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Calculating boson and fermion loops in 3+1 dimensions and the derivative expansion

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We extend our work on the numerical calculation of one-loop quantum corrections from two to four space-time dimensions, using a Green's-function technique. Studying time-independent, spherically symmetric background fields, we illustrate this method by calculating one-boson-loop and one-fermion-loop quantum corrections to the energy. We use our results to test derivative expansions and illuminate their convergence properties.

I. INTRODUCTION

Over the past few years, there has been much interest in the derivative expansion of the effective action.[1–6] This is motivated by attempts to study quantum corrections of solitons and by effective theories that contain terms with higher powers of the derivative of fields (e.g., the Skyrmion model). The derivative expansion is an approximation that allows one to extract low-energy and long-wavelength physics from local quantum fluctuations. Previously, work had been mainly focused on the leading term in the derivative expansion, the effective potential.[7] Simple methods have been developed[1–3] in the past few years to obtain higher-order terms of the expansion for the one-boson-loop and one-fermion-loop effective actions. However, little work[8] has addressed the validity of the derivative expansion in the study of quantum corrections to solitons in four space-time dimensions, or even in two dimensions.[9] This is partly due to the fact that the one-loop corrections are nonlocal, and usually cannot be calculated analytically.

Solitons are widely used to model hadrons and nuclei and it would greatly simplify the problem of understanding quantum corrections if a local expansion, such as the derivative expansion, works. In this paper, we extend our work on the boson and fermion one-loop corrections in 1 + 1 space-time dimensions[9] to study the validity of the derivative expansion in 3 + 1 space-time dimensions. In the following, we present our study of the one-boson loop in Sec. II, and in Sec. III the study of the one-fermion loop.

II. THE ONE-BOSON LOOP

We start with a renormalizable boson theory, whose Lagrangian is

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-U(\phi)\ . \tag{1}$$

From the loop expansion,[7] the one-boson-loop effective action is given by

$$\Gamma_B(\phi_c)=\frac{i}{2}\mathrm{Tr}\ln D^{-1}(\phi_c)\ , \tag{2}$$

with $\phi_c=\langle\phi\rangle$, the expectation value of ϕ, and

$$D^{-1}(\phi_c)=-\partial^2-W(\phi_c)\ , \tag{3}$$

$$W(\phi_c)=\frac{d^2U(\phi_c)}{d\phi_c^2}\ . \tag{4}$$

Throughout this paper we will assume that ϕ_c is time independent and spherically symmetric. The trace is taken on the space-time coordinates. Γ_B is divergent and renormalization is needed. We make a functional Taylor-series expansion of Γ_B around the vacuum to identify and isolate the divergences, and first subtract the divergences from Γ_B. Define

$$D^{-1}(\phi_c)=D_0^{-1}-\tilde{W}\ , \tag{5}$$

$$D_0^{-1}=-\partial^2-W_0\ , \tag{6}$$

where $W_0=W(\phi_0)$, $\phi_0=\phi(\infty)$, and $\tilde{W}=W-W_0$. The Taylor-series expansion gives

$$\mathrm{Tr}\ln D^{-1}=\mathrm{Tr}\left[\ln D_0^{-1}-\sum_{n=1}^{\infty}\frac{1}{n}(D_0\tilde{W})^n\right]$$
$$=\mathrm{Tr}(\ln D_0^{-1}-D_0\tilde{W}-\tfrac{1}{2}D_0^2\tilde{W}^2+\cdots)\ , \tag{7}$$

where in the second line we display only the terms that are divergent. Subtracting these divergent terms, one obtains a renormalized one-loop effective action:

$$\Gamma_{BR}(\phi_c)=\frac{i}{2}\mathrm{Tr}[\ln D^{-1}(\phi_c)-\ln D_0^{-1}+D_0\tilde{W}+\tfrac{1}{2}D_0^2\tilde{W}^2-\tfrac{1}{6}D_0^3(W_0')^2(\partial_\mu\phi_c)^2]\ , \tag{8}$$

where $W_0'=dW_0/d\phi_0$. Note that in Eq. (8), we also subtract a finite term, $\frac{1}{6}D_0^3(W_0')^2(\partial_\mu\phi_c)^2$. After taking the trace this latter term becomes $\int d^4x\ \frac{1}{2}z_0(\partial_\mu\phi_c)^2$, where z_0 is a constant, so that this subtraction corresponds to a wave-function renormalization.

Expanding $\Gamma_{BR}(\phi_c)$ in terms of the derivatives of ϕ_c (Ref. 7), one has

$$\Gamma_{BR}(\phi_c)=\int d^4x[-V_B(\phi_c)+\tfrac{1}{2}Z_B(\phi_c)(\partial_\mu\phi_c)^2+\cdots]\ , \tag{9}$$

where $V_B(\phi_c)$ is the familiar effective potential. The finite subtraction in Eq. (8) was made so that $Z_B(\phi_0)=0$. From Eq. (8) one obtains

$$V_B(\phi_c)=\frac{1}{64\pi^2}\left[W(r)^2\ln\frac{W(r)}{W_0}-\tfrac{1}{2}[W(r)-W_0][3W(r)-W_0]\right] \quad (10)$$

and

$$Z_B(\phi_c)=\frac{1}{192\pi^2}\left[\frac{1}{W}\left[\frac{dW}{d\phi_c}\right]^2-\frac{1}{W_0}\left[\frac{dW_0}{d\phi_0}\right]^2\right] . \quad (11)$$

We first discuss how to calculate Γ_{BR} exactly by decomposing it into partial waves, and then discuss how to decompose the various terms in the derivative expansion into partial waves.

We make the Fourier transformations

$$\tilde{D}^{-1}(\omega)=\int dt\, e^{-i\omega t}D^{-1}(\phi_c) ,$$
$$\tilde{D}_0^{-1}(\omega)=\int dt\, e^{-i\omega t}D_0^{-1} . \quad (12)$$

For a static ϕ_c, it is more convenient to study the energy E, which is simply related to the action through

$$E=-\Gamma\Big/\int dt . \quad (13)$$

From Eq. (8), integrating by parts, and making the rotation $\omega=iy$, one obtains the renormalized one-boson-loop energy

$$E_{BR}=\int_{-\infty}^{+\infty} y^2\frac{dy}{2\pi}\mathrm{Tr}'\left[\tilde{D}(iy)-\tilde{D}_0(iy)-\tilde{D}_0^{2}(iy)\tilde{W}-\tilde{D}_0^{3}(iy)\tilde{W}^2-\tfrac{1}{2}\tilde{D}_0^{4}(iy)(W_0')^2\left[\frac{d\phi_c}{dr}\right]^2\right] , \quad (14)$$

where Tr′ differs from Tr by excluding the time integration. One then makes the partial-wave decomposition of $\tilde{D}(iy,\mathbf{r},\mathbf{r}')\equiv\langle\mathbf{r}\,|\,\tilde{D}(iy)\,|\,\mathbf{r}'\rangle$:

$$\tilde{D}(iy,\mathbf{r},\mathbf{r}')=\frac{1}{rr'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\tilde{D}_l(y,r,r')Y_{lm}(\theta,\phi)Y_{lm}^{*}(\theta',\phi') , \quad (15)$$

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-y^2-W(r)\right]\tilde{D}_l(y,r,r')=\delta(r-r') . \quad (16)$$

One can decompose

$$\tilde{D}_0(iy,\mathbf{r},\mathbf{r}')\equiv\langle\mathbf{r}\,|\,\tilde{D}_0(iy)\,|\,\mathbf{r}'\rangle$$

in a similar fashion, leading to $\tilde{D}_{0l}(y,r,r')$, which satisfy the same equation as (16) with W_0 replacing $W(\phi_c)$. These Green's functions can now be accurately constructed numerically.[10,11]

After the partial-wave decomposition, the energy becomes

$$E_{BR}=\int_{-\infty}^{+\infty} y^2\frac{dy}{2\pi}\int_0^{\infty}dr\sum_{l=0}^{\infty}(2l+1)\left[\tilde{D}_l(y,r,r)-\tilde{D}_{0l}(y,r,r)-\tilde{D}_{0l}^{2}(y,r,r)\tilde{W}(r)-\tilde{D}_{0l}^{3}(y,r,r)\tilde{W}^2(r)-\tfrac{1}{2}\tilde{D}_{0l}^{4}(y,r,r)(W_0')^2\left[\frac{d\phi_c}{dr}\right]^2\right]\equiv\sum_{l=0}^{\infty}(2l+1)E_l , \quad (17)$$

where

$$\tilde{D}_{0l}^{n}(y,r,r')\equiv\langle r\,|\,\tilde{D}_{0l}^{n}(y)\,|\,r'\rangle . \quad (18)$$

The evaluation of matrix elements of powers of $\tilde{D}_{0l}$ is discussed elsewhere.[11]

One can make a partial-wave decomposition for the derivative expansion as well. The derivative expansion for the energy that includes up to second order is given by[4,12]

$$E_{BR}^{D}=-\int_{-\infty}^{+\infty} y^2\frac{dy}{2\pi}\int d^3r\frac{d^3p}{(2\pi)^3}\left\{(\Delta-\Delta_0+\Delta_0^2\tilde{W}-\Delta_0^3\tilde{W}^2)-\frac{1}{2}\left[\Delta^4\left[\frac{dW}{dr}\right]^2-\Delta_0^4(W_0')^2\left[\frac{d\phi_c}{dr}\right]^2\right]\right\} , \quad (19)$$

where $\Delta=1/[y^2+\mathbf{p}^2+W(r)]$ and $\Delta_0=1/(y^2+\mathbf{p}^2+W_0)$. One has the relation[12]

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{(y^2+\mathbf{p}^2+W_0)^n}=(-1)^n\langle\mathbf{r}\,|\,\tilde{D}_0^{n}(iy)\,|\,\mathbf{r}\rangle=(-1)^n\sum_{l=0}^{\infty}(2l+1)\frac{1}{4\pi r^2}\tilde{D}_{0l}^{n}(y,r,r) , \quad (20)$$

and similarly the relation

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{(y^2+\mathbf{p}^2+W)^n}=(-1)^n\sum_{l=0}^{\infty}(2l+1)\frac{1}{4\pi r^2}\Delta_l^n(y,r,r) , \quad (21)$$

where

$$\Delta_l^n(y,r,r)=\frac{1}{(n-1)!}\left[\frac{1}{2\alpha_y}\frac{d}{d\alpha_y}\right]^{n-1}[-\alpha_y r^2 i_l(\alpha_y r)k_l(\alpha_y r)] , \quad (22)$$

with $\alpha_y=(y^2+W)^{1/2}$. i_l and k_l are the modified spherical Bessel functions of the first and second kind. Using these relations, one arrives at[12]

$$E_{BR}^D=\sum_{l=0}^{\infty}(2l+1)E_l^D \quad (23)$$

with

$$E_l^D=\int_{-\infty}^{+\infty}y^2\frac{dy}{2\pi}\int_0^{\infty}dr\left\{(\Delta_l-\tilde{D}_{0l}-\tilde{D}_{0l}^2\tilde{W}-\tilde{D}_{0l}^3\tilde{W}^2)+\frac{1}{2}\left[\Delta_l^4\left[\frac{dW}{dr}\right]^2-\tilde{D}_{0l}^4(W_0')^2\left[\frac{d\phi_c}{dr}\right]^2\right]\right\} , \quad (24)$$

where all Green's functions have the arguments $\tilde{D}_{0l}(y,r,r)$. After summing over all the partial waves, the terms in the first set of parentheses give $\int d^3r\, V_B(\phi_c)$, and the terms in the large square brackets give $\int d^3r \frac{1}{2} Z_B(\phi_c)(\nabla\phi_c)^2$.

To study the convergence of the derivative expansion, for simplicity we take $U(\phi)=\frac{1}{4}(\phi^2-\phi_0^2)^2$, and consequently

$$W=3\phi_c^2-\phi_0^2 . \quad (25)$$

We also employ a convenient parametrized background field:

$$\phi_c=\phi_0-\frac{\phi_b}{1+\exp[(r-R)/T]} , \quad (26)$$

which we loosely refer to as a soliton. R controls the radius, T the surface thickness, and ϕ_b the depth of the soliton at the origin. Because of scaling, without losing generality, we can set $\phi_0=1$ for the convenience of our investigation. Table I presents the results for E_{BR} and the first two terms in its derivative expansion for various partial waves using the parameter set $R=10$, $\phi_b=0.2$, and $T=2$. From the results, we see that each partial-wave component shows the behavior we found for a two-dimensional boson theory.[9] That is to say, the energy lies between the first and second derivative expansion approximations, and the second improves the approximation for this set of parameters. Table II gives the results for the $l=1$ partial-wave component for various sets of parameters that represent different shapes of the background-field configuration. In the first six entries of Table II we change the thickness T of the soliton while keeping R and ϕ_b fixed. One can clearly see that as the surface thickness increases, causing the magnitude of the field derivatives to decrease, the derivative expansion improves. In the next three entries we vary R and T at the same time, but keep R/T and ϕ_b fixed. By dimensional arguments, this corresponds to scaling the effective mass, $W^{1/2}$. Since $\hbar c=1$, multiplying all lengths by α and masses by $1/\alpha$ in the definition of W will lead to a trivial scaling of E by $1/\alpha$. On the other hand, if one scales lengths by α while fixing the masses, one arrives at the same results as scaling all masses by $1/\alpha$ and fixing the lengths. As R and T simultaneously decrease, the boson effective mass decreases and the expansion begins to fail.[9] In the example shown the "failure" is not dramatic, but is more so in other cases. In the last three entries of Table II we vary ϕ_b while keeping R and T fixed. As ϕ_b increases, the approximation worsens, as one would expect.

From the results we see that for a smooth soliton the first two terms in the derivative expansion approximate the full energy well, with the second term improving the approximation. Also, for each partial wave, the full energy always falls between $\int d^3r\, V_B(\phi_c)$ and

TABLE I. The boson-loop energy and the two leading terms in its derivative expansion for various partial waves with the parameter set $R=10$, $T=2$, and $\phi_b=0.2$. All results should be multiplied by 10^{-2}.

l	E_l	$\int d^3r\, V_B$	$\int d^3r \frac{1}{2} Z_B(\nabla\sigma_c)^2$	$\int d^3r[V_B+\frac{1}{2}Z_B(\nabla\phi_c)^2]$
0	−1.793 36	−1.799 86	0.006 58	−1.793 28
1	−1.386 72	−1.392 66	0.006 06	−1.386 62
2	−1.073 33	−1.078 50	0.005 27	−1.073 23
3	−0.833 04	−0.837 34	0.004 38	−0.832 96
4	−0.649 20	−0.652 66	0.003 54	−0.649 12
5	−0.508 52	−0.511 22	0.002 78	−0.508 44
6	−0.400 64	−0.402 70	0.002 14	−0.400 58
7	−0.317 61	−0.319 18	0.001 63	−0.317 55
8	−0.253 44	−0.254 62	0.001 22	−0.253 40
9	−0.203 57	−0.204 44	0.000 91	−0.203 53

TABLE II. The boson-loop energy and the two leading terms in its derivative expansion for the $l=1$ partial wave for various ϕ_c. All results should be multiplied by 10^{-2}.

R	T	ϕ_b	E_l	$\int d^3r\, V_B$	$\int d^3r[V_B+\frac{1}{2}Z_B(\nabla\phi_c)^2]$
10	0.3	0.2	−2.050 76	−2.073 70	−2.031 74
10	0.5	0.2	−1.967 74	−1.987 28	−1.962 22
10	1	0.2	−1.762 70	−1.774 32	−1.762 31
10	2	0.2	−1.386 72	−1.392 66	−1.386 62
10	3	0.2	−1.115 47	−1.119 24	−1.115 45
10	4	0.2	−0.942 25	−0.944 86	−0.942 25
7	1.4	0.2	−0.813 48	−0.821 10	−0.813 16
5	1	0.2	−0.457 72	−0.466 48	−0.456 92
3	0.6	0.2	−0.159 18	−0.166 46	−0.156 58
10	2	0.1	−0.174 18	−0.174 76	−0.174 18
10	2	0.3	−4.711 82	−4.741 20	−4.711 22
10	2	0.4	−11.519 6	−11.650 5	−11.513 4

$$\int d^3r[V_B(\phi_c)+\tfrac{1}{2}Z_B(\phi_c)(\nabla\phi_c)^2] .$$

This indicates that the derivative expansion for the boson loop, as in two dimensions,[9] may be asymptotic. These features, however, depend on the renormalization scheme that one chooses. For a different renormalization scheme, a finite polynomial that has the same form as U as well as a term $\frac{1}{2}z(\nabla\phi_c)^2$, with z a constant, can be added to the energy density. Adding a finite polynomial would not change the fact that the energy falls between $\int d^3r\, V_B$ and $\int d^3r[V_B+\frac{1}{2}Z_B(\nabla\phi_c)^2]$. However, it may change the fact that $\int d^3r\, V_B$ alone is a good approximation to the energy in the above tables. Adding a term $\frac{1}{2}z(\nabla\phi_c)^2$, however, would modify the contribution from the second derivative term in the boson-loop energy, and could make $E_{BR}+\int d^3r\frac{1}{2}z(\nabla\phi_c)^2$ less than $\int d^3r\, V_B$ for some z, thus making the expansion appear nonasymptotic.

III. THE ONE-FERMION LOOP

For fermion fields coupled to a scalar field through a simple Yukawa interaction, the fermion-loop action is given by

$$\Gamma_F(\phi_c)=-i\,\mathrm{Tr}\ln S^{-1}(\phi_c) , \tag{27}$$

where

$$S^{-1}(\phi_c)=-(i\partial\!\!\!/-g\phi_c) , \tag{28}$$

ϕ_c is a scalar external field, and g is the Yukawa coupling constant. The trace is taken on the space-time configuration and on the Dirac indices. As for the boson case, we assume that ϕ_c is time independent and spherically symmetric, $\phi_c=\phi_c(r)$. Making the Fourier transformation

$$\tilde{S}^{-1}(\phi_c)=\int dt\, e^{-i\omega t}S^{-1}(\phi_c) , \tag{29}$$

integrating by parts and disposing of an unimportant constant, we have the one-fermion-loop energy

$$E_F=i\int_C \omega\frac{d\omega}{2\pi}\mathrm{Tr}'[\gamma^0\tilde{S}(\omega)] . \tag{30}$$

The contour is the conventional one that encloses the entire negative real axis. The physical meaning of the fermion-loop energy is clear from the following relations:

$$i\int_C \omega\frac{d\omega}{2\pi}\mathrm{Tr}'[\gamma^0\tilde{S}(\omega)]=\sum_{\epsilon_i<0}\epsilon_i , \tag{31}$$

$$(-i\boldsymbol{\alpha}\cdot\nabla+\beta g\phi_c)\psi_i=\epsilon_i\psi_i . \tag{32}$$

That is, the fermion-loop energy gives the energy shift of the Dirac sea in the presence of the background field ϕ_c.

In order to calculate $\tilde{S}(\omega)$ we make a partial-wave decomposition.[13] Making the rotation $\omega=iy$ and defining

$$\tilde{G}(y,\mathbf{r},\mathbf{r}')=-\gamma_0\tilde{S}(iy,\mathbf{r},\mathbf{r}') , \tag{33}$$

where

$$\tilde{S}(iy,\mathbf{r},\mathbf{r}')\equiv\langle\mathbf{r}\,|\,\tilde{S}(iy)\,|\,\mathbf{r}'\rangle , \tag{34}$$

one expands $\tilde{G}$ as

$$\tilde{G}(y,\mathbf{r},\mathbf{r}')=\frac{1}{rr'}\sum_{\kappa,m}\tilde{G}_\kappa(y,r,r')\otimes\mathcal{Y}_{\kappa m}(\theta,\phi)\mathcal{Y}^\dagger_{\kappa m}(\theta',\phi') , \tag{35}$$

where

$$\mathcal{Y}_{\kappa m}(\theta,\phi)\equiv\mathcal{Y}^l_{jm}(\theta,\phi)=\sum_{m_l m_s}\langle l_l m\tfrac{1}{2}m_s\,|\,jm\rangle Y_{lm_l}(\theta,\phi)\chi_{m_s} . \tag{36}$$

The $\langle lm_l\frac{1}{2}m_s\,|\,jm\rangle$ are Clebsch-Gordan coefficients, while

$$\chi_{1/2}=\begin{bmatrix}1\\0\end{bmatrix},\quad \chi_{-1/2}=\begin{bmatrix}0\\1\end{bmatrix} \tag{37}$$

are the two-component Pauli spinors, and j is the total angular momentum. κ is related to j; $\kappa=(-1)^{j-l+1/2}(j+\frac{1}{2})$. $\tilde{G}_\kappa(y,r,r')$ satisfies

$$\begin{bmatrix} g\phi_c(r)-iy & -d/dr-\kappa/r \\ d/dr-\kappa/r & -g\phi_c(r)-iy \end{bmatrix} \tilde{G}_\kappa(y,r,r')=\delta(r-r') \ . \tag{38}$$

After the partial-wave decomposition, the energy becomes

$$E_F=-i\int_{-\infty}^{+\infty} y\frac{dy}{2\pi}\int_0^\infty dr \sum_{\kappa=-\infty}^{+\infty} 2|\kappa| \,\mathrm{tr}[\tilde{G}_\kappa(y,r,r)] \ . \tag{39}$$

The remaining trace is taken on the matrix space of $\tilde{G}_\kappa$. Observe that $\tilde{G}_\kappa$ has the following property:

$$\tilde{G}_{-\kappa}(y,r,r')=-\sigma_1\tilde{G}_\kappa(-y,r,r')\sigma_1 \ , \tag{40}$$

where σ_1 is the first Pauli matrix. Therefore, we have

$$\mathrm{tr}\tilde{G}_{-\kappa}(y,r,r')=-\mathrm{tr}\tilde{G}_\kappa(-y,r,r') \ . \tag{41}$$

From this one obtains

$$E_F=\sum_{\kappa=1}^{\infty}\kappa E_\kappa \ , \tag{42}$$

where

$$E_\kappa=4\int_{-\infty}^{+\infty} y\frac{dy}{2\pi}\int_0^\infty dr \,\mathrm{tr}\,\mathrm{Im}\tilde{G}_\kappa(y,r,r) \ . \tag{43}$$

E_F here is divergent and renormalization is needed. One follows the same procedure as for the boson case, makes a functional Taylor-series expansion of $\mathrm{tr}\,\mathrm{Im}\tilde{G}_\kappa(y,r,r)$ around the vacuum, isolates all the divergent terms in E_F, and subtracts them. The partial-wave components for the renormalized energy are then given by[11,12]

$$E_{\kappa R}=\int_{-\infty}^{+\infty} y^2\frac{dy}{2\pi}\int_0^\infty dr \,\mathrm{tr}\left[\frac{1}{y}\mathrm{Im}\tilde{G}_\kappa(y,r,r)+G_{0\kappa}+G_{0\kappa}^2[(g\phi_c)^2-(g\phi_0)^2] +G_{0\kappa}^3\{[(g\phi_c)^2-(g\phi_0)^2]^2+[g\phi_c'(r)]^2\}+\cdots\right] , \tag{44}$$

where

$$G_{0\kappa}=\begin{bmatrix}\tilde{D}_{0\kappa-1} & 0 \\ 0 & \tilde{D}_{0\kappa}\end{bmatrix} , \tag{45}$$

$$\tilde{D}_{0\kappa}^{-1}=\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}-(g\phi_0)^2-y^2 \ . \tag{46}$$

In the above, the ellipsis represents further finite subtractions that go beyond the minimal subtractions and are required in order to satisfy certain renormalization conditions.

The derivative expansion allows one to expand the renormalized energy E_{FR} in powers of the derivatives of ϕ_c. The derivative expansion for E_{FR}, up to $O(\partial^2)$, is given by[12]

$$E_{FR}^D=4\int_{-\infty}^{+\infty} y^2\frac{dy}{2\pi}\int d^3r\frac{d^3p}{(2\pi)^3}(\{\Delta_F-\Delta_{0F}-\Delta_{0F}^2[(g\phi_c)^2-(g\phi_0)^2]-\Delta_{0F}^3[(g\phi_c)^2-(g\phi_0)^2]^2\}+[(\Delta_F^3-\Delta_{0F}^3)(g\phi_c')^2]) \ . \tag{47}$$

Here $\Delta_F=1/[y^2+\mathbf{p}^2+(g\phi_c)^2]$, $\Delta_{0F}=1/[y^2+\mathbf{p}^2+(g\phi_0)^2]$, $\phi_c'=d\phi_c/dr$, and $\phi_c''=d^2\phi_c/dr^2$. If one performs the $\mathbf{p}$ integration and the y integration, the above would collapse into a rather simple form that is similar to Eq. (9):

$$E_{FR}^D=\int d^3r[V_F(\phi_c)+\tfrac{1}{2}Z_F(\phi_c)(\nabla\phi_c)^2] \ , \tag{48}$$

where $V_F(\phi_c)$ is the familiar effective potential, given by

$$V_F(\phi_c)=-\frac{1}{16\pi^2}\left[(g\phi_c)^4\ln\frac{\phi_c^2}{\phi_0^2}-\tfrac{1}{2}[(g\phi_c)^2-(g\phi_0)^2][3(g\phi_c)^2-(g\phi_0)^2]\right] , \tag{49}$$

and

$$Z_F(\phi_c)=\frac{g^2}{2\pi^2}\ln\frac{\phi_0^2}{\phi_c^2} \ . \tag{50}$$

In Eq. (47), we have chosen to subtract a finite term so that $Z_F(\phi_0)=0$. Since we calculate the full fermion-loop energy partial wave by partial wave, we need to make partial-wave decompositions of the terms in the derivative expansion.

This is accomplished by using the following formulas:[12]

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{[y^2+\mathbf{p}^2+(g\phi_c)^2]^n}=(-1)^n\sum_{\kappa=1}^{\infty}\frac{\kappa}{4\pi r^2}[\Delta^n_{\kappa-1}(y,r,r)+\Delta^n_{\kappa}(y,r,r)] , \tag{51}$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{[y^2+\mathbf{p}^2+(g\phi_0)^2]^n}=(-1)^n\sum_{\kappa=1}^{\infty}\frac{\kappa}{4\pi r^2}[\tilde{D}^{\,n}_{0\kappa-1}(y,r,r)+\tilde{D}^{\,n}_{0\kappa}(y,r,r)] , \tag{52}$$

where

$$\Delta^n_{\kappa}(y,r,r)=\frac{1}{(n-1)!}\left[\frac{1}{2\beta_y}\frac{d}{d\beta_y}\right]^{n-1}[-\beta_y r^2 i_\kappa(\beta_y r)k_\kappa(\beta_y r)] , \tag{53}$$

with $\beta_y=[y^2+(g\phi_c)^2]^{1/2}$. After the partial wave decomposition, from Eq. (47) the derivative expansion of E_{FR} is given by

$$E^D_{FR}=-4\int_{-\infty}^{+\infty}y^2\frac{dy}{2\pi}\int_0^{\infty}dr\sum_{\kappa=1}^{\infty}\kappa(\{\Delta_{\kappa F}-\Delta_{0\kappa F}-\Delta^2_{0\kappa F}[(g\phi_c)^2-(g\phi_0)^2]$$
$$-\Delta^3_{0\kappa F}[(g\phi_c)^2-(g\phi_0)^2]^2\}+[(\Delta^3_{\kappa F}-\Delta^3_{0\kappa F})(g\phi'_c)^2]) , \tag{54}$$

where

$$\Delta^n_{\kappa F}\equiv\Delta^n_{\kappa-1}(y,r,r)+\Delta^n_{\kappa}(y,r,r) , \tag{55}$$

$$\Delta^n_{0\kappa F}\equiv\tilde{D}^{\,n}_{0\kappa-1}(y,r,r)+\tilde{D}^{\,n}_{0\kappa}(y,r,r) . \tag{56}$$

For convenience, we take ϕ_c to have the form found in Eq. (26). Again, we set $\phi_0=1$ and also set $g=1$. Tables III and IV contain our numerical results. In Table III we present E_{FR} and the first two terms in its derivative expansion for various partial waves with $R=10$, $T=2$, and $\phi_b=1$. The second term in the derivative expansion notably improves the approximation for this case. Table IV gives the results for the $\kappa=2$ partial-wave component for various sets of parameters that represent different shapes of the background-field configuration. Similar to the boson loop, in the first five entries of Table IV we change the thickness T of the soliton while keeping R and ϕ_b fixed. As T increases and derivatives decrease, the approximation improves. In the next three entries we vary R and T at the same time, but keep R/T and ϕ_b fixed. By dimensional arguments, this corresponds to scaling the effective mass, $g\phi_c$. As the effective mass decreases, the approximation starts to break down.[9] In the last three entries of Table IV we vary ϕ_b while keeping R and T fixed. Naturally, as the depth of the soliton decreases, the derivatives of the background field decrease and the approximation improves.

TABLE III. The fermion-loop energy and the two leading terms in its derivative expansion for various partial waves with $R=10$, $T=2$, and $\phi_b=1$.

κ	E_κ	$\int d^3r\, V_F$	$\int d^3r[V_F+\frac{1}{2}Z_F(\nabla\phi_c)^2]$
1	0.930 88	0.876 42	0.939 92
2	0.618 62	0.579 20	0.622 02
3	0.424 09	0.398 21	0.425 82
4	0.297 05	0.279 21	0.298 04
5	0.211 83	0.199 16	0.212 44
6	0.153 52	0.144 33	0.153 92
7	0.112 95	0.106 16	0.113 22
8	0.084 30	0.079 20	0.084 49
9	0.063 75	0.059 88	0.063 90
10	0.048 84	0.045 86	0.048 94

From the results of Table III we see that for the particular ϕ_c the first two terms in the derivative expansion approximate the full energy well for each partial-wave component, and the full energy always falls between $\int d^3r\, V_F(\phi_c)$ and $\int d^3r[V_F(\phi_c)+\frac{1}{2}Z_F(\phi_c)(\nabla\phi_c)^2]$. This indicates that the derivative expansion for the fermion loop, as in two dimensions,[9] may be asymptotic. The above finding, however, should only apply to smooth solitons, since we do find that for sharp solitons the energy is less than $\int d^3r\, V_F(\phi_c)$ (Ref. 12). In fact, when the derivatives become extremely large, the energy will approach negative infinity, but $\int d^3r\, V_F$ will remain positive.[14] This is in contrast with what we found for the one-boson loop, in which each partial wave is rather similar to a two-dimensional boson loop. As in the case for the boson loop, the above features are also subject to change should we alter the particular renormalization scheme used here.

In summary, we present a Green's function and partial-wave decomposition method to study the convergence and validity of the derivative expansion for the one-boson and one-fermion loop. For the cases presented here, the expansion for each partial-wave component behaves similarly to the one-loop derivative expansion in two space-time dimensions. Our study also indicates that

TABLE IV. The fermion-loop energy and the two leading terms in its derivative expansion for the $\kappa=2$ partial wave for various ϕ_c.

R	T	ϕ_b	E_κ	$\int d^3r\, V_F$	$\int d^3r[V_F+\frac{1}{2}Z_F(\nabla\phi_c)^2]$
10	0.1	1	1.240 05	0.692 38	1.888 86
10	0.5	1	0.827 82	0.665 04	0.889 18
10	1	1	0.719 77	0.633 26	0.736 09
10	2	1	0.618 54	0.579 20	0.622 02
10	4	1	0.529 34	0.514 94	0.529 97
7	1.4	1	0.334 19	0.296 26	0.338 90
5	1	1	0.176 91	0.143 57	0.182 90
3	0.6	1	0.060 19	0.039 04	0.068 23
10	2	1.5	0.813 46	0.624 02	0.872 46
10	2	0.5	0.137 30	0.134 90	0.137 32
10	2	0.1	0.001 47	0.001 46	0.001 47

the expansion may be asymptotic. This means that one can estimate the error of the derivative expansion at a given order by calculating the next term. The error will be less than the magnitude of this term. The questions of how the expansion depends on the renormalization scheme and how the higher-order (more than two) derivative terms behave require further investigation. The ability to numerically calculate energies for different ϕ_c should aid in finding alternative expansions when the derivative expansion starts to fail.

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