Positivity of Bondi mass for $R+R^2$ gravity

T. Zannias
Department of Mathematics, Statistics, and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5
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We show, by invoking the current positivity-of-energy theorems, that asymptotically flat solutions of $R+(1/2\beta^2)R^2$ theories of gravity possess a non-negative Bondi mass. This conclusion is reached provided that $R>-\beta^2$ everywhere. These results, in combination with Strominger's, indicate that at least for a class of asymptotically flat solutions of higher-derivative gravity the Arnowitt-Deser-Misner mass as well as the Bondi mass are non-negative.

Attempts to quantize gravity, or to study quantum fields propagating on a curved background, indicate that the familiar Einstein-Hilbert action might have to be enlarged by the inclusion of higher-order curvature terms. Further higher-derivative theories appear to enjoy better renormalizability properties than general relativity.[1] As a consequence, lately, some efforts have been made to understand their classical properties.

In a recent Rapid Communication Strominger[2] has examined the stability and positivity of the Arnowitt-Deser-Misner (ADM) mass for a restricted class of higher derivative gravity described by the action functional

$$S(g)=\int\left[R+\frac{1}{2\beta^2}R^2\right]\sqrt{-g}\,d^4x\ . \tag{1}$$

There it has been shown that, provided there exists a spacelike hypersurface where $R>-\beta^2$ everywhere, the currently existing positivity-of-energy theorems guarantee a non-negative ADM mass. In this paper, we demonstrate that for the same class of theories the Bondi mass is also non-negative.

The equation of motion derived from (1) in the presence of matter that we let be arbitrary for the moment described by $T_{\mu\nu}$ reads

$$G_{\mu\nu}=T_{\mu\nu}+\beta^{-2}(\nabla_\mu\nabla_\nu R-g_{\mu\nu}\Box R-R_{\mu\nu}R+\tfrac{1}{4}R^2g_{\mu\nu}) \tag{2}$$

from which we infer that R satisfies

$$\frac{3}{\beta^2}\Box R-R=T\ ; \tag{3}$$

i.e., it behaves as a massive scalar field, dynamically determined by the trace of $T_{\mu\nu}$.

Suppose now $(M,g_{\mu\nu})$ is an asymptotically flat spacetime at future null infinity, $\mathcal{I}^+$, solution of (2) and (3), representing presumably an isolated gravitating system. Asymptotic flatness at $\mathcal{I}^+$ for the moment is taken to be the standard definition; i.e., $(M,g_{\mu\nu})$ is asymptotically simple in the Penrose sense.[3] The asymptotic behavior of such nonvacuum space-times in the vicinity of $\mathcal{I}^+$ is already known. Exton, Newman, and Penrose, as well as Kozarzewski[4] have analyzed the asymptotic behavior of Einstein-Maxwell space-times while Ludwig[5] has worked out the general case of arbitrary nonvacuum spaces. Besides the fact that the five complex Weyl scalars ψ_i exhibit the peeling property, the allowed leading behavior of the Ricci scalars ϕ_{ij} and scalar curvature R are

$$\begin{aligned}
\phi_{00}&=\frac{\phi_{00}^{(0)}}{r^4}+O\left[\frac{1}{r^5}\right],\\
\phi_{01}&=\frac{1}{2}\frac{\eth\phi_{00}^{(0)}}{r^4}+O\left[\frac{1}{r^5}\right],\\
\phi_{11}&=\frac{\phi_{11}^{(0)}}{r^3}+O\left[\frac{1}{r^4}\right],\\
\phi_{02}&=\frac{\phi_{02}^{(0)}}{r^4}+O\left[\frac{1}{r^5}\right],\\
\phi_{22}&=\frac{\phi_{12}^{(0)}}{r^3}+O\left[\frac{1}{r^4}\right],\\
R&=\frac{R^{(0)}}{r^3}+O\left[\frac{1}{r^4}\right]=-\frac{1}{3}\frac{\phi_{11}^{(0)}}{r^3}+O\left[\frac{1}{r^4}\right],
\end{aligned} \tag{4}$$

and of course

$$\psi_i=\frac{\psi_i^{(0)}}{r^{5-i}},\quad i=0,1,2,3,4\ . \tag{5}$$

Here r stands for an affine parameter along the null geodesics, generators of the $u=\text{const}$ null hypersurfaces extending out to $\mathcal{I}^+$ (Ref. 6). The Bondi mass function $M(u)$ at retarded time u has many equivalent definitions. For example, it can be given as a two-dimensional surface integral taken at the intersection of the $u=\text{const}$ null hypersurface with $\mathcal{I}^+$, i.e.,

$$M(u)=-\frac{1}{2\pi}\int(\psi_2^{(0)}+\sigma^0\dot{\bar{\sigma}}^0+2R^{(0)})ds \tag{6}$$

with σ^0 being the leading term in the asymptotic form of the shear of the null geodesics generating the $u=\text{const}$ hypersurface. It can be shown[7] that (6) is equivalent to a Geroch-Winicour linkage[8] associated with an asymptotic

BMS (Bondi-Matzner-Sachs) time translation.

It is clear from (2) and (3) that R and its gradients would exhibit an exponential fall of rate along the null generators of the $u=$const hypersurface. Therefore if $T_{\mu\nu}$ is well behaved at $\mathcal{J}^+$ (see Ref. 5) one expects solutions of (2) and (3) to be asymptotically flat. Positivity of $M(u)$ for these solutions is essential, because if the system is allowed to have negative mass it could radiate more energy than it had, perhaps an infinite amount.

The currently known positivity-of-energy theorems[9] and their extensions[10] cannot be straightforwardly applied to Eqs. (2) and (3) since the right-hand side of (2) does not satisfy *a priori* the dominant energy condition. However following Strominger[2] we define a new metric $\bar{g}$, conformally related to g, i.e.,

$$\bar{g}_{\mu\nu}=\left[1+\frac{R}{\beta^2}\right]g_{\mu\nu}\ . \tag{7}$$

Positivity and not singular conformal factor requires $R>-\beta^2$ everywhere. In that case (2) takes the form

$$\bar{G}_{\mu\nu}=\tfrac{3}{2}(\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi-\tfrac{1}{2}g_{\mu\nu}\bar{\nabla}_a\phi\bar{\nabla}^a\phi)-\tfrac{1}{2}\bar{g}_{\mu\nu}V(\phi) \tag{8}$$

with

$$\phi=\ln(1+\beta^{-2}R)\ ,\qquad V(\phi)=\tfrac{1}{2}\beta^2(1-e^{-\phi})^2\ . \tag{9}$$

[In (8) we have taken $T_{\mu\nu}\equiv 0$.] Because of the fall rate of R, and the transformation properties of the integrand in (6) under conformal transformations the Bondi mass of the two metrics is the same. However, now, the right-hand side of (8) satisfies the dominant energy condition and thus $M(u)$ is non-negative by the positivity of energy theorem as applied to the case of Bondi mass in general relativity.[11]

We should point out that the positivity of $M(u)$, like the case of the ADM mass, has been reached under the assumption that $\Omega^2=1+\beta^{-2}R$ is nowhere singular (actually in the proof of Ref. 2, it required to be nonsingular only on a spacelike hypersurface). This of course does not imply that solutions with $R<-\beta^2$ locally or globally necessarily have a negative energy, simply the proof breaks down.

Another point worth mentioning is that positivity of $M(u)$ is actually independent of the weakly asymptotic simplicity of space-time. One could have arrived at the same conclusion provided that the underlying space-time admits finite $M(u)$. Space-times admitting a finite Bondi mass have been studied in Ref. 12 and in general they do not satisfy the peeling property. In summary, we have seen that solutions of $R+(1/2\beta^2)R^2$ theories, admitting finite $M(u)$ and having $R>-\beta^2$ everywhere, necessarily have a non-negative Bondi mass.

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