Induced fermionic charge in background gauge theories in odd space-time dimensions

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We have shown by using gauge- and parity-invariant regularization schemes such as dimensional and higher-covariant-derivative methods that the induced fermionic charge in a U(1) gauge theory in odd space-time dimensions is zero when the fermion mass is strictly equal to zero and recedes towards the boundary when one starts with a nonzero fermion mass and allows it to approach zero. In the latter case the induced charge is exactly the Chern-Simons index.

It has been shown by several authors[1,2] that the induced fermionic current in odd space-time dimensional background gauge theories contains a local parity-violating term. In the U(1) case the induced charge corresponding to this term is the Chern-Simons index that describes the topology of the external gauge field configuration and the effective action provides a mechanism for generating a gauge-invariant mass for the gauge particles.[3]

We would like to stress, however, that these results depend crucially on the method of regularization to remove the ultraviolet divergence and, indeed, such an "anomaly" term does not exist when the fermion mass is strictly equal to zero and the regularization procedure respects the gauge and parity symmetry. The popular Pauli-Villars regularization does not fall in this category. ζ-function regularization,[4] the gauge-invariant point-splitting method,[5] dimensional regularization,[5,6] and the higher-covariant-derivative method preserve the parity symmetry and they all point towards the nonexistence of such an anomalous term. On the other hand, if the fermion mass is nonzero to start with (as is the case in Ref. 2), the parity symmetry is already broken at the classical level and, not suprisingly, the induced fermionic current regularized by any of the methods mentioned above will have a parity-violating piece. But it is, in general, nonlocal and vanishes smoothly everywhere within a finite region as the fermion mass goes to zero, provided the external field tensors fall off sufficiently fast at infinity. The induced charge, however, turns out to be independent of the fermion mass and is precisely the Chern-Simons index $(g/4\pi)\int F_{12}d^2x$, if the field is static. The nonlocal expression of the charge density also leads to the interesting observation that the induced charge recedes towards the boundary as the fermion mass tends to zero. On the other hand, if the fermion mass is strictly equal to zero, no induced charge is produced at all. The purpose of this note is to elaborate these points and to emphasize the subtle difference between the $m=0$ and $m\to 0$ theory of fermions in odd-dimensional gauge theories by the use of one of the gauge- and parity-invariant regularization schemes (the dimensional regularization).

The common starting point in any of the regularization schemes mentioned above is to express the induced fermionic current in terms of the fermionic propagators in a gauge-invariant way:

$$\langle j^\mu(x)\rangle = \mathrm{tr}[\gamma^\mu G(x,x)]\ ,$$
$$G(x,y)=\left\langle x\left|\frac{1}{\mathcal{D}}\right|y\right\rangle\ , \tag{1}$$
$$\mathcal{D}=\gamma^\mu D_\mu,\quad D_\mu=P_\mu-gA_\mu\ .$$

To first order in g, the vacuum expectation value of the current is given by the expression

$$\langle j^\mu(x)\rangle = \mathrm{tr}\gamma^\mu\left[S_F(x,x) + ig\int d^3y\, S_F(x,y)\gamma^\nu S_F(y,x)A_\nu(y)\right]\ . \tag{2}$$

We first consider the fermion mass to be strictly equal to zero. The first term

$$S_F(x,x)=\lim_{D\to 3}\int d^Dp\frac{\not p}{p^2}$$

is identically equal to zero when dimensional regularization is used. The second term gives a contribution[5,6]

$$\sim g\int\frac{\partial_\nu F^{\mu\nu}(y)}{|x-y|^2}d^3y$$

and no parity violation takes place. Convergent terms arising out of a higher number of vertex insertions (to order g^2 and above) will also be nonlocal and involve derivatives of the field tensors. Because of the occurrence of these derivatives the net induced charge (integrated charge density over an infinite volume) will be zero if the field falls off sufficiently fast at infinity. The induced current can be thought of as a dielectric phenomenon arising out of vacuum polarization.

For a massive fermion, in momentum space the expression (2) looks like

$$ig\int \frac{d^3q}{(2\pi)^3}e^{-iq\cdot x}\tilde{A}_\nu(q)\mathrm{tr}\left[\gamma^\mu \int \frac{d^3p}{(2\pi)^3}\frac{\not p+m}{p^2-m^2}\gamma^\nu \frac{\not p-\not q+m}{(p-q)^2-m^2}\right] . \tag{3}$$

Using Feynman parameters and the identity $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha)=2\epsilon^{\mu\nu\alpha}$ in three dimensions, the integral (3) can be evaluated to give us

$$\langle j^\mu(x)\rangle = \frac{img}{4\pi}\epsilon^{\mu\nu\alpha}\int \frac{d^3q}{(2\pi)^3}e^{-iq\cdot x}\tilde{F}_{\nu\alpha}(q)\frac{1}{(q^2)^{1/2}} \times \arcsin\left[\frac{q^2}{q^2+4m^2}\right]^{1/2} . \tag{4}$$

The dimensional regularization suffers from the ambiguities regarding the meaning of $\epsilon^{\mu\nu\alpha}$ in dimensions other than three. Nevertheless, that the result (4) is correct can also be verified by the method of higher covariant derivatives.[7]

The expression (4) for the induced fermionic current is nonlocal and vanishes smoothly as m goes to zero. The remarkable thing to notice is that for the static field, i.e.,

$$\tilde{F}_{\mu\nu}(q)=\delta(q^0)\tilde{F}_{\mu\nu}(q_1,q_2) ,$$

the charge $Q=\int \langle j^0\rangle d^2x$ is immediately calculable, since

$$\lim_{q^2\to 0}\left[\frac{1}{(q^2)^{1/2}}\arcsin[q^2/(q^2+4m^2)]^{1/2}\right]=\frac{i}{2|m|} . \tag{5}$$

We thus obtain

$$Q=-\frac{g}{4\pi}\frac{m}{|m|}\int F_{12}(x)d^2x = \pm\frac{g}{4\pi}\int F_{12}(x)d^2x . \tag{6}$$

This is independent of m and is precisely the Chern-Simons index for the gauge field configuration. However, in order to obtain such an expression for Q it is absolutely necessary to start with a nonzero value of the fermion mass.

The comment made above becomes clearer when one considers the configuration space. The induced charge density can be found, by direct integration over the q variables in (4), to yield

$$\langle j^0(x)\rangle = \frac{gm}{8\pi^2}\int d^3y \frac{F_{12}(y)}{|x-y|^2}e^{-2m|x-y|} . \tag{7}$$

Consequently, the induced charge

$$Q=\int \langle j^0(x)\rangle d^2x = \frac{gm}{8\pi^2}\int d^2x\int d^3y\frac{F_{12}(y)}{|x-y|^2}e^{-2m|x-y|} . \tag{8}$$

In the static background, when F_{12} depends only on y_1 and y_2, (8) can be exactly evaluated when the domain of x_1,x_2 is infinity,

$$Q=\frac{gm}{2\pi}\int d^2y\, F_{12}(y)\int_0^\infty dr\, e^{-2|m|r} = \pm\frac{g}{4\pi}\int d^2y\, F_{12}(y) , \tag{9}$$

showing, once again, that Q is independent of m. The crucial thing here is the integration over the x variables, which is linearly divergent if one had set $m=0$. It is the nonzero m that damps out such a potential divergence at the cost of $1/m$, which cancels with m sitting outside the integral in (8).

The most interesting point to notice here is that the induced charge evaluated from (8) over any finite volume (over x) will be zero as $m\to 0$. In that case the x integration in (9) will be free from any linear divergence and consequently the factor m sitting outside will make the induced charge zero in the massless limit. This, coupled with the fact that Q over infinite volume is nonzero even for $m\to 0$, implies that in this limit the charge is residing on the surface of infinite sphere (in two space dimensions).

Therefore, the two cases $m=0$ and $m\to 0$ are different. In the first case there is no net induced charge at all and in the second case the induced charge stays at the boundary only. That the two cases are basically different have been realized by the authors of Ref. 8. Since in $(2+1)$-dimensional theories the fermion mass term breaks the parity symmetry of the Lagrangian, the above observation reflects the fact that the parity violation cannot be continuously switched off.

Before concluding we would like to point out a similarity of the phenomenon observed in this note with the work of Bell and Rajaraman[9] in the context of the observation of fractional charges in $(1+1)$-dimensional field theories. Their argument reveals that in some cases, although the total charge when integrated over the whole volume remains an integer, for any volume excluding infinity the charge is fractional; the compensating fractional charge stays at the boundary. We seem to have obtained an integer analogue of it in the odd space-time dimensions.

[1]A. N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); Phys. Rev. D **29**, 2366 (1984); L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234**, 269 (1984).

[2]A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).

[3]S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).

[4]H. Banerjee, G. Bhattacharya, and J. S. Bhattacharyya, Phys. Lett. B **189**, 431 (1987).

[5]H. Banerjee, G. Bhattacharya, and J. S. Bhattacharyya, SINP Report No. SINP-TNP-85-27, 1985 (unpublished).

[6]S. Rao and R. Yahalom, Phys. Lett. B **172**, 227 (1986).

[7]L. D. Fadeev and A. A. Slavnov, *Gauge Fields, Introduction to Quantum Theory* (Benjamin, New York, 1980).

[8]D. Boyanovsky and R. Blankenbecler, Phys. Rev. D **31**, 3234 (1985); R. Akhoury and A. Comtet, Ann. Phys. (N.Y.) **172**, 245 (1986).

[9]J. S. Bell and R. Rajaraman, Phys. Lett. **116B**, 151 (1982); Nucl. Phys. **B220**, 1 (1983).