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Novel perturbative scheme in quantum field theory

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A novel perturbative technique for solving quantum field theory is proposed. In this paper we explore this scheme in the context of self-interacting scalar field theory. For a ϕ^{2p} theory the method consists of expanding a $\phi^{2(1+\delta)}$ theory in powers of δ. A diagrammatic procedure for computing the terms in this series is given. We believe that for any Green's function the radius of convergence of this series is finite and is, in fact, 1. Moreover, while the terms in the unrenormalized series are individually divergent, they are considerably less so than in the standard weak-coupling perturbation series. In simple, low-dimensional quantum-field-theory models, the δ expansion gives excellent numerical results. We hope this new technique will ultimately shed some light on the question of whether a $(\phi^4)_4$ theory is free.

I. INTRODUCTION

This paper is an elaboration of a recent note[1] in which we introduced a new perturbative approach to quantum field theory. Throughout this paper we consider only self-interacting scalar field theory, although we believe that the method is fully applicable to fields of arbitrary spin. The fundamental idea described in this paper is very simple: given a self-interaction term of the form $\lambda\phi^{2p}$ we rewrite it as $\phi^{2(1+\delta)}$ and consider δ to be a small positive perturbation parameter. We find that the Green's functions are formal power series in δ. Specifically, if $G^{(n)}(x_1,\ldots,x_n;\delta)$ is the n-point Green's function then it has an expansion of the form

$$G^{(n)}(x_1,\ldots,x_n;\delta)=\sum_{k=0}^{\infty}\delta^k g_k^{(n)}(x_1,\ldots,x_n)\ . \tag{1.1}$$

We will describe diagrammatic rules for calculating $g_k^{(n)}$ in any dimension d of spacetime.

We will see that the coefficients $g_k^{(n)}$ possess a number of advantageous properties. They are complicated and nontrivial functions of the Lagrangian parameters such as coupling constants and masses. Thus, the perturbation expansion in δ achieves what nonperturbative computational schemes in quantum field theory attempt. Furthermore, in a theory with divergences (space-time dimension $d\geq 2$) the coefficients $g_k^{(n)}$ are much less divergent than the coefficients of a conventional weak-coupling perturbation expansion in powers of the coupling constant λ. Finally, we present evidence that the series (1.1) has a finite radius of convergence, in contrast with conventional weak-coupling expansions, which are at best asymptotic series (they have a zero radius of convergence). We believe that for $0\leq d\leq 4$, for any n, and for all values of $x_1,\ldots,x_n$, $G^{(n)}(x_1,\ldots,x_n;\delta)$ in (1.1) has a radius of convergence of 1. Thus, a $\lambda\phi^4$ theory sits on the circle of convergence. Moreover, we have numerical evidence that Padé theory provides an accurate analytic continuation to theories that are well outside the circle of convergence.

It is apparent that the new perturbation theory we introduce here is unconventional and potentially awkward. This is because the interaction term $\lambda\phi^{2(1+\delta)}$, when expanded in powers of δ, generates a formidable nonpolynomial Lagrangian containing all powers of $\ln(\phi^2)$:

$$\lambda\phi^{2(1+\delta)}=\lambda\phi^2\left[1+\delta\ln(\phi^2)+\frac{\delta^2}{2!}\ln^2(\phi^2)+\frac{\delta^3}{3!}\ln^3(\phi^2)+\cdots\right]. \tag{1.2}$$

The method proposed in this paper is based on the observation that derivatives of exponents produce logarithms:

$$\frac{d}{dx}a^x=a^x\ln a\ . \tag{1.3}$$

The computational scheme for obtaining the coefficients $g_k^{(n)}$ in (1.1) is an elaborate combinatorial generalization of (1.3). The coefficients $g_k^{(n)}$ emerge as the result of applying a derivative operator D to the Green's functions of a specially constructed polynomial Lagrangian $\tilde{L}$ which, because it is a polynomial Lagrangian, can be solved by ordinary, weak-coupling diagrammatic methods. Furthermore, the coefficients of the polynomial interaction terms are proportional to δ, so that only a finite number of diagrams need be evaluated to compute $g_k^{(n)}$.

We have organized this paper as follows. In Sec. II we discuss perturbation theory and distinguish between two different classes of perturbation expansion: natural and artificial. We give illustrative examples of both classes of perturbation expansion. We argue that artificial expansions, such as the one used in this paper, can have many inherent advantages. In Sec. III we develop the perturbative machinery used to obtain the expansion (1.1). In particular, we give formulas for the Lagrangian $\tilde{L}$ and the differential operator D. We illustrate the properties of the δ expansion in Sec. IV by considering an extremely simple model field theory in zero space-time dimension. Because this theory can be solved exactly, it is easy to identify the singularities in the complex-δ plane which determine that the radius of convergence is 1. We show how, using the diagrammatic method of Sec. III, to reproduce the terms in the δ expansion in (1.1) correct to order δ^2. We also show how Padé approximants can be used to sum the δ expansion outside its circle of convergence. Field theory in one-dimensional space-time (quantum mechanics) is treated in Sec. V. We compute the δ expansion to order δ^2 in two ways. First, we use ordinary Rayleigh-Schrödinger perturbation theory. Then we verify the resulting series using the method of Sec. III. In Sec. VI we examine the δ expansion within the context of the well-known $1/N$ approximation. Here, the scalar field ϕ is replaced by an N-component object $\boldsymbol{\phi}$ and the interaction term is taken to be $\lambda(\boldsymbol{\phi}\cdot\boldsymbol{\phi})(\boldsymbol{\phi}\cdot\boldsymbol{\phi}/N)^{\delta}$. In the limit $N\rightarrow\infty$ this theory can be solved exactly. We then expand the exact solution to third order as a series in powers of δ. Alternatively, we use the diagrammatic method described in Sec. III and keep only the leading graphs ("cactus" graphs) in $1/N$. We observe that for the two-, four-, and six-point Green's functions, we get the same δ series as were obtained previously. This shows that the $1/N$ approximation commutes with the δ expansion. In Sec. VII the diagrammatic method of Sec. III is applied to d-dimensional field theory through first order in δ, to determine the pole of the two-point function, and to determine the four- and six-point functions.

II. NATURAL VERSUS ARTIFICIAL PERTURBATION EXPANSIONS

Perturbation theory involves three distinct steps. First, one identifies or inserts a perturbation parameter ϵ which is treated as a small, positive number. Second, one seeks a solution as a perturbation series involving powers in ϵ and computes, using iterative methods, as many terms as possible in this series. Third, one attempts to extrapolate to the sum of the series from the limited number of terms that have been computed. (In cases where the series is divergent, powerful summation procedures such as Padé approximation must be used in this third step.)

In the initial step in this procedure one is faced with a fundamental choice, whether to use a naturally occurring parameter in the theory, such as a coupling constant, or to insert an entirely new expansion parameter into the theory. When a physical parameter is used to expand the theory, we shall call such an expansion a *natural* perturbation expansion. When a newly inserted parameter is used we shall call the resulting series an *artificial* perturbation expansion. In quantum field theory weak-coupling expansions in powers of λ and semiclassical (loop) expansions in powers of $\hbar$ are examples of natural perturbation expansions, while large N or $d-4$ expansions are artificial.

In this paper we advocate the position that natural expansions are inferior to artificial ones in several respects. First, most natural perturbation expansions in quantum field theory are divergent and some are so divergent that summation procedures are useless.[2] Even more important, natural perturbation expansions reveal only limited information about the dependence of the theory on the physical parameters, because one of the physical parameters is being used as an expansion parameter. Although the exact solution of the theory depends on the perturbing parameter in a complicated way, this dependence may be inaccessible if the solution is forced to have the form of a power series in the parameter.[3] For example, if a theory with an instanton is expanded in a weak-coupling series, the instanton sector is invisible to all orders in perturbation theory.

It is for this reason that deeper insights into quantum field theory have been sought in the form of nonperturbative numerical calculations, such as lattice simulations and large-N expansion. The artificial expansion proposed in this paper is nonperturbative in the sense that the expansion coefficients $g_k^{(n)}$ in (1.1) have a very complicated functional dependence on the Lagrangian parameters of the theory.

Perturbation theory is an art as well as a science; the first step, inserting an artificial perturbation parameter, requires insight and intuition. One can illustrate this point by discussing a model with a poorly chosen perturbation parameter. Consider, for example, the possibility of solving a $\lambda\phi^{2p}$ theory by expanding in powers of $\epsilon=1/p$. This approach is interesting because the leading term in the perturbation expansion ($p=\infty,\epsilon=0$) corresponds in the coordinate-space representation of the theory to solving a free field theory in an infinite-dimensional box $|\phi|<1$. That is to say, the range of functional integration at each lattice site i is reduced from $-\infty<\phi_i<\infty$ to $-1\leq\phi_i\leq 1$. However, this perturbative method has several drawbacks. Although in most perturbative schemes, once the leading-order approximation is computed, higher-order approximations are routine,[4] here it is not at all clear how to compute higher-order corrections. Furthermore, expanding about the point $p=\infty$ corresponds to approaching the renormalizable theory $\lambda\phi^4$ along a path of nonrenormalizable theories with $2p>4$.

These difficulties are readily apparent in quantum mechanics (dimension $d=1$). In Fig. 1 we have plotted

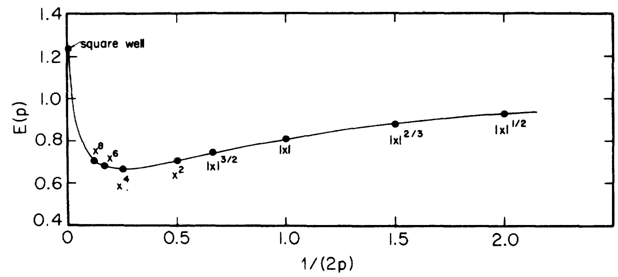

FIG. 1. The ground-state energy $E(p)$ for the Hamiltonian H in (2.1) as a function of the parameter $1/(2p)$. Observe that $E(\infty)$ is a very poor approximation to $E(p)$ for $p \lesssim 4$. The values of $E(p)$ are taken from Ref. 5.

the ground-state energy $E(p)$ of the Hamiltonian

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+x^{2p} . \tag{2.1}$$

When $p=\infty$, this Hamiltonian describes a free particle confined to an infinite square well, $|x| \leq 1$. Thus, to leading order, the ground-state energy $E(\infty)=\pi^2/8$. However, it is a difficult calculation to determine the higher-order terms in the series in powers of $1/p$. Moreover, from Fig. 1 we see that $E(\infty)$ is a terrible approximation to the ground-state energy $E(p)$ for $p<4$ (Ref. 5). (It is wrong by almost a factor of 2.) Thus, it is not numerically promising to approximate quartic or parabolic wells by a square well.

On the other hand, Fig. 1 lends strong numerical support for the utility of the δ-expansion method outlined in Sec. I. The energy function $E(p)$ is quite flat between $p=1$ and $p=4$. The ground-state energy varies by only 5% over this range, on which δ changes from 0 to 3.

III. LOGARITHMIC-EXPANSION METHOD

The theory we are considering in this paper is defined by the Lagrangian

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\mu^2\phi^2+\lambda M^2\phi^2(M^{2-d}\phi^2)^{\delta} , \tag{3.1}$$

where d is the space-time dimension, μ is the bare mass, and M is a mass parameter introduced to give the correct dimensions to the interaction term with λ the dimensionless coupling constant. The exponent δ is the expansion parameter which we regard as small and positive.

As we observed in Sec. I, the Lagrangian L in (3.1) has an awkward expansion in powers of δ:

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2 +\lambda M^2\phi^2\sum_{k=1}^{\infty}\frac{\delta^k}{k!}[\ln(\phi^2M^{2-d})]^k . \tag{3.2}$$

The perturbative terms in this form of the Lagrangian are nonpolynomial and therefore there is no obvious simple diagrammatic procedure for computing the Green's functions as power series in δ as given in (1.1).

The main progress that we have to report in this paper is a diagrammatic recipe for calculating the coefficients $g_k^{(n)}$ in (1.1). The recipe consists of introducing a new provisional Lagrangian $\tilde{L}$ which has polynomial interaction terms and whose Green's functions $\tilde{G}^{(n)}$ we can relate to $g_k^{(n)}$. We compute the Green's functions $\tilde{G}^{(n)}$ for this Lagrangian using ordinary diagrammatic methods and apply a linear derivative operator D to $\tilde{G}^{(n)}$. The result of this computation is a finite number of terms of the expansion (1.1) for the Green's function $G^{(n)}$.

More precisely, the recipe proceeds as follows. The leading term in the series (1.1) (the coefficient of δ^0) is simply the n-point Green's function of a free theory whose bare mass is $\mu^2+2\lambda M^2$. Next we must decide in advance to what order in powers of δ we want to calculate $G^{(n)}$. We specify this order by the integer K. Given K we can specify the derivative operator D and the provisional Lagrangian $\tilde{L}$. We emphasize that for each value of K there is a corresponding D and $\tilde{L}$.

The general form of $\tilde{L}$ is

$$\tilde{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2 +\lambda M^d\sum_{k=1}^{K}(\phi^2M^{2-d})^{\alpha_k+1}P_k , \tag{3.3}$$

where the coefficients P_k are polynomials in α_1, $\alpha_2,\ldots,\alpha_k$, and δ. Observe that $\tilde{L}$ describes a theory having K different self-interaction terms of the form $(\phi^2)^{\alpha_k+1}$. We regard $\alpha_1,\alpha_2,\ldots,\alpha_K$, in the initial stages of the calculation, as integers so that simple diagrammatic rules apply to $\tilde{L}$. The polynomials P_k do not have a simple form. We give the first four sets of polynomials below. For $K=1$,

$$P_1=\delta ; \tag{3.4}$$

for $K=2$,

$$P_1=\delta+\delta^2 \text{ and } P_2=-\delta+\delta^2 ; \tag{3.5}$$

for $K=3$,

$$\begin{aligned} P_1&=\delta+\frac{1+\alpha_1}{2}\delta^2+\delta^3 ,\\ P_2&=\delta\omega+\frac{\omega^2+\alpha_2}{2}\delta^2+\delta^3 ,\\ P_3&=\delta\omega^2+\frac{\omega+\alpha_3}{2}\delta^2+\delta^3 , \end{aligned} \tag{3.6}$$

where $\omega=\exp(2\pi i/3)$; for $K=4$,

$$\begin{aligned}
P_1&=\delta+[\tfrac{1}{3}+\tfrac{1}{6}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\\
&\quad+\tfrac{1}{2}(\alpha_1^2-i\alpha_2^2-\alpha_3^2+i\alpha_4^2)]\delta^2\\
&\quad+\tfrac{1}{9}(4+5\alpha_1)\delta^3+\delta^4\ ,\\
P_2&=i\delta+[-\tfrac{1}{3}-\tfrac{1}{6}i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\\
&\quad+\tfrac{1}{2}(i\alpha_1^2+\alpha_2^2-i\alpha_3^2-\alpha_4^2)]\delta^2\\
&\quad+\tfrac{1}{9}(-4i+5\alpha_2)\delta^3+\delta^4\ ,\\
P_3&=-\delta+[\tfrac{1}{3}-\tfrac{1}{6}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\\
&\quad+\tfrac{1}{2}(-\alpha_1^2+i\alpha_2^2+\alpha_3^2-i\alpha_4^2)]\delta^2\\
&\quad+\tfrac{1}{9}(-4+5\alpha_3)\delta^3+\delta^4\ ,\\
P_4&=-i\delta+[-\tfrac{1}{3}+\tfrac{1}{6}i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\\
&\quad+\tfrac{1}{2}(-i\alpha_1^2-\alpha_2^2+i\alpha_3^2+\alpha_4^2)]\delta^2\\
&\quad+\tfrac{1}{9}(4i+5\alpha_4)\delta^3+\delta^4\ .
\end{aligned}\tag{3.7}$$

Observe that every polynomial P_k contains at least one power of δ. Thus, in the Feynman rules for $\tilde{L}$ every vertex is proportional to δ. As a result, a calculation exact to order δ^K requires diagrams having at most K vertices, and only a finite number of diagrams are required.

Having computed all diagrams contributing to $\tilde{G}^{(n)}$, we now regard the parameters α_k as continuous, and we define a derivative operator D which acts on α_k. Unlike $\tilde{L}$, D has a simple and general formula for all values of K:

$$D=\frac{1}{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\frac{\exp[2\pi ij(1-k)/K]}{j!}\left[\frac{\partial}{\partial\alpha_k}\right]^j .\tag{3.8}$$

After applying this derivative operator to $\tilde{G}^{(n)}$ we evaluate the result at $\alpha_1=\alpha_2=\cdots=\alpha_K=0$. This procedure yields $G^{(n)}$. Note that the Green's functions $G^{(n)}$ of the Lagrangian L are derivatives of the Green's functions $\tilde{G}^{(n)}$ on $\tilde{L}$ at the point where $\tilde{L}$ is a free Lagrangian. We illustrate the use of the above recipe for field theories in various dimensions in the next four sections of this paper.

We close this section with a few remarks on the derivation of $\tilde{L}$ and D. One can verify the recipe given above in the context of a functional integral representation of the Green's functions $G^{(n)}$. The computation is extremely lengthy and we do not present it here. The structures of $\tilde{L}$ and D given in (3.3)–(3.8) were initially found by considering general forms and requiring that for all n, $D\tilde{G}^{(n)}|_{\alpha_i=0}=G^{(n)}$, correct to order δ^K.

It is not clear whether the forms of D and $\tilde{L}$ given above are unique. However, we have tried to find other forms and have failed. The form for the derivative operator D in (3.8) is so simple that we were able to guess its general form at $K=3$ and verify it at $K=4$. On the other hand, we have been unable to find a general form for the polynomials P_k.

IV. ZERO-DIMENSIONAL FIELD THEORY

To illustrate the structure of the δ expansion and the diagrammatic recipe given in Sec. III, we consider an extremely simple field theory in zero space-time dimension. The partition function in this model field theory is

$$Z=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-(x^2)^{1+\delta}} .\tag{4.1}$$

A. Radius of convergence of δ series

The integral in (4.1) can be evaluated exactly in terms of Γ functions because there is no mass term. The result is

$$Z=\frac{2}{\sqrt{\pi}}\Gamma\left[\frac{3+2\delta}{2+2\delta}\right] .\tag{4.2}$$

Let us consider the free energy $E(\delta)$ which is defined as

$$E(\delta)=-\ln Z\ .\tag{4.3}$$

It is easy to expand the energy in (4.3) as a series in powers of δ. The terms in this series involve the ψ function $\psi(x)\equiv\Gamma'(x)/\Gamma(x)$ and its derivatives:

$$\begin{aligned}
E(\delta)&=\frac{\delta}{2}\psi(\tfrac{3}{2})\delta^2[-\tfrac{1}{2}\psi(\tfrac{3}{2})-\tfrac{1}{8}\psi'(\tfrac{3}{2})]\\
&\quad+\delta^3[\tfrac{1}{2}\psi(\tfrac{3}{2})+\tfrac{1}{4}\psi'(\tfrac{3}{2})+\tfrac{1}{48}\psi''(\tfrac{3}{2})]\\
&\quad+\delta^4[-\tfrac{1}{2}\psi(\tfrac{3}{2})-\tfrac{3}{8}\psi'(\tfrac{3}{2})-\tfrac{1}{16}\psi''(\tfrac{3}{2})\\
&\quad-\tfrac{1}{384}\psi'''(\tfrac{3}{2})]+\cdots\ .
\end{aligned}\tag{4.4}$$

To determine the radius of convergence of this series it is necessary to determine the singularities of $E(\delta)$ in (4.2) and (4.3). From (4.2) we see that branch-point singularities occur whenever $(3+2\delta)/(2+2\delta)=-m$, where $m=0,1,2,3,\ldots$. Thus the singularities of $E(\delta)$ are located at

$$\delta_m=-\frac{2m+3}{2m+2} ,\tag{4.5}$$

which form a sequence lying between $-\frac{3}{2}$ and -1 on the real δ axis and having an accumulation point at $\delta=-1$ (see Fig. 2). Thus, the radius of convergence of the Taylor series in (4.4) is 1.

B. Diagrammatic calculation (K = 1)

Next we show how to use the machinery described in Sec. III to derive the series (4.4). The Lagrangian L for this zero-dimensional theory is just $(x^2)^{1+\delta}$. This corresponds to the general Lagrangian (3.1) in zero space-time dimensions, with bare mass $\mu=0$, $\lambda=1$, and $M^2\phi^2=x^2$.

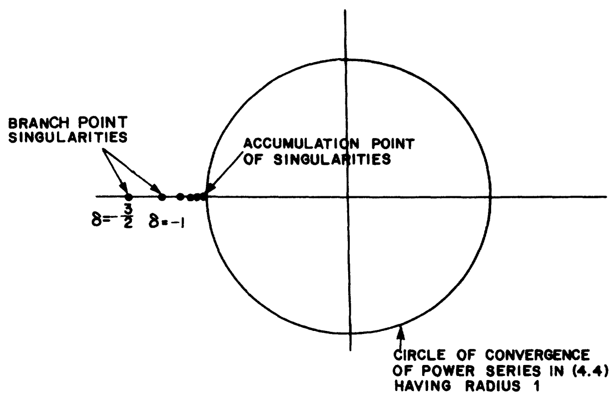

FIG. 2. The branch-point singularities of $E(\delta)$ in (4.3) in the complex-δ plane. Note that $E(\delta)$ is analytic in a circle of radius 1 about the origin.

First we consider the case $K=1$. For this case (3.3) becomes

$$\tilde{L}_{K=1}=x^2+\delta(x^2)^{\alpha+1}\ , \tag{4.6}$$

where we have written α in place of α_1. For this Lagrangian we determine the Feynman rules by writing the vacuum amplitude $\tilde{Z}$ in the presence of an external source:

$$\tilde{Z}=\int\frac{dx}{\sqrt{\pi}}e^{-x^2-\delta(x^2)^{\alpha+1}+Jx}\ . \tag{4.7}$$

The vertex amplitude at the $(2\alpha+2)$-point vertex is the coefficient of $x^{2\alpha+2}/(2\alpha+2)!$ in the exponent:

$$\text{Vertex},\ -(2\alpha+2)!\delta\ . \tag{4.8a}$$

To determine the line amplitude we set $\delta=0$, evaluate the resulting Gaussian integral, and obtain $\exp(J^2/4)$. From this we read off the amplitude for a line which is the coefficient of $J^2/2$:

$$\text{Line},\ \tfrac{1}{2}\ . \tag{4.8b}$$

The energy $\tilde{E}(\delta)$ is defined as the negative of the sum of the connected vacuum graphs (the Green's function with no external legs, $n=0$). Because we are calculating $\tilde{E}(\delta)$ to order $\delta(K=1)$, and because the vertex amplitude in (4.8a) is of order δ, only one graph is required. This is the one-vertex graph shown in Fig. 3. The amplitude for this graph is the product of the vertex amplitude in (4.8a), a factor of $2^{-\alpha-1}$ (because there are $\alpha+1$ lines in the graph) and the symmetry number of the graph, which

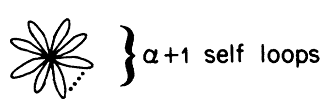

FIG. 3. The only vacuum graph in a $\phi^{2\alpha+2}$ theory having one vertex. This graph has $\alpha+1$ self-loops. It is of order δ.

is $2^{-\alpha-1}/(\alpha+1)!$. Thus,

$$\tilde{E}(\delta)=\frac{\delta(2\alpha+2)!}{2^{2\alpha+2}(\alpha+1)!}\ . \tag{4.9}$$

We use the Legendre duplication formula to simplify (4.9) slightly:

$$\tilde{E}(\delta)=\frac{\delta}{2}\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\frac{3}{2})}\ . \tag{4.10}$$

Having derived the expression (4.10) we no longer regard α as an integer but rather as a continuous variable. From (3.8) the derivative operator corresponding to $K=1$ is

$$D_{K=1}=\frac{\partial}{\partial\alpha}\ . \tag{4.11}$$

Following the recipe of Sec. III, we apply (4.11) to (4.10) and evaluate the result at $\alpha=0$ to obtain

$$E(\delta)=\frac{\delta}{2}\psi(\tfrac{3}{2})+O(\delta^2)\ , \tag{4.12}$$

which agrees with the first term in (4.4).

C. Diagrammatic calculation ($K=2$)

Now we derive the first two terms in the series (4.4). From (3.3) and (3.5) we have

$$\tilde{L}_{K=2}=x^2+(\delta+\delta^2)(x^2)^{\alpha+1}+(-\delta+\delta^2)(x^2)^{\beta+1}\ . \tag{4.13}$$

The Feynman rules for this Lagrangian are determined from the vacuum amplitude in the presence of an external source just as before:

$$\begin{aligned}&(2\alpha+2)\text{-point vertex},\ -(\delta+\delta^2)(2\alpha+2)!\ ;\\&(2\beta+2)\text{-point vertex},\ -(-\delta+\delta^2)(2\beta+2)!\ ;\\&\text{Line},\ \tfrac{1}{2}\ .\end{aligned} \tag{4.14}$$

There are five connected vacuum graphs which contribute to $\tilde{E}(\delta)$ to order δ^2. These are shown in Fig. 4. We have already seen how to evaluate graphs (a) and (b); the only change from the result in (4.10) comes from the vertex factor. The contribution of these two graphs to $\tilde{E}(\delta)$ is

$$\frac{(\delta+\delta^2)}{2}\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\frac{3}{2})}+\frac{(-\delta+\delta^2)}{2}\frac{\Gamma(\beta+\frac{3}{2})}{\Gamma(\frac{3}{2})}\ . \tag{4.15}$$

Next we consider graph (c). This graph is actually not one graph but a large class of graphs parametrized by the integer l, $l=1,2,\ldots,\alpha+1$, where $2l$ is the number of lines connecting the two vertices. For each graph of type (c) the symmetry number is

$$\frac{1}{2(2^{\alpha+1-l})^2(2l)![(\alpha+1-l)!]^2}\ , \tag{4.16a}$$

the contribution from the vertices is

$$\delta^2[(2\alpha+2)!]^2\ , \tag{4.16b}$$

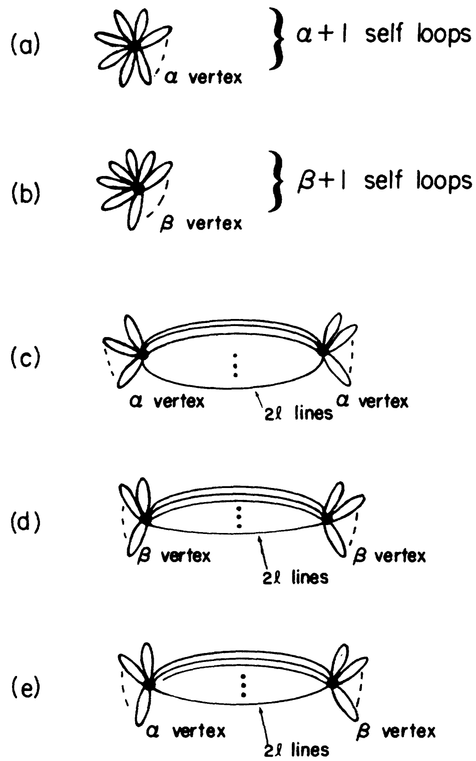

FIG. 4. The five types of connected vacuum graphs which have one or two vertices. Graphs of type (c), (d), and (e) are actually large classes of graphs parametrized by the integer $2l$, the number of lines joining the two vertices.

where we have dropped terms of order δ^3, and the contribution from the lines is

$$\frac{1}{2^{2\alpha+2}} . \tag{4.16c}$$

To find the contribution of all graphs of type (c), we multiply together the expressions in (4.16), multiply by -1, and sum on l:

$$-\sum_{l=1}^{\alpha+1} \frac{\delta^2[(2\alpha+2)!]^2}{2^{4\alpha+5-2l}(2l)![(\alpha+1-l)!]^2} . \tag{4.17}$$

To evaluate the sum on l, we use the following identity:

$$\sum_{l=1}^{\min(\alpha+1,\beta+1)} \frac{4^l}{(2l)!(\alpha+1-l)!(\beta+1-l)!} = \frac{1}{(\alpha+1)!(\beta+1)!}\left[\frac{\Gamma(\alpha+\beta+\frac{5}{2})\sqrt{\pi}}{\Gamma(\alpha+\frac{3}{2})\Gamma(\beta+\frac{3}{2})}-1\right] . \tag{4.18}$$

Specializing this identity to the case $\beta=\alpha$, we evaluate (4.17) to be

$$-\frac{\delta^2}{2\pi}[\Gamma(2\alpha+\tfrac{5}{2})\sqrt{\pi}-\Gamma^2(\alpha+\tfrac{3}{2})] . \tag{4.19}$$

The contribution from the graphs (d) is obtained from (4.19) by replacing α with β:

$$-\frac{\delta^2}{2\pi}[\Gamma(2\beta+\tfrac{5}{2})\sqrt{\pi}-\Gamma^2(\beta+\tfrac{3}{2})] . \tag{4.20}$$

Graph (e) represents a class of graphs again parametrized by an integer l, $l=1,2,\ldots,\ \min(\alpha+1,\beta+1)$, where $2l$ is the number of lines connecting the two vertices. For each graph of type (e), the symmetry number is

$$\frac{1}{(2^{\alpha+1-l})(2^{\beta+1-l})(2l)!(\alpha+1-l)!(\beta+1-l)!} , \tag{4.21a}$$

the vertex factor is

$$-\delta^2(2\alpha+2)!(2\beta+2)! , \tag{4.21b}$$

and the line contribution is

$$\frac{1}{2^{\alpha+\beta+2}} . \tag{4.21c}$$

Multiplying the factors (4.21) together, changing the sign, and summing on l using the identity (4.18), we find the contribution from graphs (e) to be

$$\frac{\delta^2}{\pi}[\Gamma(\alpha+\beta+\tfrac{5}{2})\sqrt{\pi}-\Gamma(\alpha+\tfrac{3}{2})\Gamma(\beta+\tfrac{3}{2})] . \tag{4.22}$$

The final expression for $\tilde{E}(\delta)$ is obtained by adding the partial answers in (4.15), (4.19), (4.20), and (4.22). Having derived this expression, we now regard α and β as continuous variables. From (3.8) the derivative operator for $K=2$ is

$$D_{K=2}=\frac{1}{2}\left[\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta}\right]+\frac{1}{4}\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right] . \tag{4.23}$$

Applying (4.23) to $\tilde{E}(\delta)$ and setting $\alpha=\beta=0$, we obtain the energy $E(\delta)$ correct to order δ^2:

$$E(\delta)=\frac{\delta}{2}\psi(\tfrac{3}{2})-\tfrac{1}{2}\delta^2[\psi(\tfrac{3}{2})+\tfrac{1}{4}\psi'(\tfrac{3}{2})]+O(\delta^3) , \tag{4.24}$$

which agrees with the first two terms in (4.4).

The calculations that were done in Secs. IV B and IV C play an important role in establishing confidence in the recipe given in Sec. III. The one nonrigorous step is the generalization of α_k from an integer variable to a continuous variable, which subsequently is differentiated. In general, analytic continuation off the integers is a unique process only if one has precise information about the asymptotic behavior of the function, which we have not supplied. The calculations of this section, together with those presented in Secs. V and VI, where the δ expansion is already known using an independent method, lend strong support to the procedure described in Sec. III.

D. Summation of the δ series

We have already demonstrated that the series calculated in this section has a radius of convergence of 1. However, many interesting field theories correspond to values of δ lying on or outside the circle of convergence. Therefore, it is important to be able to extract information about such theories from the δ series in (4.4). The coefficients in (4.4) can be shown to alternate in sign. This suggests that Padé summation is a natural way to analytically continue the series outside of its circle of convergence. In Table I we compare the exact values of $E(\delta)$ with the [3,2] and [5,4] Padé approximants constructed from the δ series for several values of δ. We find that the numerical accuracy of the Padé approximants is extremely good.

V. ONE-DIMENSIONAL FIELD THEORY

In the previous section we illustrated and verified in the context of a trivial zero-dimensional field theory the diagrammatic recipe given in Sec. III for calculating the δ series. We consider in this section a much less trivial field theory model in one-dimensional space-time (quantum mechanics). We choose to work in quantum mechanics because, once again, we are able to calculate several terms in the δ expansion using an independent method (here, Rayleigh-Schrödinger perturbation theory) and to compare the results with a direct diagrammatic calculation of the δ series.

Specifically, we consider the Lagrangian (3.1) with $\mu=0$ and $\lambda=\frac{1}{2}$ in $d=1$ space-time dimensions. In the coordinate-space representation the ground-state energy is determined by the time-independent Schrödinger equation

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}M^{2+\delta}x^{2(1+\delta)}-E\right]\psi(x)=0 . \qquad (5.1)$$

A. Rayleigh-Schrödinger perturbation theory

In this subsection we show how to compute the ground-state energy $E(\delta)$ through order δ^2 using conventional techniques. To obtain the leading-order result, we simply set $\delta=0$ in (5.1). This gives the differential equation for the harmonic oscillator

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}M^2x^2-E_0\right]\phi_0(x)=0 . \qquad (5.2)$$

The lowest-energy solution to this equation is

$$E_0=\frac{M}{2},\quad \phi_0=e^{-Mx^2/2} . \qquad (5.3)$$

Next we seek a solution to (5.1) in the form of a series in powers of δ:

$$\psi(x)=\phi_0(x)\sum_{k=0}^{\infty}F_k(x)\delta^k,\quad E=\sum_{k=0}^{\infty}E_k\delta^k . \qquad (5.4)$$

We substitute (5.4) into (5.1) and identify the coefficient of δ^k as

$$F_k''(x)-2xMF_k'(x)-M^2x^2\sum_{j=l}^{k}\frac{1}{J!}[\ln(Mx^2)]^jF_{k-j}(x)+2\sum_{j=1}^{k}E_jF_{k-j}(x)=0 . \qquad (5.5)$$

This equation is easy to solve using the integrating factor ϕ_0^2. It is best to take as boundary conditions

$$F_n(0)=0,\quad n\geq 1,\quad F_n'(0)=0,\quad n\geq 1 . \qquad (5.6)$$

From (5.5) and (5.6) we obtain an expression which determines the expansion coefficients E_k of the energy:

$$0=\int_0^{\infty}dx\,e^{-Mx^2}\sum_{j=1}^{k}\left[\frac{[\ln(Mx^2)]^j}{j!}M^2x^2-2E_j\right]F_{k-j}(x) . \qquad (5.7)$$

Computing E_1 and E_2 from the above formula requires the evaluation of very complicated triple integrals. We do not present the details of the calculation here, but merely give the results:

$$E_1=\frac{M}{4}\psi(\tfrac{3}{2}) , \qquad (5.8)$$

$$F_1(x)=\int_0^x dt\,e^{Mt^2}\int_0^t ds\,e^{-Ms^2}[M^2s^2\ln(Ms^2)-2E_1] , \qquad (5.9)$$

$$E_2=\frac{M}{2}[-\tfrac{1}{32}\psi''(\tfrac{3}{2})-\tfrac{1}{4}\psi'(\tfrac{3}{2})\ln 2+\tfrac{1}{4}\psi(\tfrac{3}{2})^2-\tfrac{1}{2}\psi(\tfrac{3}{2})+1-\ln 2] . \qquad (5.10)$$

We note a provocative similarity between the series for E given by (5.4), (5.8), and (5.10) and zero-dimensional series given in (4.4); both involve the ψ function and its derivative evaluated at $\frac{3}{2}$.

TABLE I. Comparison between the 10-term power series (p_{10}), 20-term power series (p_{20}), [3,2] Padé approximant ($pd_{[3,2]}$), [5,4] Padé approximant ($pd_{[5,4]}$, and the exact energy $E(\delta)$ in (4.3) for the zero-dimensional field theory given by (4.1).

δ	$p_{10}(\delta)$	$p_{20}(\delta)$	$pd_{[3,2]}(\delta)$	$pd_{[5,4]}(\delta)$	$E(\delta)$
−2.0	−1266.99	-2.0×10^6	−0.651 268	−0.693 111	−0.693 147
−0.5	−0.120 055	−0.120 781	−0.120 831	−0.120 782	−0.120 782
0.5	−0.007 817 11	−0.007 590 89	−0.007 590 95	−0.007 590 58	−0.007 590 60
1.0	−0.367 106	−0.517 356	−0.022 516 6	−0.022 510 3	−0.022 510 4
2.0	−465.831	-6.9×10^5	−0.045 814 4	−0.045 756 1	−0.045 756 2
5.0	-5.5×10^6	-7.8×10^{13}	−0.078 666 8	−0.078 172 7	−0.078 172 9

B. Diagrammatic derivation

Because we are calculating the ground-state energy, we follow exactly the calculations that were done for zero space-time dimensions in Secs. IV B and IV C and reevaluate the graphs in Figs. 3 and 4. The only change in the calculation concerns the Feynman amplitude for a line, which was 1/2 in zero dimensions and in the insertion of a factor of $M^{\alpha+2}$ in the $(2\alpha+2)$-point vertex amplitude. In one-dimensional Euclidean momentum space it is $(p^2+M^2)^{-1}$, or $\Delta(x-y)=\exp(-|x-y|M)/(2M)$ in one-dimensional Euclidean coordinate space. Therefore, each closed self-loop (or petal) in the graph in Figs. 3 and 4 gives a factor of $\Delta(0)=1/(2M)$ instead of $\frac{1}{2}$. Also, in the graphs (c)–(e) in Fig. 4 the $2l$ lines connecting the vertices at x and y give a factor of

$$\int_{-\infty}^{\infty} dx\,[\Delta(x-y)]^{2l}=\frac{1}{(2M)^{2l}}\frac{1}{lM} . \tag{5.11}$$

Thus, for example, the overall contribution from the lines in graph (a) of Fig. 4 is $(2M)^{-\alpha-1}$ instead of $2^{-\alpha-1}$. In graph (c) the lines contribute $(2M)^{-2\alpha-2}/(lM)$ instead of $2^{-2\alpha-2}$ and in graph (e) the lines give $(2M)^{-\alpha-\beta-2}/(lM)$ instead of $2^{-\alpha-\beta-2}$. Therefore, for example, the zero-dimensional amplitude (4.17) corresponding to all graphs of type (c) is replaced, in one dimension, by

$$-\sum_{l=1}^{\alpha+1}\frac{\delta^2[(2\alpha+2)!]^2M^{2\alpha+4}}{2^{2\alpha+5-2l}(2M)^{2\alpha+2}(lM)(2l)![(\alpha+1-l)!]^2} . \tag{5.12}$$

We now continue to follow the procedure in Sec. IV; we again apply the differential operator $D_{k=2}$ in (4.23) to the sum of the amplitudes for the graphs in Fig. 4 now evaluated in one-dimensional space-time. The results (5.8) and (5.10) for the expansion of the ground-state energy are reproduced exactly.

VI. VERIFICATION OF THE δ-EXPANSION METHOD IN THE LARGE-N APPROXIMATION

The large-N method provides a technique for obtaining an approximation to a quantum field theory of N interacting scalar fields with an O(N) symmetry.[6,7] For a theory whose interaction has the form $(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^p$, where $\boldsymbol{\phi}$ has N components, one can readily obtain for each of the Green's functions a closed-form expression to leading order in powers of $1/N$. These leading-order large-N Green's functions provide an excellent testing ground for verifying the δ-expansion diagrammatic method.

There is a well-defined procedure for calculating the leading-order large-N Green's functions. It consists of selecting from the set of all weak-coupling graphs only those graphs (called cactus graphs) that contribute in the limit as $N\rightarrow\infty$. In this section we find exact expressions for the large-N limit of the two-, four-, and six-point Green's functions in a $(\boldsymbol{\phi}^2)^{1+\delta}$ theory using these cactus graphs. Then, we expand each of these Green's functions as series in powers of δ. We show that the radius of convergence of each of these series in the complex-δ plane is 1. Next, we recalculate these δ series using the δ-expansion diagrammatic rules for the Lagrangian $\tilde{L}$ given in Sec. III together with the restriction that we retain only those graphs (the cactus graphs) that dominate the $1/N$ expansion for large N. We calculate these Green's functions to third order in powers of δ. The results agree completely with the expansions of the exact large-N Green's functions. This indicates that the process of expanding in δ commutes with the large-N limit and further confirms the validity of the recipe given in Sec. III.

A. The graphical rules for the large-N expansion

The large-N approximation applies to the d-dimensional Euclidean Lagrangian

$$L=\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}\mu^2\boldsymbol{\phi}^2+\lambda M^2\boldsymbol{\phi}^2\left[\frac{M^{2-d}}{N}\boldsymbol{\phi}^2\right]^{\delta} . \tag{6.1}$$

In the limit $N\rightarrow\infty$ with μ, λ, and M fixed, a variational calculation, employing a Gaussian ground-state wave functional, gives the exact solution to the theory. This solution can also be obtained by a set of graphical rules (summing only cactus graphs) that can be derived from a saddle-point expansion of the functional-integral representation of the vacuum amplitude.

The large-N graphical rules for the Lagrangian (6.1) are the usual Feynman rules augmented as follows.

(1) Only "cactus" graphs are selected. These are graphs in which any pair of internal loops cannot have more than one vertex in common. For example, in Fig. 4, the only cactus graphs of type (c), (d), or (e) are those with $l=1$.

(2) The vertex factor is

$$-\lambda M^2M^{\delta(2-d)}2^{\delta+1}(\delta+1)! , \tag{6.2}$$

where we have set $N=1$. The crucial point here is that the factor $(2\delta+2)!$, which would occur in ordinary Feynman rules, is replaced by $2^{\delta+1}(\delta+1)!$. This is a combinatoric device whose purpose is to exclude graphs whose loops contain propagators belonging to different components of $\boldsymbol{\phi}$. The excluded graphs are suppressed by a factor of at least $1/N$.

B. Two-point function

Consider first the calculation of the two-point function. The simplest one-particle-irreducible (OPI) graph contributing to the two-point function is shown in Fig. 5(a). From the Lagrangian in (6.1) it follows that in momentum space each line is represented by $(p^2+\mu^2)^{-1}$ and each petal by

$$I(\mu^2)=\frac{1}{(2\pi)^d}\int\frac{d^dp}{p^2+\mu^2} . \tag{6.3}$$

To evaluate the graph in Fig. 5(a) we multiply together the vertex factor (6.2), the symmetry number $2^{-\delta}/\delta!$ for the graph, and the value of the Feynman integral, which is $[I(\mu^2)]^\delta$. The result is

$$-2\lambda M^2[M^{2-d}I(\mu^2)]^\delta(\delta+1) . \tag{6.4}$$

The effect of including all higher-order diagrams, such as those in Figs. 5(b) and 5(c) is to replace μ in (6.4) by the renormalized mass m_R. That is, the sum of all one-

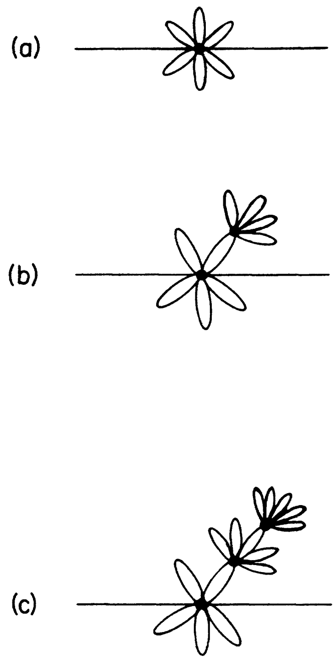

FIG. 5. One-particle-irreducible "cactus" graphs contributing to the two-point Green's function.

particle-irreducible graphs is

$$\mathrm{OPI}=-2\lambda M^2[M^{2-d}I(m_R^2)]^{\delta}(\delta+1)\ . \tag{6.5}$$

Thus, the exact result for the leading $1/N$ approximation to the two-point function $G^{(2)}$ is

$$G^{(2)}(p)=\frac{1}{p^2+\mu^2-\mathrm{OPI}}\ . \tag{6.6}$$

We identify the renormalized mass squared as $\mu^2-\mathrm{OPI}$. Thus, the renormalized mass satisfies the implicit gap equation

$$m_R^2=\mu^2+2\lambda M^2[M^{2-d}I(m_R^2)]^{\delta}(\delta+1)\ . \tag{6.7}$$

C. Four-point function

We turn next to the calculation of the four-point Green's function. The four-point function is the sum of the amplitudes corresponding to the graphs in Fig. 6, where it is understood that internal lines carry the renormalized mass m_R, and are represented by the amplitude $(p^2+m_R^2)^{-1}$. We define

$$J(m_R^2)=\frac{1}{(2\pi)^d}\int\frac{d^dp}{(p^2+m_R^2)^2}\ . \tag{6.8}$$

The sum of the geometric series represented by Fig. 6 is straightforward and gives the four-point function at zero momentum for all external legs:

$$G^{(4)}(0)=\frac{2A}{J(m_R^2)}\frac{1}{1-A}\ , \tag{6.9a}$$

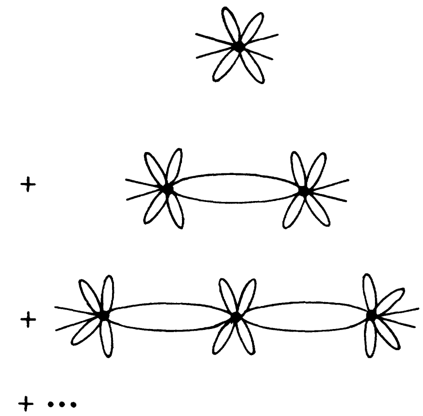

FIG. 6. Classes of "cactus" graphs contributing to the four-point Green's function.

where

$$A=-2\lambda M^2(M^{2-d})^{\delta}\delta(\delta+1)J(m_R^2)[I(m_R^2)]^{\delta-1}\ . \tag{6.9b}$$

D. Six-point function

All graphs contributing to the six-point function lie in two topologically distinct classes shown in Figs. 7(a) and 7(c). A more complicated member of class (a) is shown in

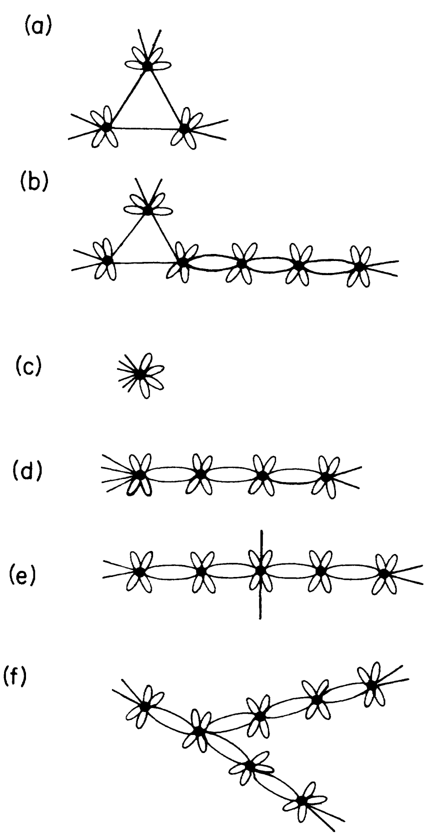

FIG. 7. "Cactus" graphs contributing to the six-point Green's function.

Fig. 7(b) and more complicated members of class (c) are shown in Figs. 7(d)–7(f). The sum of all such graphs can be evaluated in closed form. We define the functions

$$K(m_R^2)=\frac{1}{(2\pi)^d}\int\frac{d^dp}{(p^2+m_R^2)^3}\,,$$
$$B=-8\lambda M^2(M^{2-d})^{\delta}\delta(\delta^2-1)[I(m_R^2)]^{\delta-2}\,, \tag{6.10}$$

in terms of which the six-point function for all external momenta equal to zero is

$$G^{(6)}(0)=\frac{1}{(1-A)^3}\left[B+K(m_R^2)\left[\frac{2A}{J(m_R^2)}\right]^3\right]. \tag{6.11}$$

E. Expansion in δ

The implicit formula for m_R^2 in (6.7) and the expression for $G^{(4)}(0)$ and $G^{(6)}(0)$ in (6.9) and (6.11) can be expanded in powers of δ with coefficients expressed in terms of the parameters μ, λ, and M^2. The results are

$$\begin{aligned}m_R^2=&\mu^2+2\lambda M^2+\delta[2\lambda M^2(1+L)]+\delta^2\left[\lambda M^2L(L+2)-4\lambda^2M^4(1+L)\frac{J(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}\right]\\
&+\delta^3\left[\lambda M^2L^2(\tfrac{1}{3}L+1)-2\lambda^2M^4(3L^2+6L+2)\frac{J(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}\right.\\
&\left.+8\lambda^3M^6(L+1)^2\frac{K(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}-4\lambda^3M^6(L^2-1)\left[\frac{J(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}\right]^2\right]\\
&+\delta^4\left[\frac{\lambda}{12}M^2L^3(L+4)-\lambda^2\frac{J(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}M^4(\tfrac{14}{3}L^3+14L^2+8L)+8\lambda^3\frac{K(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}M^6(2L^3+6L^2+5L+1)\right.\\
&-4\lambda^3\left[\frac{J(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}\right]^2M^6(2L^3-7L-4)-8\lambda^4\left[\frac{J(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}\right]^3M^8(\tfrac{2}{3}L^3-L^2-2L-\tfrac{1}{3})\\
&\left.-16\lambda^4\frac{Q(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}M^8(L+1)^3+16\lambda^4\frac{K(\mu^2+2\lambda M^2)J(\mu^2+2\lambda M^2)}{[I(\mu^2+2\lambda M^2)]^2}M^8(L^3-3L-2)\right],\end{aligned} \tag{6.12}$$

where

$$L=\ln[M^{2-d}I(\mu^2+2\lambda M^2)] \tag{6.13}$$

and

$$Q(m_R^2)=\frac{1}{(2\pi)^d}\int\frac{d^dp}{(p^2+m_R^2)^4}\,;$$

$$\begin{aligned}G^{(4)}(0)=&-\frac{4\delta\lambda M^2}{I(\mu^2+2\lambda M^2)}+\delta^2\left[-\frac{4\lambda M^2(L+1)}{I(\mu^2+2\lambda M^2)}-\frac{8\lambda^2M^4LJ(\mu^2+2\lambda M^2)}{[I(\mu^2+2\lambda M^2)]^2}\right]\\
&+\delta^3\left[-\frac{\lambda M^22L(L+2)}{I(\mu^2+2\lambda M^2)}-\frac{4\lambda^2M^4(3L^2-4)J(\mu^2+2\lambda M^2)-16\lambda^3M^6(L^2-1)K(\mu^2+2\lambda M^2)}{[I(\mu^2+2\lambda M^2)]^2}\right.\\
&\left.-\frac{16\lambda^3M^6(L^2-L-1)[J(\mu^2+2\lambda M^2)]^2}{[I(\mu^2+2\lambda M^2)]^3}\right],\end{aligned} \tag{6.14}$$

and

$$\begin{aligned}G^{(6)}(0)=&\delta\frac{8\lambda M^2}{[I(\mu^2+2\lambda M^2)]^2}+\delta^2\left[\frac{8\lambda M^2L^2}{[I(\mu^2+2\lambda M^2)]^2}+\frac{16\lambda^2M^4J(\mu^2+2\lambda M^2)}{[I(\mu^2+2\lambda M^2)]^3}L(2L+1)\right]\\
&+\delta^3\frac{8\lambda M^2L}{[I(\mu^2+2\lambda M^2)]^2}\left[\frac{L^2}{2}-1+\frac{2\lambda M^2J(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}(3L^2-3L-4)-\frac{8\lambda^2M^4K(\mu^2+2\lambda M^2)}{I(\mu^2+2\lambda M^2)}(L^2-L-1)\right.\\
&\left.+\frac{4\lambda^2M^4[J(\mu^2+2\lambda M^2)]^2}{[I(\mu^2+2\lambda M^2)]^2}(3L^2-5L-2)\right]+\cdots\,.\end{aligned} \tag{6.15}$$

Let us examine the circles of convergence of these series. We set $\mu=0$ and use the method of dimensional regularization to evaluate the momentum integrals:

$$I(m_R^2)=\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}m_R^{2-d}}\ , \tag{6.16a}$$

$$J(m_R^2)=\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}m_R^{4-d}}\ . \tag{6.16b}$$

We can now solve (6.7) explicitly for m_R:

$$m_R=M\exp\left[\frac{\ln[2\lambda(1+\delta)\Gamma(1-d/2)^{\delta}(4\pi)^{-d\delta/2}]}{2+\delta(2-d)}\right]\ . \tag{6.17}$$

From (6.7) we see that, as a function of complex δ, m_R is singular at $\delta=-1$ and at $\delta=2/(d-2)$. Thus, in the range $0\le d\le 4$, m_R is analytic in a circle of radius 1 about the origin in the δ plane.

When we examine $G^4(0)$ and $G^{(6)}(0)$ in (6.9) and (6.11) we see that they are singular when $A=1$. From (6.7) we see that this singularity occurs when $\delta=-I(1)/J(1)$. Using (6.16), we can simplify this condition to read $\delta=2/(d-2)$, which is located at the same point in the δ plane as the dimension-dependent singularity in m_R. Apparently, for all the Green's functions, when $\mu=0$ and $0\le d\le 4$, the radius of convergence of the δ expansion is 1. When the bare mass μ is not zero, the functional-integral representation for any Green's function has the additional convergence factor $\exp(-\frac{1}{2}\int dx\,\mu^2\phi^2)$. Thus, the region of analyticity in the δ plane is at least as large as in the $\mu=0$ case.

F. Recalculation of the renormalized mass using the δ expansion

We begin by calculating the one-particle-irreducible contribution to the two-point function to order δ, using the δ-expansion diagrammatic procedure given in Sec. III, as modified for large N. The provisional Lagrangian $\tilde{L}$ appropriate to calculating to order δ is

$$\tilde{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2+\delta\lambda M^d(M^{2-d}\phi^2)^{\alpha+1}\ . \tag{6.18}$$

The Feynman rules appropriate for calculating Green's functions in the large-N limit are, in momentum space, as follows:

$$\text{Line,}\ \frac{1}{p^2+\mu^2+2\lambda M^2}\ ; \tag{6.19a}$$

$$\text{Vertex,}\ -\delta\lambda M^{d+(2-d)(\alpha+1)}2^{\alpha+1}(\alpha+1)!\ . \tag{6.19b}$$

There is just one diagram in the one-particle-irreducible contribution to the two-point function. It is given by the graph in Fig. 5(a), where now there are α petals. The total amplitude for this graph is

$$-2\delta(\alpha+1)\lambda M^2[I(\mu^2+2\lambda M^2)M^{2-d}]^{\alpha}\ . \tag{6.20}$$

Applying $D=\partial/\partial\alpha$ to (6.20) and evaluating at $\alpha=0$, we find, to order δ,

$$\text{OPI}=-2\delta\lambda M^2(L+1)\ , \tag{6.21}$$

where L is given by (6.13). This reproduces the order-δ term in the expansion of M_R^2 in (6.12).

To compute the one-particle-irreducible graphs to order δ^3 we must include the graphs shown in Fig. 8; these are all the graphs with up to three vertices. The Feynman rules for this calculation are as follows. The amplitude for a line is just the same as in (6.19a); however, from (3.6), we obtain the following three vertex amplitudes:

$$-\left[\delta+\delta^2\frac{1+\alpha}{2}+\delta^3\right]2^{\alpha+1}(\alpha+1)!\lambda M^2M^{(2-d)\alpha}\ ,$$
$$-\left[\delta\omega+\delta^2\frac{\omega^2+\beta}{2}+\delta^3\right]2^{\beta+1}(\beta+1)!\lambda M^2M^{(2-d)\beta}\ ,$$
$$-\left[\delta\omega^2+\delta^2\frac{\omega+\gamma}{2}+\delta^3\right]2^{\gamma+1}(\gamma+1)!\lambda M^2M^{(2-d)\gamma}\ , \tag{6.22}$$

where $\omega=e^{2\pi i/3}$. Note that, in each of these graphs, each vertex can be of α, β, or γ type [see (3.3) and (3.6)] and one must sum over all possible choices of vertices. This

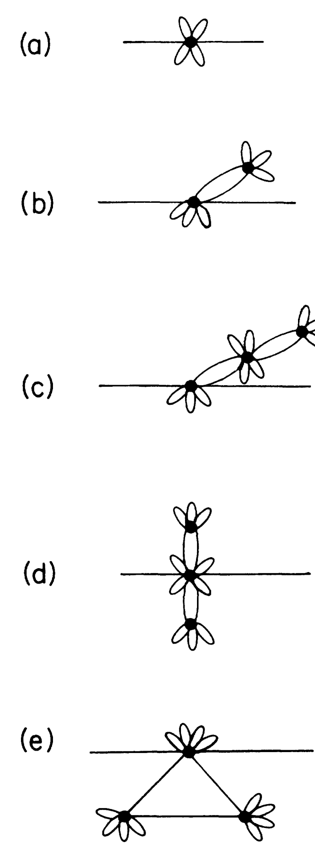

FIG. 8. One-particle-irreducible "cactus" graphs to order δ^3 contributing to the two-point Green's function.

means that there are 3 graphs of type (a), 9 graphs of type (b), 27 graphs of type (c), 18 graphs of type (d), and 18 graphs of type (e) in Fig. 8. After computing the sum of these graphs, one must apply the derivative operator $D_{K=3}$ in (3.8):

$$D_{K=3}=\frac{1}{3}\left[\frac{\partial}{\partial\alpha}+\omega^2\frac{\partial}{\partial\beta}+\omega\frac{\partial}{\partial\gamma}\right]+\frac{1}{6}\left[\frac{\partial^2}{\partial\alpha^2}+\omega\frac{\partial^2}{\partial\beta^2}+\omega^2\frac{\partial^2}{\partial\gamma^2}\right]+\frac{1}{18}\left[\frac{\partial^3}{\partial\alpha^3}+\frac{\partial^3}{\partial\beta^3}+\frac{\partial^3}{\partial\gamma^3}\right]. \tag{6.23}$$

The result reproduces (6.12) exactly. The full calculation is tedious but very simple, and we do not give it here.

G. Recalculation of the four- and six-point functions using δ expansion

To calculate $G^{(4)}(0)$ and $G^{(6)}(0)$ we calculate $\tilde{G}^{(4)}(0)$ and $\tilde{G}^{(6)}(0)$ from the graphs in Figs. 9 and 10 using the Feynman rules (6.19a) and (6.22). We then apply the differential operator $D_{K=3}$ in (6.23) and evaluate at $\alpha=\beta=\gamma=0$. The results are precisely (6.14) and (6.15).

VII. CALCULATION OF THE GREEN'S FUNCTIONS TO LEADING ORDER IN POWERS OF δ

In the previous section we showed how to calculate the diagrams contributing to the δ expansion of the Green's functions in the limit of large N. In this section we calculate the first term in the δ expansion of the Green's functions, but without making the large-N approximation. We postpone to a paper reserved for a discussion of renormalization the more difficult calculation of the higher-order terms.

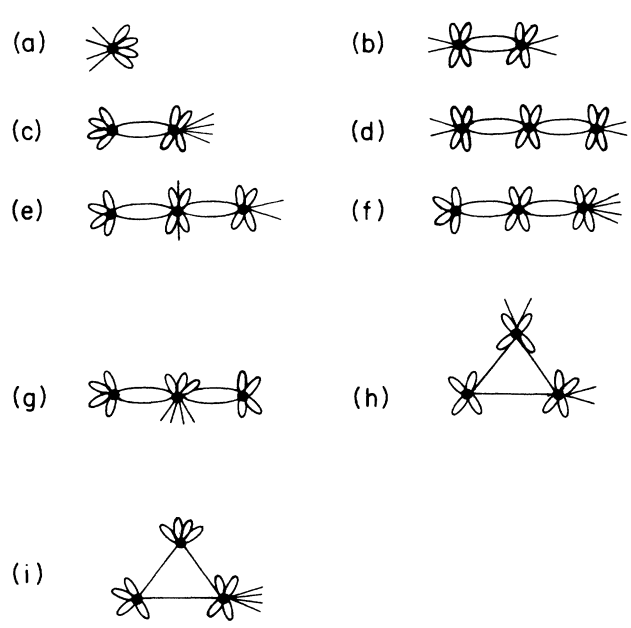

FIG. 9. "Cactus" graphs to order δ^3 contributing to the four-point Green's function.

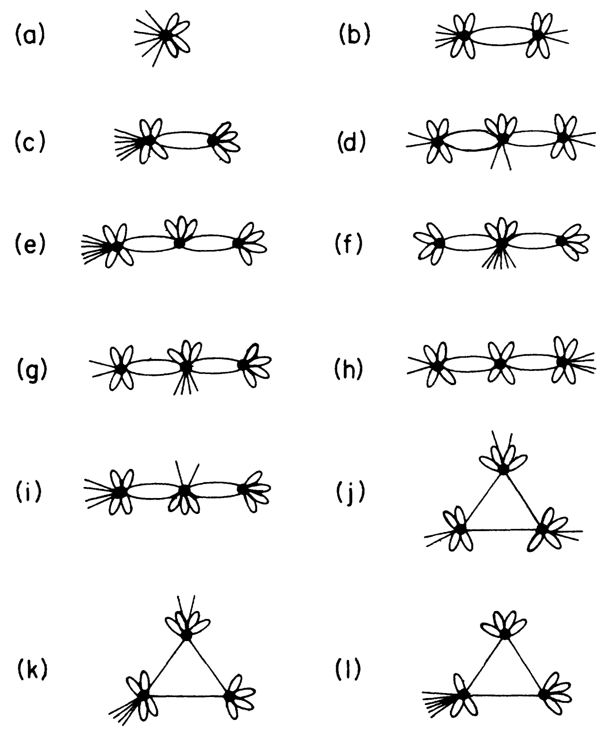

FIG. 10. "Cactus" graphs to order δ^3 contributing to the six-point Green's function.

The Feynman rules for this calculation are obtained from the provisional Lagrangian $\tilde{L}$ correct to order δ:

$$\tilde{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2+\delta\lambda M^d(M^{2-d})^{\alpha+1}(\phi^2)^{\alpha+1}. \tag{7.1}$$

From this Lagrangian the amplitudes for a line and a vertex are the following:

$$\text{line, } \frac{1}{p^2+\mu^2+2\lambda M^2}; \tag{7.2a}$$

$$(2\alpha+2)\text{-point vertex, } -\delta\lambda M^{d+(2-d)(\alpha+1)}(2\alpha+2)!. \tag{7.2b}$$

We emphasize again the difference between the exact vertex amplitude (7.2b) and the large-N vertex amplitude (6.19b); in the later the factor $(2\alpha+2)!$ is replaced by $2^{\alpha+1}(\alpha+1)!$.

Only one graph contributes to the one-particle-irreducible connected $2n$-point Green's function; this

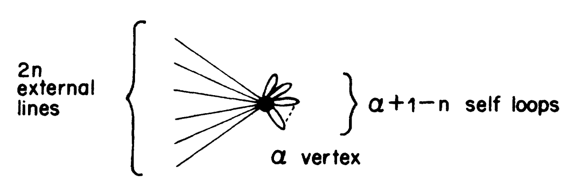

FIG. 11. The graph contributing to the one-particle-irreducible connected $2n$-point Green's function to leading order in δ.

graph is shown in Fig. 11. There are $\alpha+1-n$ self loops (petals) in this graph. Therefore, the symmetry number is $(\alpha+1-n)!2^{\alpha+1-n}$. Each petal contributes $I(\mu^2+2\lambda M^2)$. Thus, to leading order in δ,

$$\tilde{G}^{(2n)}=-\delta\lambda M^{d+(2-d)(\alpha+1)} \times\frac{(2\alpha+2)![I(\mu^2+2\lambda M^2)]^{\alpha+1-n}}{2^{\alpha+1-n}(\alpha+1-n)!} . \quad (7.3)$$

Next, we apply the derivative operator $D_{K=1}=\partial/\partial\alpha$ to $\tilde{G}^{(2n)}$ and evaluate the result at $\alpha=0$ to obtain

$$G^{(2n)}=\frac{(-1)^{n+1}\lambda\delta M^{d+n(2-d)}2^n(n-2)!}{[M^{2-d}I(\mu^2+2\lambda M^2)]^{n-1}} . \quad (7.4)$$

As a special case of this formula, we let $n=1$. The renormalized mass m_R is obtained by summing all powers of the one-particle-irreducible graphs contributing to $G^{(2)}$ as a geometric series and finding the pole of this sum. The result is

$$m_R^2=\mu^2+2\lambda M^2+2\delta\lambda M^2\{1+\psi(\tfrac{3}{2}) +\ln[2M^{2-d}I(\mu^2+2\lambda M^2)]\} . \quad (7.5)$$

This formula is remarkable because it is far less divergent than the corresponding formula in the weak-coupling perturbation series. In weak-coupling perturbation theory the corresponding formula for m_R^2 reads

$$m_R^2=\mu^2+\lambda M^2\frac{(2\delta+2)!}{2^\delta\delta!}[M^{2-d}I(\mu^2)]^\delta+O(\lambda^2) . \quad (7.6)$$

Note that, when $d=2$, the integral I is logarithmically divergent but only $\ln I$ appears in (7.5). Thus, if Λ is a momentum cutoff (7.6) diverges like $(\ln\Lambda)^\delta$ while (7.5) diverges like $\ln(\ln\Lambda)$. When $d=4, I$ is quadratically divergent, so (7.6) diverges like $\Lambda^{2\delta}$ while (7.5) diverges like $\ln\Lambda$. The expressions for the higher Green's functions to order δ are not even divergent.

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[1]C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. Lett. **58**, 2615 (1987).

[2]For example, in a simple quantum-mechanical system with tunneling, when tunneling occurs rapidly because of a low barrier potential, the Wentzel-Kramers-Brillouin (WKB) method gives a very poor approximation to the tunneling amplitude.

[3]For example, in electrodynamics $g-2$ is an unknown, but surely complicated, function of α. Its weak-coupling expansion $g-2=c_1\alpha+c_2\alpha^2+\cdots$ only makes sense in the limit $\alpha\to 0$. This expansion does not even begin to suggest how $g-2$ depends on the parameter α when α is not small.

[4]C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978), Chap. 7.

[5]Figure 1 is drawn from data taken from C. M. Bender, L. R. Mead, and L. M. Simmons, Jr., Phys. Rev. D **24**, 2674 (1981), Table II.

[6]K. Wilson, Phys. Rev. D **7**, 2911 (1973). For a review, see S. Coleman, in *Pointlike Structures Inside and Outside of Hadrons*, edited by A. Zichichi (Plenum, New York, 1982).

[7]W. A. Bardeen and M. Moshe, Phys. Rev. D **28**, 1372 (1983); **34**, 1229 (1986).