PHYSICAL REVIEW D VOLUME 37, NUMBER 6 15 MARCH 1988

Schwinger model on a finite-element lattice

Ted Grose and Kimball A. Milton
Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma 73019
(Received 8 October 1987)

The mass gap in the Schwinger model is calculated directly using a method which involves expansion in the lattice spacing. On the lattice the mass in the model is given by $\mu^2=e^2/M\sin(\pi/M)$, where M is the number of spatial lattice sites. Current correlation functions and matrix elements are also examined.

I. INTRODUCTION

In a series of papers[1–12] a new procedure for implementing quantum field theory on a Minkowski space-time lattice has been developed. This procedure, called the finite-element method, consists in defining operator fields at lattice sites, with interpolating values given by operator-valued polynomials in the space-time coordinates. Continuity only is demanded at the lattice sites; the operator equations of motion are imposed at the Gauss points. This results in a lattice theory that is exactly unitary in that the canonical commutation relations are preserved at each lattice site.[1,4] Moreover, when the procedure is applied to fermions, there is no species doubling.[2]

We have applied the linear finite-element method to two two-dimensional field theory models—to the Schwinger model[3] (massless, two-dimensional quantum electrodynamics) and to the sine-Gordon model.[11] The accuracy of the method is very good; the relative error of the mass gap in the Schwinger model, for example, is $O(1/M^2)$, where M is the number of spatial-lattice points. Here, we wish to treat the Schwinger model more fully in this context, and, in particular, to apply the expansion-in-the-lattice-spacing method developed in Refs. 4 and 11. Our goal, of course, is to gain more experience in exploiting this new lattice technique, which will ultimately be applied to realistic four-dimensional non-Abelian theories such as quantum chromodynamics.

The plan of this paper is as follows. In Sec. II we will remind the reader of the lattice equations of motion and introduce the Fock-space decomposition of the fermion fields. Using the latter, in Sec. III we extract the dispersion relation for the physical boson state in the model by examining matrix elements of the Maxwell equations. In Sec. IV we evaluate current correlation functions which reveal the boson mass in a different context. Finally, we see in Sec. V how, in an external field, the vector current acquires a vacuum expectation value proportional to the lattice mass-gap parameter μ^2.

II. THE EQUATIONS OF MOTION

In paper I we showed how to formulate the Maxwell and Dirac equations on a $(1+1)$-dimensional lattice, which equations are exactly invariant under Abelian gauge transformations. For zero bare mass, and with h being the lattice spacing, the equations are (here the first index stands for the spatial coordinate, and the second for the time coordinate)

$$-\frac{1}{2h}(E_{m+1,n+1}+E_{m+1,n}-E_{m,n+1}-E_{m,n})=J^0_{m,n}\ , \tag{2.1}$$

$$\frac{1}{2h}(E_{m+1,n+1}+E_{m,n+1}-E_{m+1,n}-E_{m,n})=J^1_{m,n}\ , \tag{2.2}$$

$$\tilde{E}_{m,n}\equiv\tfrac{1}{4}(E_{m+1,n+1}+E_{m+1,n}+E_{m,n+1}+E_{m,n})=\frac{1}{h}(C_{m+1,n+1}-C_{m+1,n})-\frac{1}{h}(B_{m+1,n+1}-B_{m,n+1})\ , \tag{2.3}$$

$$\begin{aligned}\gamma^0\Bigg[&(\phi_{m,n+1}-\phi_{m,n})-2\sum_{n'=1}^{n}(-1)^{n+n'}\exp\left[ieh\sum_{n''=n'+1}^{n}B_{m,n''}\right](e^{iehB_{m,n'}}-1)\phi_{m,n'}\\&-2(-1)^n\exp\left[ieh\sum_{n''=1}^{n}B_{m,n''}\right](e^{iehB_{m,1}/2}-1)\phi_{m,0}\Bigg]\\&+\gamma^1\Bigg[(\theta_{m+1,n}-\theta_{m,n})+\sec\left[\frac{eh}{2}\sum_{m'=1}^{M}C_{m',n}\right]\sum_{m''=1}^{M}\mathrm{sgn}(m''-m)(-1)^{m+m''}\\&\times\exp\left[\frac{ieh}{2}\sum_{m'''=1}^{M}\mathrm{sgn}(m'''-m)\mathrm{sgn}(m'''-m'')\mathrm{sgn}(m''-m)C_{m''',n}\right](e^{iehC_{m'',n}}-1)\theta_{m'',n}\Bigg]=0\ ,\end{aligned} \tag{2.4}$$

where $E_{m,n}$ is the electric field, the corresponding scalar and vector potentials are

$$B_{m,n}=\tfrac{1}{2}[(A_0)_{m,n}+(A_0)_{m,n-1}]\ ,$$
$$C_{m,n}=\tfrac{1}{2}[(A_1)_{m,n}+(A_1)_{m-1,n}]\ , \tag{2.5}$$

and the Dirac fields are

$$\phi_{m,n}=\tfrac{1}{2}(\psi_{m,n}+\psi_{m+1,n})\ ,$$
$$\theta_{m,n}=\tfrac{1}{2}(\psi_{m,n}+\psi_{m,n+1})\ . \tag{2.6}$$

Here M is the number of spatial lattice points which we take to be even so that $\psi_{m,n}$ is antiperiodic in m. We take the gauge-invariant electric current to be

$$J^\mu=e\tfrac{1}{2}(\phi_{m,n}+\phi_{m,n+1})^\dagger\gamma^0\gamma^\mu\tfrac{1}{2}(\phi_{m,n}+\phi_{m,n+1})$$
$$=e\tfrac{1}{2}(\theta_{m,n}+\theta_{m+1,n})^\dagger\gamma^0\gamma^\mu\tfrac{1}{2}(\theta_{m,n}+\theta_{m+1,n})\ . \tag{2.7}$$

In paper I we showed that J^μ is conserved in the sense that

$$\langle\text{“}\partial_\mu J^\mu\text{”}\rangle=0\ , \tag{2.8}$$

where the angular brackets stand for vacuum expectation value and the quotation marks for lattice derivative, while the dual current possesses the axial anomaly,

$$\langle\text{“}\partial_\mu J_5^\mu\text{”}\rangle=-\frac{e^2}{M\sin\frac{\pi}{M}}\tilde{E}_{m,n}\ , \tag{2.9}$$

which we took as evidence for the mass gap

$$\mu^2=\frac{e^2}{M\sin\frac{\pi}{m}}\ . \tag{2.10}$$

This result was derived by using the gauge[13] $A_0=0$, in which case the Dirac equation (2.4) could be solved by

$$\phi^{(+)}_{m,n+1}=e^{iehC_{m,n}}\phi^{(+)}_{m-1,n}\ ,$$
$$\phi^{(-)}_{m-1,n+1}=e^{-iehC_{m,n}}\phi^{(-)}_{m,n}\ , \tag{2.11}$$

where the superscripts denote the chirality, the eigenvalues of $\gamma_5=\gamma^0\gamma^1$. We took $\phi^{(\pm)}_{m,n}$ to be the canonical fermion variables, which for free fields had the Fock-space expansion

$$\phi_{m,n}=\sum_{k=1}^{M}e^{ip_k mh}(e^{-ip_k nh}v^{(+)}a_k^{(+)}+e^{ip_k nh}v^{(-)}a_k^{(-)})\ , \tag{2.12}$$

where $p_k=(2k+1)\pi/(Mh)$, $v^{(\pm)}$ are the eigenvectors of γ_5, and

$$[a_k^{(\pm)},a_{k'}^{(\pm)\dagger}]_+=\frac{1}{Mh}\delta_{kk'}\ , \tag{2.13}$$

the other anticommutators being zero. The physical interpretation of $a_k^{(\pm)},a_k^{(\pm)\dagger}$ as creation and annihilation operators is as follows:

$$\text{for } 0\le k\le\frac{M}{2}-1\ ,\quad a_k^{(+)}|0\rangle=a_k^{(-)\dagger}|0\rangle=0\ ,$$
$$\text{for } -\frac{M}{2}\le k<0\ ,\quad a_k^{(+)\dagger}|0\rangle=a_k^{(-)}|0\rangle=0\ ; \tag{2.14}$$

this construction then implies the correct lattice fermion Green's function.

III. THE DISPERSION RELATION

The only physical particle in the Schwinger model is a boson, which we denote by B, of mass μ. We can obtain the dispersion relation for this particle in a manner analogous to that employed in Ref. 11 by taking matrix elements of the equations of motion between the vacuum and a B state of momentum $q_l=2\pi l/Mh$ (Ref. 14). Using (2.1) and (2.2) we find

$$\langle B,l\,|\,J^{(+)}_{m,n}\,|\,0\rangle=\tfrac{1}{2}\langle B,l\,|\,J^0_{m,n}+J^1_{m,n}\,|\,0\rangle$$
$$=\frac{1}{4h}[-(e^{i\omega_l h}+1)(e^{iq_l h}-1)+(e^{i\omega_l h}-1)(e^{-iq_l h}+1)]\langle B,l\,|\,E_{m,n}\,|\,0\rangle$$
$$\approx\frac{i}{2}(q_l+\omega_l)\langle B,l\,|\,E_{m,n}\,|\,0\rangle\ , \tag{3.1}$$

as $h\to 0$. On the other hand, we evaluate the current matrix element using the solution of the Dirac equation (2.11) and the Fock-space expansion at the initial time $n=0$. That is, since

$$\phi^{(+)}_{m,n+1}=\exp\left[ieh\sum_{r=0}^{n}C_{m-n+r,r}\right]\phi^{(+)}_{m-n-1,0} \tag{3.2}$$

and

$$\phi^{(+)}_{m,0}=\sum_{k=1}^{M}e^{ip_k mh}a_k^{(+)}\ , \tag{3.3}$$

we have

$$J^{(+)}_{m,n}=\frac{e}{4}\sum_{k,k'=1}^{M} a_k^{(+)\dagger}e^{i(p_{k'}-p_k)(m-n)h}\left[1+e^{i(p_k-p_{k'})h}+e^{ip_kh}\exp\left[-ieh\sum_{r=0}^{n}(C_{m-n+r,r}-C_{m-n+1+r,r})\right]e^{-iehC_{m+1,n}}\right.$$
$$\left.+e^{-ip_{k'}h}\exp\left[ieh\sum_{r=0}^{n}(C_{m-n+r,r}-C_{m-n+1+r,r})\right]e^{iehC_{m+1,n}}\right]a_{k'}^{(+)} . \quad (3.4)$$

We now assume that the B states are *not* created by fermion operators, that is, for all $m/2-1\geq k\geq 0$,

$$\langle B,l\mid a_k^{(+)\dagger}=0 , \quad (3.5)$$

and that the commutator of $a_k^{(+)}$ and $C_{m,n}$ is negligible as $h\to 0$ (Ref. 15). Then, using the canonical relation (2.13) together with the vacuum definition (2.14), we find the matrix element of (3.4) is

$$\langle B,l\mid J^{(+)}_{m,n}\mid 0\rangle=\frac{e}{4}\frac{1}{Mh}ieh\sum_{k=-M/2}^{-1}(e^{-ip_kh}-e^{ip_kh})\left\langle B,l\left|C_{m+1,n}+\sum_{r=0}^{n}(C_{m-n+r,r}-C_{m-n+1+r,r})\right|0\right\rangle , \quad (3.6)$$

where an expansion in h has been carried out. The sum on k in (3.6) is immediately evaluated as $2i/\sin(\pi/M)$, while the remaining matrix element is

$$\langle B,l\mid C_{m,n}\mid 0\rangle\left[e^{-iq_lh}+iq_lh\sum_{r=0}^{n}e^{ih(q_l-\omega_l)r}\right]\approx\frac{\omega_l}{\omega_l-q_l}\langle B,l\mid C_{m,n}\mid 0\rangle , \quad (3.7)$$

where in the last summation on r we have deleted a rapidly oscillating term $\sim e^{-i(\omega_l-q_l)nh}\to 0$ ($nh\to\infty$). Finally, from (2.3) we learn, as $h\to 0$, that

$$i\omega_l\langle B,l\mid C_{m,n}\mid 0\rangle=\langle B,l\mid E_{m,n}\mid 0\rangle . \quad (3.8)$$

When we put (3.1), (3.6), (3.7), and (3.8) together, we obtain the desired dispersion relation,

$$\langle B,l\mid E_{m,n}\mid 0\rangle=\frac{2}{i}\frac{1}{\omega_l+q_l}\langle B,l\mid J^{(+)}_{m,n}\mid 0\rangle$$
$$=\frac{2}{i}\frac{1}{\omega_l+q_l}\frac{ie^2}{4M}\frac{2i}{\sin(\pi/M)}\frac{\omega_l}{\omega_l-q_l}\langle B,l\mid C_{m,n}\mid 0\rangle$$
$$=\frac{1}{\omega_l^2-q_l^2}\frac{e^2}{M\sin(\pi/M)}\langle B,l\mid E_{m,n}\mid 0\rangle , \quad (3.9)$$

or

$$\omega_l^2=q_l^2+\mu^2 , \quad (3.10)$$

where the mass μ is

$$\mu^2=\frac{e^2}{M\sin(\pi/M)} , \quad (3.11)$$

just as found in paper I.

IV. CURRENT CORRELATION FUNCTION

The current correlation function provides another unambiguous signal of the dynamical mass generation mechanism in the Schwinger model. It is best to start with the equal-time current commutator, since that can be evaluated using the free-field commutation relations. Doing so, we find, first of all,

$$[J^{(+)}_{m,n},J^{(+)}_{m',n}]=\frac{e^2}{16h}(2\delta_{m,m'}+\delta_{m,m'+1}+\delta_{m+1,m'})[(\phi^{(+)}_{m,n}+\phi^{(+)}_{m,n+1})^\dagger(\phi^{(+)}_{m',n}+\phi^{(+)}_{m',n+1})$$
$$-(\phi^{(+)}_{m',n}+\phi^{(+)}_{m',n+1})^\dagger(\phi^{(+)}_{m,n}+\phi^{(+)}_{m,n+1})] . \quad (4.1)$$

The vacuum expectation value of this is worked out using (2.14):

$$\langle 0|\phi^{(+)\dagger}_{m,n}\phi^{(+)}_{m',n}|0\rangle=\frac{1}{Mh}\sum_{k=-M/2}^{-1}e^{ip_k(m'-m)h}\,, \tag{4.2}$$

which in turn is evaluated as

$$\sum_{k=-M/2}^{-1}e^{ip_k(m'-m)h}=\begin{cases}M/2\,, & m'=m\,,\\ 0\,, & m'-m \text{ even},\neq 0,\\ [i\sin\pi(m'-m)/M]^{-1}\,, & m'-m \text{ odd}\,.\end{cases} \tag{4.3}$$

The unequal-time field products in (4.1) do not contribute, according to the free-field version of (2.11), and we find for the vacuum expectation value

$$\langle 0|[J^{(\pm)}_{m,n},J^{(\pm)}_{m',n}]|0\rangle=\pm\frac{ie^2}{4Mh^2}\frac{1}{\sin\pi/M}(\delta_{m,m'+1}-\delta_{m+1,m'})\,, \tag{4.4}$$

where we have given the result for the similar calculation involving $J^{(-)}$. Equivalently, we have

$$\langle 0|[J^0_{m,n},J^1_{m',n}]|0\rangle=i\frac{e^2}{M\sin\pi/M}\frac{1}{2h^2}(\delta_{m,m'+1}-\delta_{m+1,m'})\,. \tag{4.5}$$

This is the lattice equivalent of the continuum result[16]

$$\langle 0|[J^0(r,t),J^1(r',t)]|0\rangle=-i\mu^2\frac{\partial}{\partial r}\delta(r-r')\,. \tag{4.6}$$

Next, we consider the general correlation function. For this we use the normal-ordered current, and assume, as above, that the fermion operators a_k are independent of the boson operators. A short calculation then gives the form

$$\langle 0|:J^{(+)}_{m,n}:\,:J^{(+)}_{m',n'}:|0\rangle=-\frac{e^2}{16M^2h^2}\frac{1}{\sin^2\pi N/M}\langle 0|(2+\Sigma_{m,n}\Sigma^*_{m',n'}+\Sigma_{m',n'}\Sigma^*_{m,n})|0\rangle\,, \tag{4.7}$$

where $N=(m-m')-(n-n')$ is assumed to be odd, and

$$\Sigma_{m,n}=\exp\left[ieh\sum_{r=0}^{n}(C_{m-n+r,r}-C_{m-n+1+r,r})+iehC_{m+1,n}\right]. \tag{4.8}$$

Consider the short-distance limit, in which $m-m'$ and $n-n'$ are small compared to M and n. Then $\Sigma_{m,n}\Sigma^*_{m',n'}$ may be replaced by 1 in (4.7) and

$$\langle :J^{(+)}_{m,n}:\,:J^{(+)}_{m',n'}:\rangle\approx-\frac{e^2}{4\pi^2N^2h^2}\,. \tag{4.9}$$

This may be compared with the continuum result[16] ($\mu^2=e^2/\pi$)

$$\begin{aligned}\langle J^{(+)}(x)J^{(+)}(0)\rangle&=-\tfrac{1}{4}\partial_+^2\mu^2\Delta^{(+)}(x,\mu^2)\\&=-\frac{i\mu^4}{16}\frac{x_+}{x_-}H_2^{(1)}(-\mu\sqrt{x_+x_-})\,,\end{aligned} \tag{4.10}$$

in terms of light-cone variables

$$x_+=x^0+x^1\,,\quad x_-=x^0-x^1\,. \tag{4.11}$$

The limit $x_+x_-\to 0$ of (4.10) is $-e^2/(4\pi^2x_-^2)$, which precisely agrees with (4.9).

V. VACUUM CURRENT IN EXTERNAL FIELD

The vacuum expectation value of the normal-ordered current considered in the previous section is, of course, zero. However, there is another definition of the lattice current which has a nonzero vacuum expectation value in the presence of a spatially constant external field C. Instead of the current defined by (2.7), this current is given by the commutator

$$j^\mu_{m,n}=\frac{e}{2}[\tfrac{1}{2}(\phi_{m,n}+\phi_{m,n+1})^\dagger,\gamma^0\gamma^1\tfrac{1}{2}(\phi_{m,n}+\phi_{m,n+1})]\,, \tag{5.1}$$

a construction which agrees with normal ordering in the continuum. If we compare with (3.4) and use the definitions (2.14) we find

$$\begin{aligned}\langle j^{(+)}_{m,n}\rangle=-\frac{e}{8}\frac{1}{Mh}&\left[\sum_{k=0}^{M/2-1}-\sum_{k=-M/2}^{-1}\right]\\&\times(2+e^{ip_kh}\Sigma^*_{m,n}+e^{-ip_kh}\Sigma_{m,n})\,,\end{aligned} \tag{5.2}$$

where we have used the symbols defined in (4.8). When we use (4.3), we find

$$\langle j^{(+)}_{m,n}\rangle=-\frac{ie}{8Mh}\frac{2}{\sin\pi/M}(\Sigma^*_{m,n}-\Sigma_{m,n})\,. \tag{5.3}$$

If we regard $C_{m,n}$ in (4.8) as a constant in space, (5.3) becomes

$$\langle j_{m,n}^{(\pm)} \rangle = \mp \frac{e}{2Mh} \frac{1}{\sin\pi/M} \sin ehC_n$$
$$\approx \mp \left[\frac{e^2}{2M \sin\pi/M} \right] C_n \,, \tag{5.4}$$

where we have included the result for a similar calculation for $j_{m,n}^{(-)}$.

Thus, although the vacuum charge density is zero,

$$\langle j_{m,n}^{0} \rangle = \langle j_{m,n}^{(+)} + j_{m,n}^{(-)} \rangle = 0 \,, \tag{5.5}$$

the current density is nonzero,

$$\langle j_{m,n}^{1} \rangle = \langle j_{m,n}^{(+)} - j_{m,n}^{(-)} \rangle = -\frac{e^2}{M \sin\pi/M} C_n \,. \tag{5.6}$$

Effectively, then, we see the appearance of a mass μ in the corresponding continuum equation,

$$\partial_0^2 A_1 + \mu^2 A_1 = 0 \,; \tag{5.7}$$

the lattice value of μ^2 is again (2.10). According to (2.3), (5.6) implies the axial anomaly (2.9), as it must.

ACKNOWLEDGMENTS

We thank the U.S. Department of Energy and the Vice Provost for Research Administration, University of Oklahoma, for partial support of this work.

[1] C. M. Bender and D. H. Sharp, Phys. Rev. Lett. **50**, 1535 (1983).
[2] C. M. Bender, K. A. Milton, and D. H. Sharp, Phys. Rev. Lett. **51**, 1815 (1983).
[3] C. M. Bender, K. A. Milton, and D. H. Sharp, Phys. Rev. D **31**, 383 (1985). We refer to this paper as I.
[4] C. M. Bender, K. A. Milton, D. H. Sharp, L. M. Simmons, Jr., and R. Stong, Phys. Rev. D **32**, 1476 (1985).
[5] C. M. Bender, F. Cooper, V. P. Gutschick, and M. M. Nieto, Phys. Rev. D **32**, 1486 (1985).
[6] F. Cooper, K. A. Milton, and L. M. Simmons, Jr., Phys. Rev. D **32**, 2056 (1985).
[7] C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. D **33**, 1692 (1986).
[8] C. M. Bender, L. M. Simmons, Jr., and R. Stong, Phys. Rev. D **33**, 2362 (1986).
[9] C. M. Bender and M. L. Green, Phys. Rev. D **34**, 3255 (1986).
[10] C. M. Bender, L. R. Mead, and S. S. Pinsky, Phys. Rev. Lett. **56**, 2445 (1986).
[11] C. M. Bender and K. A. Milton, Phys. Rev. D **34**, 3149 (1986).
[12] C. M. Bender, F. Cooper, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. D **35**, 3081 (1987).
[13] We choose this gauge, rather than the more pathological Coulomb gauge, $A_1 = 0$. Recall B. Zumino, Phys. Lett. **10**, 224 (1964).
[14] We assume that the boson fields are periodic.
[15] Presumably, these statements are only true to $O(h^2)$.
[16] L. S. Brown, Nuovo Cimento **29**, 617 (1963).