac{l^2\Sigma^2}{(l^2+\Sigma^2)^2}\left[3+2\Sigma\Sigma'+3l^2(\Sigma')^2+l^2\Sigma\Sigma''-\frac{l^2}{l^2+\Sigma^2}(1+2\Sigma\Sigma')^2\right]. \quad \text{(A2)}$$

$$I_{\text{mix}}=\int_0^\infty dl^2\frac{l^2\Sigma}{(l^2+\Sigma^2)^2}(\Sigma-\tfrac{1}{2}l^2\Sigma')\ , \quad \text{(A3)}$$

$$I_{\pi^0}=\int_0^\infty dl^2\frac{l^2\Sigma}{(l^2+\Sigma^2)^3}(\Sigma-l^2\Sigma')\ . \quad \text{(A4)}$$

A prime denotes differentiation in the above expressions, (A1)–(A4). Summarizing the formulas for f_π in terms of the integrals, we have the following.

(1) $\pi^0\to\gamma\gamma$,

$$f_\pi=\left[\frac{N_c}{2}\right]^{1/2}\frac{1}{\pi}\left[\frac{\sqrt{I_N}}{4I_{\pi^0}}\right]; \quad \text{(A5)}$$

(2) W-π mixing,

$$f_\pi=\left[\frac{N_c}{2}\right]^{1/2}\frac{1}{\pi}\left[\frac{I_{\text{mix}}}{2}\right]^{1/2}\left[\frac{2I_{\text{mix}}}{I_N}\right]^{1/2}, \quad \text{(A6)}$$

(3) Pagels-Stokar (Ref. 21),

$$f_\pi=\left[\frac{N_c}{2}\right]^{1/2}\frac{1}{\pi}\left[\frac{I_{\text{mix}}}{2}\right]^{1/2}. \quad \text{(A7)}$$

We note that when $2I_{\text{mix}}=I_N$, then our f_π calculated from the W-π effective interaction is the same as that of Pagels and Stokar. In fact, $2I_{\text{mix}}=I_N$ numerically to the precision of the numerical integration in every model we studied (except in the Nambu–Jona-Lasinio model), so (2) and (3) agree in all of these models (see Table I). The reason for this behavior is that the last term in expression (A2) for I_N can be integrated by parts giving the relationship

$$I_n = 2I_{\text{mix}} + \left[\frac{1}{2} \frac{(l^2)^2\Sigma^2(1+2\Sigma\Sigma')}{(l^2+\Sigma^2)^2} \right]_0^\infty . \quad \text{(A8)}$$

The last term vanishes for all the models in which $\Sigma(l^2)$ vanishes when $l^2 \to \infty$ thus leading to the equality of expressions (A6) and (A7) for f_π. The specific form of the equation by Pagels and Stokar for f_π requires the proper normalization of the Bethe-Salpeter wave function for the pion, while our equation for f_π derives from the normalization of the pion's Lagrangian, which includes the use of the same wave function for the pion (up to a constant), thus showing the consistency of the two normalizations. For the Nambu–Jona-Lasinio model, Σ is a constant and therefore the last term in Eq. (A8) does not vanish leading to the discrepancy shown in the table.

(iii) Models for $\Sigma(q)$. Specific parametrized forms of $\Sigma(q)$ in the three models we use are the following.

(a) Chang and Chang[22] and Munczek[23] model:

$$\frac{1}{\mu}\Sigma(x) = \frac{1}{\sqrt{e}} - \tfrac{1}{2}x + \left[\frac{1}{2} - \frac{1}{\sqrt{2}} \right] x^2, \quad 0 \le x \le 1$$

$$= 0, \quad x > 1 ,$$

where $x = q^2/\mu^2$, and μ is the typical mass scale of the problem ($\mu \simeq 500$ GeV in the analysis of Munczek).

(b) Cahill and Roberts[4] model:

$$\frac{1}{\mu}\Sigma(x) = \sqrt{1-4x} , \quad x < \tfrac{1}{6} ,$$

$$\frac{1}{\mu}\Sigma(x) = 4(0.29)^3/x, \quad x > \tfrac{1}{6}$$

($\mu \simeq 0.9$ GeV in the analysis of Cahill and Roberts).

(c) "Low-q^2-improved" Appelquist and Wijewardhana[24] model:

$$\frac{1}{\mu}\Sigma(x) = \sqrt{1-4x} , \quad x < \tfrac{1}{8} ,$$

$$\frac{1}{\mu}\Sigma(x) = 1/4\sqrt{x} , \quad x > \tfrac{1}{8}$$

($\mu \simeq 350$ GeV in the technicolor application of Appelquist and Wijewardhana).

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