Self-consistency equation for the order parameter and restoration of chiral symmetry

Liu Bao-hua* and Li Jia-rong
Institute of Particle Physics, Huazhong Normal University, Wuhan, China
(Received 5 May 1987)

An analytical method is used to investigate the chiral-symmetry-restoring phase transition at finite temperature and density. The self-consistency equation for the order parameter at zero temperature is extended and the corresponding equation at finite temperature is established. The order parameter as a function of temperature and the phase diagram of the theory are obtained.

I. INTRODUCTION

One of the most important symmetries of strong interactions is chiral symmetry, which plays an essential part in explaining strong-interaction phenomena at low energy. As is well known, this symmetry is spontaneously broken at zero temperature. In recent years, many authors investigated the behavior of the strongly interacting system at finite temperature and density by using Monte Carlo methods.[1] It was found that a chiral-symmetry-restoring transition will appear as long as temperature or density is high enough. The purpose of this paper is to study this phase transition by an analytical method.

Lurie studied chiral-symmetry breaking at $T=0$ using the self-consistency equation for the order parameter.[2] In this paper, we will extend his method to the case in which temperature and density are finite.

We review briefly the self-consistency equation at zero temperature in Sec. II. Using this equation and considering the fact that the physical nucleon mass is not zero, we determine the momentum cutoff Λ in the theory. In Sec. III, the self-consistency equation at finite temperature and density is established and the phase structure of the theory is analyzed in detail.

II. THE MODEL AND THE SELF-CONSISTENCY EQUATION AT $T=0$

We consider the following Lagrangian with broken chiral symmetry:[2]

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I \ , \tag{1}$$

$$\mathcal{L}_0=-\bar{\psi}\gamma_\mu\partial_\mu\psi-\tfrac{1}{2}\partial_\mu\phi_S\partial_\mu\phi_S-\tfrac{1}{2}m^2\phi_S^2 -\tfrac{1}{2}\partial_\mu\phi_P\partial_\mu\phi_P-\tfrac{1}{2}m^2\phi_P^2 \ , \tag{2}$$

$$\mathcal{L}_I=g\bar{\psi}\psi\phi_S+ig\bar{\psi}\gamma_5\psi\phi_P \ , \tag{3}$$

where ϕ_S is a scalar field, ϕ_P a pseudoscalar field, and ψ the nucleon field. The Lagrangian is invariant under the following chiral transformation:

$$\psi\to\exp(i\gamma_5\alpha)\psi \ , \tag{4}$$

$$\phi_S\to\phi_S\cos2\alpha+\phi_P\sin2\alpha \ , \tag{5}$$

$$\phi_P\to\phi_P\cos2\alpha-\phi_S\sin2\alpha \ . \tag{6}$$

Introducing the conserved charge X associated with the chiral invariance and the vacuum expectation values of ϕ_S and ϕ_P,

$$\varphi_S=\langle 0\,|\,\phi_S\,|\,0\rangle, \quad \varphi_P=\langle 0\,|\,\phi_P\,|\,0\rangle \ , \tag{7}$$

one can easily prove that

$$\exp(i\alpha X)\,|\,0\rangle=|\,0\rangle \quad \text{for } \varphi_S=0 \ , \tag{8}$$

$$\exp(i\alpha X)\,|\,0\rangle\neq|\,0\rangle \quad \text{for } \varphi_S\neq0 \ . \tag{9}$$

In the second case, chiral symmetry is spontaneously broken. Therefore, φ_S is the "order parameter" describing chiral-symmetry breaking at zero temperature.

Lurie established the following self-consistency equations for φ_S and φ_P:

$$m^2\varphi_S=-g\,\mathrm{Tr}S_F(0) \ , \tag{10}$$

$$m^2\varphi_P=-g\,\mathrm{Tr}[i\gamma_5S_F(0)] \ , \tag{11}$$

where $S_F=\langle 0\,|\,T\psi(x)\bar{\psi}(y)\,|\,0\rangle$ is the fermion propagator. Using the functional method and neglecting the contribution of the fermion self-energy, one can obtain the approximate expression for $S_F(0)$ as follows:

$$S_F(0)=-\int\frac{d^4p}{(2\pi)^4}\frac{1}{\gamma\cdot p+ig\varphi_S} \ . \tag{12}$$

From this equation we know that the physical nucleon mass M is $g\varphi_S$. Substitution of $S_F(0)$ into Eq. (10) gives

$$\varphi_S\left[m^2+4ig^2\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2+g^2\varphi_S^2}\right]=0 \ . \tag{13}$$

The integration in the equation is divergent. In order to make the discussion meaningful, a momentum cutoff Λ should be introduced. For the sake of convenience, we adopt the three-momentum cutoff scheme in this paper; thus Eq. (13) reads

$$\varphi_SF(\varphi_S)=0 \ , \tag{14}$$

$$F(\varphi_S)=m^2-\frac{g^2}{2\pi^2}\left[\Lambda(\Lambda^2+g^2\varphi_S^2)^{1/2}-g^2\varphi_S^2\ln\frac{\Lambda+(\Lambda^2+g^2\varphi_S^2)^{1/2}}{g\varphi_S}\right]. \quad (15)$$

Equation (14) has two possible solutions: $\varphi_S=0$ or $F(\varphi_S)=0$. The first trivial one corresponds to the unbroken chiral symmetry. The second, nontrivial solution will determine φ_S in terms of Λ, m, and g. If the corresponding equation at finite temperature and density can be established, then the order parameter can be determined as a function of temperature and the chiral-symmetry-restoring phase transition can be analyzed by this method. This problem will be discussed in the next section.

For future convenience, the momentum cutoff Λ in the theory has to be determined. Noting that the physical nucleon mass M is $g\varphi_S$, we know that the equation $F(\varphi_S)=0$ has a nontrivial solution $\varphi_S=M/g$. Substituting $\varphi_S=M/g$ into $F(\varphi_S)=0$, we obtain an equation for Λ. Setting $M=1000$ MeV, $m=2000$ MeV, and $g=15$ (Ref. 3), we find

$$\Lambda=860 \text{ MeV} . \quad (16)$$

The order parameter used in this paper is φ_S, whereas the order parameter used in most articles is $\langle 0|\bar{\psi}(x)\psi(x)|0\rangle$. Let us look at how they are connected. According to Eq. (10) and the identity

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle=-\mathrm{Tr}S_F(x-x)=-\mathrm{Tr}S_F(0) , \quad (17)$$

we have

$$\varphi_S=\frac{g}{m^2}\langle 0|\bar{\psi}(x)\psi(x)|0\rangle . \quad (18)$$

Therefore, φ_S is different from $\langle 0|\bar{\psi}(x)\psi(x)|0\rangle$ by a constant factor.

III. THE SELF-CONSISTENCY EQUATION AT FINITE TEMPERATURE AND THE PHASE STRUCTURE OF THE THEORY

We consider the case in which the temperature T is finite and the chemical potential μ is zero. To get the self-consistency equation, we introduce the external sources coupled to ϕ_S and ϕ_P according to

$$\mathcal{L}_J=J_S\phi_S(x)+J_P\phi_P(x) , \quad (19)$$

where J_S and J_P are spacetime constants, and define

$$Z^J=\mathrm{Tr}e^{-\beta H^J} , \quad (20)$$

$$S_{F\beta}^J(x-y)=\frac{1}{Z^J}\mathrm{Tr}e^{-\beta H^J}T\psi(x)\bar{\psi}(y) , \quad (21)$$

$$\varphi_{S\beta}^J=\frac{1}{Z^J}\mathrm{Tr}e^{-\beta H^J}\phi_S , \quad (22)$$

$$\varphi_{P\beta}^J=\frac{1}{Z^J}\mathrm{Tr}e^{-\beta H^J}\phi_P . \quad (23)$$

Here and afterwards, the time arguments are continued[4] to the interval $0\le ix_0,iy_0\le\beta$. Using a well-known trick,[5] we can rewrite Eqs. (20)–(23) as

$$Z^J=\int[d\phi_S][d\phi_P][d\bar{\psi}][d\psi]\times\exp\left[-\int_0^\beta d\tau\int d\mathbf{x}(\mathcal{L}+J_S\phi_S+J_P\phi_P)\right] , \quad (24)$$

$$S_{F\beta}^J(x-y)=\frac{1}{Z^J}\int[d\phi_S][d\phi_P][d\bar{\psi}][d\psi]\psi(x)\bar{\psi}(y)\times\exp\left[-\int_0^\beta d\tau\int d\mathbf{x}(\mathcal{L}+J_S\phi_S+J_P\phi_P)\right], \quad (25)$$

$$\varphi_{S\beta}^J=\frac{1}{Z^J}\int[d\phi_S][d\phi_P][d\bar{\psi}][d\psi]\phi_S\times\exp\left[-\int_0^\beta d\tau\int d\mathbf{x}(\mathcal{L}+J_S\phi_S+J_P\phi_P)\right]=-\frac{1}{Z^J}\frac{\delta Z^J}{\delta J_S} , \quad (26)$$

$$\varphi_{P\beta}^J=\frac{1}{Z^J}\int[d\phi_S][d\phi_P][d\bar{\psi}][d\psi]\phi_P\times\exp\left[-\int_0^\beta d\tau\int d\mathbf{x}(\mathcal{L}+J_S\phi_S+J_P\phi_P)\right]=-\frac{1}{Z^J}\frac{\delta Z^J}{\delta J_P} . \quad (27)$$

From Eqs. (26) and (27) we know that $\varphi_{S\beta}^J$ and $\varphi_{P\beta}^J$ are independent of x because Z is only the functional of J_S and J_P, which are spacetime constants.

Taking the statistical average value of the equations of motion

$$(\Box-m^2)\phi_S=-g\bar{\psi}\psi-J_S , \quad (28)$$

$$(\Box-m^2)\phi_P=-ig\bar{\psi}\gamma_5\psi-J_P , \quad (29)$$

then setting $J_S=0$ and $J_P=0$, we obtain

$$m^2\varphi_{S\beta}=-g\,\mathrm{Tr}S_{F\beta}(0) , \quad (30)$$

$$m^2\varphi_{P\beta}=-g\,\mathrm{Tr}[i\gamma_5S_{F\beta}(0)] . \quad (31)$$

In order to solve these equations, $S_{F\beta}(0)$ has to be determined. Applying the operator $\gamma\cdot\partial$ on both sides of Eq. (21), we have

$$\gamma\cdot\partial S^J_{F\beta}(x-y)=-i\delta^{(4)}(x-y)+\frac{1}{Z^J}\mathrm{Tr}e^{-\beta H^J}T[g\psi(x)\bar\psi(y)\phi_S(x)+ig\gamma_5\psi(x)\bar\psi(y)\phi_P(x)]$$
$$=-i\delta^{(4)}(x-y)+\frac{1}{Z^J}\int[d\phi_S][d\phi_P][d\bar\psi][d\psi][g\psi(x)\bar\psi(y)\phi_S(x)+ig\gamma_5\psi(x)\bar\psi(y)\phi_P(x)]$$
$$\times\exp\left[-\int_0^\beta d\tau\int d\mathbf{x}(\mathcal{L}+J_S\phi_S+J_P\phi_P)\right]$$
$$=-i\delta^{(4)}(x-y)+\frac{1}{Z^J}\left[-g\frac{\delta}{\delta J_S}-ig\gamma_5\frac{\delta}{\delta J_P}\right][Z^JS^J_{F\beta}(x-y)]$$
$$=-i\delta^{(4)}(x-y)+\left[g\varphi^J_{S\beta}+ig\gamma_5\varphi^J_{P\beta}-g\frac{\delta}{\delta J_S}-ig\gamma_5\frac{\delta}{\delta J_P}\right]S^J_{F\beta}(x-y)\ , \tag{32}$$

where we have used the equation of motion

$$\gamma\cdot\partial\psi=g\psi\phi_S+ig\gamma_5\psi\phi_P \tag{33}$$

as well as the identity

$$T\psi(x)\bar\psi(y)=\tfrac{1}{2}[\psi(x),\bar\psi(y)]+\tfrac{1}{2}\epsilon(x_0-y_0)\{\psi(x),\bar\psi(y)\} \tag{34}$$

with

$$\epsilon(x_0-y_0)=\theta(x_0-y_0)-\theta(y_0-x_0)\ . \tag{35}$$

It has been proven that the terms involving $\delta/\delta J_S$ and $\delta/\delta J_P$ stand for the contribution of fermion self-energy to the propagator.[2] To obtain an analytical, approximate solution, as Lurie did at $T=0$, we neglect the functional derivative terms in Eq. (32). Now we have

$$(\gamma\cdot\partial-g\varphi^J_{S\beta}-ig\gamma_5\varphi^J_{P\beta})S^J_{F\beta}(x-y)=-i\delta^{(4)}(x-y)\ . \tag{36}$$

Considering that $S^J_{F\beta}(x-y)$ satisfies the condition[4]

$$S^J_{F\beta}(x-y)\,|_{x_0=0}=-S^J_{F\beta}(x-y)\,|_{x_0=-i\beta}\ , \tag{37}$$

we write down the Fourier expansion as

$$S^J_{F\beta}(x-y)=\frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{dP}{(2\pi)^3}S^J_{F\beta}(p)e^{ip\cdot(x-y)}\ , \tag{38}$$

$$p_0=\omega_n=\frac{(2n+1)\pi i}{\beta},\quad n=0,\pm1,\pm2,\ldots\ . \tag{39}$$

Setting $J_S=0$ and $J_P=0$, we finally obtain

$$S_{F\beta}(0)=\frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{-\gamma\cdot p+ig\varphi_{S\beta}+g\gamma_5\varphi_{P\beta}}{p^2+g^2\varphi_\beta^2}\ , \tag{40}$$

$$\varphi_\beta^2=\varphi_{S\beta}^2+\varphi_{P\beta}^2\ .$$

From Eq. (40) we know that the physical nucleon mass is a function of temperature. Inserting (40) into (28) and (29), we obtain the result

$$\varphi_{i\beta}F_\beta(\varphi_\beta)=0\ , \tag{41}$$

where

$$\varphi_{1\beta}=\varphi_{S\beta},\quad \varphi_{2\beta}=\varphi_{P\beta}\ ,$$
$$F_\beta(\varphi_\beta)=m^2+\frac{4g^2}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{p^2+g^2\varphi_\beta^2}\ . \tag{42}$$

Using the identity

$$\frac{i}{\beta}\sum_{n=-\infty}^{\infty}f\left[\omega_n,\omega_n=\frac{(2n+1)\pi i}{\beta}\right]=-\frac{1}{2\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon}\frac{dp_0f(p_0)}{1+e^{\beta p_0}}-\frac{1}{2\pi}\int_{-i\infty-\epsilon}^{i\infty-\epsilon}\frac{dp_0f(p_0)}{1+e^{-\beta p_0}}+\frac{1}{2\pi}\int_{-i\infty}^{i\infty}dp_0f(p_0)\ , \tag{43}$$

and introducing the three-momentum cutoff Λ, we obtain

$$F_\beta(\varphi_\beta)=m^2-\frac{g^2}{2\pi^2}\left[\Lambda(\Lambda^2+g^2\varphi_\beta^2)^{1/2}-g^2\varphi_\beta^2\ln\frac{\Lambda+(\Lambda^2+g^2\varphi_\beta)^{1/2}}{g\varphi_\beta}\right]$$
$$+\frac{2g^2}{\pi^2}\int_0^\infty\frac{d\,|\mathbf{p}|\,|\mathbf{p}|^2}{(|\mathbf{p}|^2+g^2\varphi_\beta^2)^{1/2}}\frac{1}{1+\exp[\beta(|\mathbf{p}|^2+g^2\varphi_\beta^2)^{1/2}]}\ . \tag{44}$$

The equation $F_\beta(\varphi_\beta)=0$ can be easily solved. The result is shown in Fig. 1. When the value of T is small, the equation has a nontrivial solution. As the temperature increases, the value of φ_β decreases. When T is larger than some

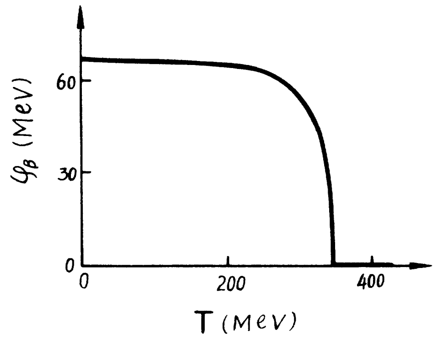

FIG. 1. The order parameter φ_β vs temperature T.

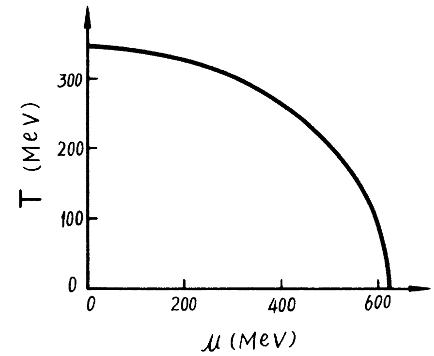

FIG. 2. The critical temperature T_c vs the critical chemical potential μ_c.

critical value, the equation has no real root and the self-consistency equations (41) have only a zero solution. In this case, the nucleon mass is zero and chiral symmetry is restored.

Setting $\varphi_\beta=0$ in $F_\beta(\varphi_\beta)=0$, we obtain an equation for the critical temperature T_c:

$$m^2-\frac{g^2\Lambda^2}{2\pi^2}+\frac{2g^2}{\pi^2}\int_0^\infty d\,|\mathbf{p}|\,|\mathbf{p}|\frac{1}{1+\exp(\beta_c|\mathbf{p}|)}=0 . \tag{45}$$

From this equation, we find

$$T_c=\frac{1}{\beta_c}=\left[6\left[\frac{\Lambda^2}{2\pi^2}-\frac{m^2}{g^2}\right]\right]^{1/2}=340\text{ MeV} . \tag{46}$$

Now we consider the effect of nonvanishing chemical potential. In this case, the self-consistency equations for the order parameter still can be written as

$$\varphi_{i\beta}F_\beta(\varphi_\beta)=0 . \tag{47}$$

Because the fermion propagator has some changes[4] [substituting ω_n by $\omega_n+\mu$ in Eq. (40), one can obtain $S_{F\beta}(0)$ at $\mu\neq0$], the function $F_\beta(\varphi_\beta)$ now reads

$$\begin{aligned}F_\beta(\varphi_\beta)=m^2-\frac{g^2}{2\pi^2}&\left[\Lambda(\Lambda^2+g^2\varphi_\beta^2)^{1/2}-g^2\varphi_\beta^2\ln\frac{\Lambda+(\Lambda^2+g^2\varphi_\beta^2)^{1/2}}{g\varphi_\beta}\right]\\&+\frac{g^2}{\pi^2}\int_0^\infty\frac{d\,|\mathbf{p}|\,|\mathbf{p}|^2}{(|\mathbf{p}|^2+g^2\varphi_\beta^2)^{1/2}}\left[\frac{1}{1+\exp\{\beta[(|\mathbf{p}|^2+g^2\varphi_\beta^2)^{1/2}-\mu]\}}\right.\\&\left.+\frac{1}{1+\exp\{\beta[(|\mathbf{p}|^2+g^2\varphi_\beta^2)^{1/2}+\mu]\}}\right] .\end{aligned} \tag{48}$$

For any fixed μ ($<\mu_c$), φ_β can be determined as a function of temperature according to equation $F_\beta(\varphi_\beta)=0$. The result is similar to that in Fig. 1. Setting $\varphi_\beta=0$ in the equation $F_\beta(\varphi_\beta)=0$, we obtain the equation for the phase curve

$$m^2-\frac{g^2\Lambda^2}{2\pi^2}+\frac{g^2}{\pi^2}\int_0^\infty d\,|\mathbf{p}|\,|\mathbf{p}|\left[\frac{1}{1+\exp[\beta_c(|\mathbf{p}|-\mu_c)]}+\frac{1}{1+\exp[\beta_c(|\mathbf{p}|+\mu_c)]}\right]=0 . \tag{49}$$

The resulting phase diagram is shown in Fig. 2. It is in good agreement with what several authors conjectured.[6]

IV. CONCLUSION

We investigated a chiral-phase-transition model by an analytical method. According to the self-consistency equation, we obtain the order parameter as a function of the temperature and phase diagram of the theory. This shows the method is a useful tool for investigating the chiral phase transition. We hope it may be extended to study the chiral-symmetry restoration in QCD.

ACKNOWLEDGMENTS

We thank Professor Liu Lian-sou and Professor Wu Ji-ming for valuable suggestions and useful discussions. This work was supported by Academia Sinica under Contract No. 85146.

*Present address: Department of Physics, Hubei University, Wuhan, China.

[1]J. Cleymans, R. V. Gavai, and E. Suhonen, Phys. Rep. **130**, 217 (1986).

[2]D. Lurie, *Particles and Fields* (Interscience, New York, 1986), p. 453.

[3]Y. Nambu and G. Jona-Lasinio, Phys. Rev. **112**, 345 (1961).

[4]P. D. Morley and M. B. Kislinger, Phys. Rep. **51**, 64 (1979).

[5]C. Bernard, Phys. Rev. D **9**, 3312 (1974).

[6]See, e.g., G. Baym, in *Quark Matter Formation and Heavy Ion Collisions,* edited by M. Jacob and H. Satz (World Scientific, Singapore, 1982).