Analytic conditions for three-neutrino resonant oscillations in matter

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Exact conditions for resonant oscillations of three neutrinos are derived analytically. These are obtained by demanding maximal mixing between states in ν_e at resonance in analogy with the two-generation case. The criteria for the adiabatic approximation in the present case are proposed.

I. INTRODUCTION

The neutrino oscillations have been thought of as a possible solution[1,2] to account for the suppression of the solar-neutrino counting rate in the ^{37}Cl experiment, the so-called solar-neutrino puzzle. The pattern of these oscillations in matter could be very different from that in vacuum when the interaction of ν_e with matter is taken into account. It was pointed out by Mikheyev and Smirnov[3] and later by Bethe[4] that this matter-induced effect could be significant and can under certain conditions lead to a complete conversion of ν_e to ν_μ accounting for the suppressed counting rate in the ^{37}Cl experiment. This conversion is brought about by the resonant amplification of even a tiny mixing angle in vacuum between two generations of neutrinos into a maximal mixing.

In the realistic case with three generations of neutrinos, extracting resonance conditions seems to be very difficult even though formal solutions exist.[5] This has led to numerical studies by various authors.[6,7] In this paper we report on an analytic study of the three-generation neutrino oscillations in matter and derive conditions for the resonance to occur in direct analogy with the two-generation problem. But unlike the two-generation case, there are more parameters in the three-generation problem such as mixing angles and masses in vacuum. Since these mixing angles undergo a correlated variation in the presence of matter, the occurrence of a resonance is not automatic, but demands certain initial conditions which lead to constraints on vacuum parameters. Interestingly, therefore, the resonance may not occur in arbitrary models even when the magnitudes of the vacuum mixing angles and the mass differences are in conformity with what is needed in the two-generation case.

For the purpose of analysis we divide the three-generation problem into two cases. First we discuss conditions under which a three-generation problem effectively reduces to that of two generations with one mass remaining constant. Most of the cases studied numerically in the literature fall approximately under this category. Next we discuss a genuine three-generation case in which the resonance occurs simultaneously between all the three generations. This case, as far as we know, has not been studied in the literature as yet. The conditions (often referred to with the prefix adiabatic) under which a complete conversion of ν_e to ν_μ takes place are obtained in the latter case. In the former case these are identical to the two-generation case and no separate discussion seems to be necessary.

II. A SPECIAL CASE

Before proceeding with the three-generation problem, let us recapitulate the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism for two generations. We shall put it in a form which is more amenable to the analysis of the three-generation MSW mechanism. The evolution of the states inside the Sun is governed by[2,4] the appropriate mass matrix which in the flavor basis can be parametrized as

$$M_A^2=\begin{pmatrix} x_1+A & y_1 \\ y_1 & x_2 \end{pmatrix} , \tag{2.1}$$

where the parameters x_i and y_i determine the vacuum mixing matrix ($A=0$) U_v and μ_i^2 through

$$M_v^2=U_v \begin{pmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix} U_v^\dagger . \tag{2.2}$$

The interaction of ν_e, of energy E, with electrons in matter gives rise[2] to an additional contribution to the mass matrix given by

$$A=2\sqrt{2}G_F N_e E ,$$

where N_e is the electron density in matter and G_F is the Fermi coupling. Since the matter density is not constant, A changes as ν_e produced near the core travels to the surface. At any given A, M_A^2 can be diagonalized by a unitary matrix U_A,

$$U_A^\dagger M_A^2 U_A=M_0^2 , \tag{2.3}$$

where M_0^2 is a diagonal matrix with A-dependent masses m_1^2 and m_2^2. In particular, we shall assume U_A to be real, which amounts to setting the CP-violating phase to zero in the three-generation analysis. Thus, for two generations,

$$U_A=R(\beta_A)=\begin{pmatrix} \cos\beta_A & \sin\beta_A \\ -\sin\beta_A & \cos\beta_A \end{pmatrix} \tag{2.4}$$

with

$$\tan 2\beta_A = -\frac{\delta_{21}\sin 2\beta_v}{A-\delta_{21}\cos 2\beta_v} ,$$

where β_v is the vacuum mixing angle and $\delta_{21}=\mu_2^2-\mu_1^2$ is the mass difference in vacuum. Differentiating M_0^2 in Eq. (2.3) with respect to A, we obtain

$$\frac{dM_0^2}{dA}+\left[U_A^T\frac{dU_A}{dA},M_0^2\right]=U_A^T\frac{dM_A^2}{dA}U_A , \qquad (2.5)$$

where $U_A^T(dU_A/dA)$ is an antisymmetric matrix given by

$$U_A^T\frac{dU_A}{dA}=\begin{bmatrix} 0 & d\beta_A/dA \\ -d\beta_A/dA & 0 \end{bmatrix} .$$

The matrix equation (2.5) can be explicitly written as

$$\frac{dm_1^2}{dA}=\frac{1}{2}\left[\frac{A-\delta_{21}\cos 2\beta_v}{[(A-\delta_{21}\cos 2\beta_v)^2+(\delta_{21}\sin 2\beta_v)^2]^{1/2}}+1\right] ,$$

$$\frac{dm_2^2}{dA}=\frac{1}{2}\left[1-\frac{A-\delta_{21}\cos 2\beta_v}{[(A-\delta_{21}\cos 2\beta_v)^2+(\delta_{21}\sin 2\beta_v)^2]^{1/2}}\right] , \qquad (2.6)$$

$$\frac{d\beta_A}{dA}=\frac{\sin 2\beta_A}{2(m_2^2-m_1^2)} = -\frac{\delta_{21}\sin 2\beta_v}{2[(A-\delta_{21}\cos 2\beta_v)^2+(\delta_{21}\sin 2\beta_v)^2]^{1/2}} .$$

Note that the right-hand sides in Eqs. (2.6) are entirely given in terms of the vacuum mixing angles and masses whose values fix the model. Consider now the situation when the two mass eigenstates are maximally mixed in ν_e. The U_A is then given by

$$U_A=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} . \qquad (2.7)$$

We now demand that at resonance the $(\text{mass})^2$ matrix given by (2.1) be diagonalized by the U_A given above. Equivalently, the mixing angle at resonance, i.e., when $A=A_R$, is $\beta_A=\pi/4$. It then immediately follows that

$$A_R=\delta_{21}\cos 2\beta_v . \qquad (2.8)$$

Thus it follows from Eqs. (2.6) that

$$\frac{dm_1^2}{dA}=\frac{dm_2^2}{dA}=\frac{1}{2},\quad \frac{d}{dA}(\Delta_{21})=0,\quad \Delta_{21}=m_2^2-m_1^2 , \qquad (2.9)$$

which implies that the mass difference becomes an extremum at resonance. Note that this is also sufficient to determine $U_A^T(dU_A/dA)$ at resonance since $d\beta_A/dA=1/2\Delta_{21}$. Equation (2.8) is in fact the resonance condition derived by Bethe[4] for the two-generation case. Conversely, we could have started by demanding that the mass difference be a minimum and obtained the maximal mixing between two mass eigenstates in ν_e at resonance.

Consider now the extension to three generations. Unlike the two-generation case where the maximal mixing between two generations can be uniquely fixed for resonance, in the three-generation case we could either choose maximal mixing between any two or all three mass eigenstates in ν_e. We shall first show that the former case trivially reduces to the two-generation problem with one mass eigenstate completely decoupled. Later we shall consider the latter case, which is what we would like to refer to as the genuine three-generation problem. If we interpret the MSW mechanism purely in terms of maximal mixing alone, these two cases exhaust all the possibilities.

The mixing at resonance when $A=A_R$ in the ideal case of any two states resonating is described by

$$U_{A=A_R}=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -c_3/\sqrt{2} & c_3/\sqrt{2} & s_3 \\ s_3/\sqrt{2} & -s_3/\sqrt{2} & c_3 \end{bmatrix} , \qquad (2.10)$$

where $c_3=\cos(\beta_{3A_R})$ and $s_3=\sin(\beta_{3A_R})$. We could have chosen $(U_A)_{e2}=0$ instead of the $(U_A)_{e3}$, but that is immaterial for what follows. We now demand that U_A in Eq. (2.10) diagonalizes M_A^2 given by

$$M_A^2=\begin{bmatrix} x_1+A & y_1 & y_2 \\ y_1 & x_2 & y_3 \\ y_2 & y_3 & x_3 \end{bmatrix} \qquad (2.11)$$

when $A=A_R$. Thus the various columns of U_A are eigenstates of M_A^2 with eigenvalues m_i^2 $(i=1,2,3)$. Writing these eigenvalue equations and eliminating m_i^2 from them we obtain the conditions

$$x_2+\frac{y_2}{y_1}y_3=x_3+\frac{y_1}{y_2}y_3 , \qquad (2.12a)$$

$$A_R=x_2-x_1+\frac{y_2}{y_1}y_3 , \qquad (2.12b)$$

$$\tan(\beta_{3A})=-y_2/y_1 . \qquad (2.12c)$$

Note that the condition (2.12a) is independent of A and hence imposes constraints on x_i and y_i which in turn can be expressed in terms of vacuum mixing angles and masses. Also by virtue of Eq. (2.12c) β_{3A} always remains independent of A and, therefore, $\beta_{3A}=\beta_{3v}$. Condition (2.12b) determines the value of A at resonance. The mass differences at resonance are given by

$$(\Delta_{21})_R=2(y_1^2+y_2^2)^{1/2} , \qquad (2.13a)$$

$$(\Delta_{32})_R=-(y_1^2+y_2^2)^{1/2}-\frac{y_3}{y_1y_2}(y_1^2+y_2^2) , \qquad (2.13b)$$

where $\Delta_{ij}=m_i^2-m_j^2$.

Thus the maximal mixing between only two states can occur in a class of models satisfying Eq. (2.12a). This condition has a simple interpretation. It follows by requiring that U_{e3} (or U_{e2}) be zero at resonance; but by virtue of its A independence U_{e3} is forced to remain zero for all A and even in vacuum. Once this is satisfied in any model the present case completely reduces to the two-generation case. The unitary matrix U_A then has the following parametrization for all A:

$$U_A=\begin{bmatrix} c_1 & s_1 & 0 \\ -s_1c_3 & c_1c_3 & s_3 \\ s_1s_3 & -s_3c_1 & c_3 \end{bmatrix}, \quad (2.14)$$

where, as before, $c_3=\cos(\beta_{3A})$, etc., and β_{3A} is independent of A given by Eq. (2.12c). The parameters x_i and y_i can be expressed in terms of β_{1v}, β_{3v} and the masses μ_i^2 in vacuum using the parametrization for U_A given above at $A=0$. Substituting them in Eq. (2.12b) we find

$$A_R=\delta_{21}\cos(2\beta_v),\quad \delta_{21}=\mu_2^2-\mu_1^2\,,$$

and

$$\tan(2\beta_{1A_R})=-\frac{\delta_{21}\sin(2\beta_{1v})}{A_R-\delta_{21}\cos(2\beta_{1v})}\,,$$

which are identical to the two-generation case, see, for example, Eq. (2.8). Two of the eigenvalues m_1^2 and m_2^2 are identical to those in the two-generation case, while

$$m_3^2=\mu_3^2 \quad (2.15)$$

and is totally decoupled from the other two mass eigenstates. This constant eigenvalue can equal one of the A-dependent eigenvalues at

$$A=A_R'=\frac{\delta_{31}\delta_{32}}{\sin^2\beta_{1v}\delta_{31}+\cos^2\beta_{1v}\delta_{32}}\,. \quad (2.16)$$

However, in the limit U_{e3} (or U_{e2}) being zero this cannot be interpreted as resonance since m_3^2 is totally decoupled from the rest. This would be the consequence if the MSW mechanism is interpreted literally as corresponding to maximal mixing and there exists a unique value of $A=A_R$ at which the resonance occurs. As a result the adiabatic condition which results in complete conversion of ν_e into ν_μ (when $U_{e3}=0$) or to ν_τ (when $U_{e2}=0$) is identical to the one derived by Bethe.[4]

If, on the other hand, U_{e3} is nonzero but small then the crossing at $A=A_R'$ does not occur and one gets a resonancelike behavior a second time with approximately equal mixing between the states which are crossing at $A=A_R'$ as, for example, found in the numerical study by Kuo and Pantaleone.[6]

It is interesting to note that if two of the masses are exactly degenerate, then one could always choose the basis in which U_{e2} or U_{e3} is zero for all A and the condition (2.12a) is automatically satisfied, reducing the problem to the two-generation case.

III. A GENUINE THREE-GENERATION CASE

We shall now consider the maximal mixing between all three mass states in ν_e. This requires

$$U_{e1}^2=U_{e2}^2=U_{e3}^2=\tfrac{1}{3} \quad (3.1)$$

at $A=A_R$. A general orthogonal matrix in three dimensions can be written as

$$U_A=R_x(\beta_{3A})R_y(\beta_{2A})R_z(\beta_{1A}) \quad (3.2)$$

in terms of rotations along x, y, and z axes. Imposing the maximal-mixing condition, as given by Eq. (3.1), fixes $\beta_{1A}=\pi/4$ and $\beta_{2A}=\pi/6$ at resonance. Explicitly,

$$U_{A=A_R}=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -\dfrac{c_3}{\sqrt{2}}-\dfrac{s_3}{\sqrt{6}} & \dfrac{c_3}{\sqrt{2}}-\dfrac{s_3}{\sqrt{6}} & \sqrt{2/3}s_3 \\ \dfrac{s_3}{\sqrt{2}}-\dfrac{c_3}{\sqrt{6}} & -\dfrac{s_3}{\sqrt{2}}-\dfrac{c_3}{\sqrt{6}} & \sqrt{2/3}c_3 \end{bmatrix}, \quad (3.3)$$

where c_3 [s_3] is, as usual, $\cos(\beta_{3A_R})$ [$\sin(\beta_{3A_R})$]. As in the two-generation case [see Eq. (2.9)] this immediately implies, using Eq. (2.5),

$$\left[\frac{d}{dA}\Delta_{ij}\right]_{A_R}=0,\quad \Delta_{ij}=m_i^2-m_j^2,\quad i,j=1,2,3\,, \quad (3.4)$$

that is, the maximal mixing corresponds to a situation where all the (mass)^2 differences attain an extremum at resonance. In the trivial three-generation case considered earlier in Sec. II this was true only for Δ_{12}. In this sense the maximal mixing between all three generations is a more direct generalization of the MSW mechanism of two generations. In fact, later we shall see that this is true for any arbitrary number of generations.

As a result of the form given in (3.3) ν_e could resonate with both ν_e and ν_τ for the same value of $A=A_R$. Again using the antisymmetry of $U_A^T(dU_A/dA)$, we find

$$\left[U_A^T\left[\frac{dU}{dA}A\right]\right]_{ij}=X_{ij}=\frac{1}{3\Delta_{ij}} \quad (3.5)$$

at resonance. Notice that this is identical to the two-generation case except for the fact that the factor $\frac{1}{2}$ has been replaced by $\frac{1}{3}$. If we now demand the U_A at $A=A_R$ diagonalize M_A^2, we obtain the conditions

$$y_1^2+y_2^2=\tfrac{1}{2}(x_2-x_3)^2+2y_3^2 \quad (3.6)$$

and

$$A_R=\tfrac{1}{2}(x_2+x_3)-x_1\,. \quad (3.7)$$

The first condition is independent of A and imposes a restriction on the vacuum parameters, which should be satisfied in order to have resonance in the sense implied by Eq. (3.3), while the second condition fixes the value of A at resonance in terms of vacuum parameters. The angle β_{3A} is, in general, dependent on A and at resonance it is given by

$$\tan(3\beta_{3A})=\frac{y_1(x_2-x_3)-2y_2y_3}{y_2(x_2-x_3)-2y_1y_3}\,. \quad (3.8)$$

The (mass)^2 differences at $A=A_R$ are given by

$$(\Delta_{21})_R = \sqrt{6}(y_2 s_3 - y_1 c_3) , \quad (3.9)$$

$$(\Delta_{32})_R = (3/\sqrt{2})(y_1 s_3 + y_2 c_3) + \sqrt{3/2}(y_2 s_3 - y_1 c_3) . \quad (3.10)$$

If the vacuum parameters are such that any of $(\Delta_{ij})_R$ is zero, then the conditions given above do not describe a resonance any longer. This follows from the fact that U_{A_R} is then arbitrary up to a rotation and the criteria for maximal mixing are no longer meaningful.

The conditions described above can be written in terms of vacuum masses and mixing angles. In particular,

$$A_R = \tfrac{1}{2}\{\delta_{21}[3(U_v^2)_{11} - 1] + \delta_{32}[1 - 3(U_v^2)_{13}]\} . \quad (3.11)$$

This condition has significantly different implications for models of the neutrino masses than the corresponding condition in the case of two generations. The numerical value of A_R is fixed by the parameters of the Sun and by requiring the $\frac{1}{3}$ reduction of solar ν_e flux on the Earth. From this Bethe[4] estimated A_R to be approximately 6×10^{-5} eV^2. Suppose the vacuum mixing angles are small, then, from Eq. (3.11),

$$A_R \simeq \delta_{21} + \delta_{32}/2 . \quad (3.12)$$

Depending on the value of δ_{32} this allows considerable latitude in δ_{21}. For $\delta_{32} > 2A_R$ the resonance can occur even when $\mu_2^2 < \mu_1^2$, unlike in the two-generation case where $\delta_{21} > 0$ and of the order of A_R. If both δ_{21} and δ_{32} are positive then each has to be $\lesssim 10^{-5}$ eV^2 in order to satisfy (3.12).

In order to study the implications of (3.6), let us consider the theoretically interesting case of small and hierarchical mixing between neutrinos. Consider

$$U_v = R_x(\beta_{3v}) R_y(\beta_{2v}) R_z(\beta_{1v})$$

and let

$$\beta_{1v} = \epsilon, \quad \beta_{2v} = \epsilon^3, \quad \beta_{3v} = \epsilon^2$$

be vacuum mixing angles for ϵ small. This hierarchy in mixing angles is analogous to what is found in the quark mixing.[8] With this U, condition (3.6) can be satisfied provided

$$\delta_{32} = \pm\sqrt{2}\delta_{21} + O(\epsilon^2) ,$$

consistent with the expected pattern $\mu_1 < \mu_2 < \mu_3$ for masses, but requiring $\mu_2/\mu_3 \approx 1$. Once the masses follow this relation, all initially small mixing angles become large at resonance.

The approximate equality of μ_2 and μ_3 is a general feature imposed by Eq. (3.6) if the mixing angles are small. This equality does not occur in the currently popular models based on the "seesaw" mechanism[9] which typically predict $\mu_2 \ll \mu_3$. The maximal mixing cannot be attained in such models if mixing angles are small. In contrast, in models with a pseudo-Dirac neutrino,[10] the two masses are nearly degenerate and maximal mixing can be achieved.

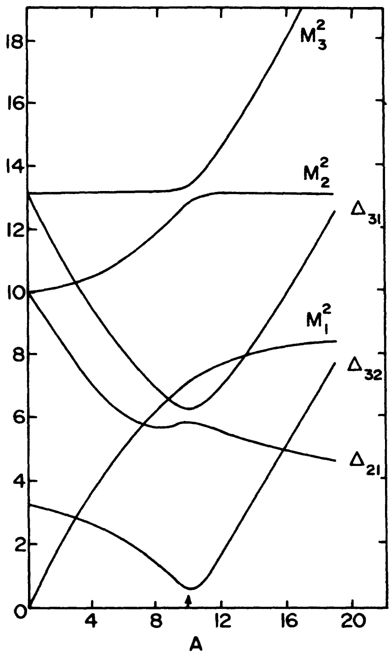

FIG. 1. Variation of M_i^2 and Δ_{ij} as a function of A in the theoretically interesting case of small but hierarchical mixing of neutrinos. The vacuum mixing angles are $\beta_{1v} = \epsilon$, $\beta_{2v} = \epsilon^3$, $\beta_{3v} = \epsilon^2$ with $\epsilon = 0.3$ and $\mu_1^2 = 0.001$, $\mu_2^2 = 10$, and μ_3^2 is fixed through the condition (3.6). The number along the A axis should be scaled by a factor 1.02. The arrow indicates the value of A at resonance.

Figure 1 displays the neutrino (mass)^2 (m_i^2) and (mass)^2 differences (Δ_{ij}) as a function of A for $\epsilon = 0.3$. We have chosen $\mu_1^2 = 0.001$ and $\mu_2^2 = 10.0$ and fixed μ_3^2 using the condition (3.6). At the value of $A = A_R = 10.2$ given by Eq. (3.7) all Δ_{ij} attain extremum. The Δ_{21} has a local maximum at $A = A_R$ unlike the two-generation case. This is obvious once the second derivative $d^2 M_0^2/dA^2$ is evaluated at $A = A_R$.

The resonance condition (3.6) itself is not sufficient to ensure (almost) complete conversion of ν_e that passes through resonance into ν_μ or ν_τ. This can happen if the passage of ν_e through resonance is adiabatic.[4] We now try to formulate conditions for adiabatic passage in analogy with the two-generation case. Adiabatic approximation assumes that the mass eigenstates of the instantaneous Hamiltonian evolve independently inside the Sun. In general, these eigenstates satisfy the evolution equation

$$i\frac{d\nu_m}{dt} = \left[\frac{M_0^2}{2E} + i\frac{A}{N_e}\frac{dN_e}{dt} U_A^T \frac{dU_A}{dA}\right]\nu_m . \quad (3.13)$$

The second term tends to mix various adiabatic states and can make the passage of ν_e nonadiabatic. Using the antisymmetry of $U_A^T dU_A/dA$ we have already seen that it can be parametrized in terms of X_{ij} [see Eq. (3.5)]. For two generations at resonance,

$$X_{12} = 1/(2\Delta_{21})_R , \quad (3.14)$$

which describes the inverse width of the resonance layer. The second term in Eq. (3.13) corresponds to the inverse length of the resonance layer,[4] $(r_{12})^{-1}$, and the adiabatic approximation is good if

$$r_{12} \gg \lambda_{12} = \frac{4\pi E}{(\Delta_{21})_R} , \quad (3.15)$$

λ_{12} being the wavelength of oscillation at resonance. In the three-generation case under consideration, X_{ij} are given by Eq. (3.5) at resonance. These define three (two of which are independent) characteristic distances representing mixing of various adiabatic states at resonance. Even though the resonance occurs at the same value of A, these distances are different at the resonance. Likewise there are three wavelengths,

$$\lambda_{ij} = \frac{4\pi E}{(\Delta_{ij})_R} , \quad (3.16)$$

at resonance. We speculate that the adiabatic condition is good if the smallest of the distances r_{ij} is much larger than the largest of the wavelengths. All the terms describing the mixing of adiabatic states in Eq. (3.13) are expected to be suppressed under this approximation.

Noting from Eq. (3.13) that

$$(r_{ij})_{\min} = \left[-\frac{A_R}{3} \frac{1}{\rho} \frac{d\rho}{dr} \right]^{-1} (\Delta_{ji})_{R,\min} , \quad (3.17)$$

the above requirement corresponds to

$$\frac{(\Delta_{ij})^2_{R,\min}}{A_R^2} \gg \frac{4\pi}{3\alpha} \frac{d}{dr} \left[\frac{1}{\rho} \right] \simeq 1.067 \times 10^{-4} , \quad (3.18)$$

where $\alpha = A_R / E_c \rho_c$, $E_c(\rho_c)$ is the critical energy (density) defined by Bethe,[4] and r is the distance from the core. In the relevant region of the Sun, $d(1/\rho)/dr$ varies between $(0.8-2)\times 10^{-2}$ cm^2/g. The higher of these values gives the number quoted in Eq. (3.18). The $(\Delta_{ij})_{R,\min}$ can be obtained from Eqs. (3.9) and (3.10) in any general model and the above conditions can be checked. For example, in Fig. 1, $(\Delta_{32})_R$ corresponds to the minimum of (mass)^2 differences at resonance and

$$(\Delta_{32})^2_R / A_R^2 = 16.8 \times 10^{-4} .$$

Thus the adiabatic condition is automatically satisfied.

In general, when the adiabatic conditions are satisfied, the probability $P_{\nu_e : \nu_e}$ can be given (when appropriate averages are performed[1]) for ν_e produced in the core by

$$P_{\nu_e : \nu_e} = |(U_v)_{ei} (U_c)_{ei}|^2 ,$$

where U_v (U_c) defines the mixing matrix in vacuum (core). This probability is given by the square of $(U_v)^2_{e3}$ and is very small if in the core ν_e coincides with the highest mass state ν_3.

IV. POSSIBLE GENERALIZATION AND COMMENTS

In principle the analysis we have carried out in Sec. III can be extended to an arbitrary number of generations. Even though explicit calculations become tedious, some general features can be easily derived. Demanding maximal mixing of states in ν_e at resonance, i.e.,

$$|\nu_e\rangle = (|\nu_1\rangle + |\nu_2\rangle + \cdots + |\nu_n\rangle)/\sqrt{n} , \quad (4.1)$$

where n is the number of generations, and using Eq. (2.5) it immediately follows that

$$d(\Delta_{ij})_R / dA = 0 \quad (4.2)$$

and

$$\left[U_A^T \frac{dU_A}{dA} \right] = X_{ij} = \frac{1}{n(\Delta_{ji})_R} \quad \text{for } i<j . \quad (4.3)$$

The value of A at resonance is then given by

$$(n-1)A_R = \sum_{i=2}^{n} x_i - x_1 , \quad (4.4)$$

where x_i, $i=1,\ldots,n$ denote the diagonal matrix elements of the (mass)^2 matrix in the absence of A in flavor basis. In addition there are $(n-2)$ sufficiency conditions, similar to Eq. (3.6), to be satisfied. Notice that Eqs. (4.3) and (4.4) reduce to Bethe's conditions for two generations and to the analysis of Sec. III for three generations. In this sense, the maximal mixing of all mass eigenstates in ν_e constitutes a generalized MSW mechanism for arbitrary generations. This does not completely exhaust all the possibilities since maximal mixing could occur with a subset of mass eigenstates in ν_e such as the one considered in Sec. II, for example. These cases, however, lead to different sets of sufficiency conditions.

In conclusion, we have given analytic conditions for resonance to occur in the case of three generations. In a sense the cases considered here represent two extreme situations corresponding to maximal mixing between three and two mass eigenstates. Each of these cases requires some sufficiency conditions to be satisfied by the vacuum parameters. If they are not satisfied, one will not get resonance in the sense implied by the MSW mechanism for two generations. In practice, it is possible that the conditions for neither of the cases considered here may be realized exactly. Nevertheless, they allow us to infer which of the two possibilities are approximately realized.

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