Null-plane formulation of Bethe-Salpeter *qqq* dynamics: Baryon mass spectra

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The Bethe-Salpeter (BS) equation for a *qqq* system is formulated in the null-plane approximation (NPA) for the BS wave function, as a direct generalization of a corresponding QCD-motivated formalism developed earlier for $q\bar{q}$ systems. The confinement kernel is assumed vector type ($\gamma_\mu^{(1)}\gamma_\mu^{(2)}$) for both $q\bar{q}$ and qq pairs, with identical harmonic structures, and with the spring constant proportional, among other things, to the running coupling constant α_s (for an explicit QCD motivation). The harmonic kernel is given a suitable Lorentz-invariant definition [*not* $\Box^2\delta^4(q)$], which is amenable to NPA reduction in a covariant form. The reduced *qqq* equation in NPA is solved algebraically in a six-dimensional harmonic-oscillator (HO) basis, using the techniques of SO(2,1) algebra interlinked with S_3 symmetry. The results on the nonstrange baryon mass spectra agree well with the data all the way up to $N=6$, thus confirming the asymptotic prediction $M \sim N^{2/3}$ characteristic of vector confinement in HO form. There are no extra parameters beyond the three basic constants (ω_0, C_0, m_{ud}) which were earlier found to provide excellent fits to meson spectra ($q\bar{q}$).

I. INTRODUCTION AND SUMMARY

The two- and three-quark problems for hadron dynamics are perhaps more closely related at this subhadronic (quark) level than are the corresponding two- and three-body problems at the successively outer (nuclear, atomic) levels of compositeness. For this reason (whose source can be traced to color and confinement) the success of any form of quark dynamics must be judged by its *simultaneous* performances on *both* fronts ($q\bar{q}$ *and* *qqq*), and this must be particularly true of QCD which is believed to be the natural language of strong interactions. However, pending its formal capacity to account for confinement in a sufficiently realistic and practical form, any hadronic application of a QCD-oriented dynamics must continue to rely heavily on an "effective" confinement program; and the close physical connection between $q\bar{q}$ and *qqq* systems offers a more comprehensive testing ground for any such confinement ansatz than would be the case if these systems were to be considered independently.

There is good evidence of rich dividends from such simultaneous studies in the past, based on the Schrödinger equation[1] or its relativistic adaptations.[2] The usual wisdom has been to consider an effective *scalar* confinement for both $q\bar{q}$ and *qq* pair interactions, with a short-range Fermi-Breit term[3] playing the crucial role of mass splittings.[1] There are two difficulties with this kind of approach. (i) While a Schrödinger form of dynamics is justified for heavy-quark systems, its adaptation to light-quark systems, albeit with relativistic adaptations such as the Todorov equation[4] or similar variants,[5,6] would be less appealing than a more natural dynamical framework such as the Bethe-Salpeter equation,[7] especially if one has to live with such an "effective" form of dynamics until such time as a practical form of the confinement emerges from the QCD Lagrangian. (ii) Secondly a "scalar" confinement changes sign[8] as between *qq* and $q\bar{q}$ pairs, unlike a "vector" confinement which preserves the same sign,[8] and is therefore unsuitable for an integrated approach to both $q\bar{q}$ and *qqq* systems at the same time. A Bethe-Salpeter (BS) framework for $q\bar{q}$ and *qqq* systems with a *vector* confinement would seem to meet both these objections *a priori.*

Now the BS equation at the quark level has had a long history.[9,10] In particular, its O(4)-like character yields the hyperspherical angular momentum K, which turns out to be "one quantum number too many" for a meaningful contact with the data[11] which continues to respect an SU(6)$\times$O(3)-like classification[12] after two decades of quark physics. The instantaneous approximation (IA) was applied in this context,[13] but applied mostly to heavy quarkonia (where its need is limited), *and* with "scalar" confinement.

Keeping these issues in view, a somewhat less orthodox, QCD-motivated, BS formalism with *vector* confinement (kernel proportional to the usual $\frac{1}{2}\lambda_1\cdot\frac{1}{2}\lambda_2$ for color, but $\gamma_\mu^{(1)}\gamma_\mu^{(2)}$ for spin) was proposed some years ago[14] for an integrated understanding of both $q\bar{q}$ and *qqq* spectra on the one hand[15] and applications to various transition amplitudes involving hadron-hadron and photon-hadron couplings on the other.[16] This required a *two-tier approach*: viz., (a) a three-dimensional (IA) reduction of the BSE's for $q\bar{q}$ and *qqq* states (for contact with spectroscopic data) which suppresses the role of virtual $q\bar{q}$, etc., effects (or higher Fock states in a three-dimensional description, much like Tamm-Dancoff amplitudes developed in the 1950s), and (b) a prior reconstruction of the *four-dimensional* BS wave function (which would help identify the hadron quark vertex function) to restore the neglected $q\bar{q}$, etc., effects on different transition amplitudes perturbatively through Feynman diagrams.[16]

The fact that such an approach yielded a fairly good overlap with the spectral data[15,17] would seem to suggest that the effect of higher Fock states[18] on the spectral calculations is presumably not large, thus *a fortiori* justifying their perturbative inclusion for the evaluation of certain transition amplitudes (which also turned out to be in fair agreement[16,19] with several data). In retrospect, however, this formulation had certain drawbacks: (a) its reliance on the IA limited its applicability to slow-moving hadrons; (b) the harmonic-oscillator (HO) kernels gave too-large spacings for $c\bar{c}$ and $b\bar{b}$ systems to match their data; and (c) the vacuum structures were ill defined, leading to varying amounts of zero-point-energy (ZPE) shortfalls in the predicted masses.[17]

These shortcomings have since been removed[20,21] in two respects. The formal limitation (a) arising from the IA was overcome through the following *Lorentz-invariant* generalization[20] of the scalar function V (coefficient of $\gamma_\mu^{(1)}\gamma_\mu^{(2)}$) (Ref. 14), representing the HO kernel for a $q\bar{q}$ or qq pair interaction in the full four-dimensional BSE:

$$\langle q \mid V \mid q' \rangle = 3\pi\omega_{qq}^2 \lim_{m\to 0} \frac{-\partial^3}{\partial m^3}[m^2+(q_\mu-q'_\mu)^2]^{-1} . \tag{1.1}$$

Then we effect a three-dimensional reduction through the null-plane approximation[22] (NPA), thus automatically ensuring "null-plane covariance" of the three-dimensional BSE. [It may be noted that (1.1) is *not* the usual four-dimensional HO kernel[10] $\Box^2\delta^4(p-p')$.] To overcome limitations (b) and (c) which are of a physical nature, an ansatz has been proposed on the flavor variation of the spring constant,[21]

$$\text{(b)}\ \omega_{qq}^2 = 4\mu_{12}\omega_0^2\alpha_s,\quad \mu_{12}=m_1m_2(m_1+m_2)^{-1} \tag{1.2}$$

and a corresponding modification on the NPA form (r^2) of the HO kernel,[21]

$$\text{(c)}\ r^2 \to r^2(1+A_0m_1m_2r^2)^{-1/2}-C_0\omega_0^{-2} , \tag{1.3}$$

where r^2 must be read *covariantly* in the NPA language[20] (viz., the third component A_3 of any NPA three-vector **A** should read $A_3 = A_+M/P_+$). The ansatz (1.2), involving the running coupling constant α_s, offers an explicit QCD motivation for the entire kernel, and the postulated constancy of ω_0 ($=158$ MeV) over *all* flavors checks extremely well with the data[11] on all meson sectors ($q\bar{q}$, $Q\bar{q}$, $Q\bar{Q}$), in conjunction with (1.3). The additive constant C_0 ($=0.296$) in (1.3) plays its intended role by filling the ZPE shortfalls rather precisely for all quarkonia.[21] Finally the smallness of the constant A_0 ($=0.0283$) in (1.3) ensures a smooth transition from a (continued) *harmonic* confinement in *uds* sectors (small m_1m_2) to an effectively *linear* one for the heaviest $b\bar{b}$ sector (large m_1m_2), and has played a major role in unifying the spectroscopic data on all the sectors.[21]

This experimental success on the $q\bar{q}$ front of the above structure of the BS dynamics has led us to examine its effect on the (dual) *qqq system,* in keeping with the interlinked nature of the physics that governs the two systems. In this paper we restrict our attention to equal-mass kinematics ($m_1=m_2=m_3$) only, which amounts essentially to the *ud* sector. Furthermore, the unlikelihood of data on *QQQ* systems in the foreseeable future warrants the assumption $A_0=0$ in (1.3) at the outset. The central question is whether or not the qqq problem of *ud* quarks with *vector* interaction for qq pairs can be understood in terms of the three basic constants (ω_0, C_0, and m_{ud}) already determined from the meson spectra,[21] so that no independent freedom of parametrization exists for the qqq system. Our results strongly suggest that this is indeed the case, thus bearing out the expectation that a parallel treatment of the $q\bar{q}$ and qqq within a common BS framework gives consistent results.

The paper is organized as follows. In Sec. II we rapidly recapitulate the main sequence of steps leading from the four-dimensional BSE for a qqq system to a three-dimensional covariant NPA form,[20] Eq. (2.22), representing the focal theme of this paper. Section III describes an algebraic solution of Eq. (2.22), first by reducing it to the form (3.16) which formally resembles a six-dimensional HO equation in two independent internal variables (ξ,η) (see Sec. III) but has its different terms appearing with nonlinearly (M,N) dependent coefficients, where M is the baryon mass and N is the total HO quantum number in the six-dimensional space (ξ,η). This is achieved through a generous use of the techniques of SO(2,1) algebra in conjunction with S_3 symmetry, which characterizes three identical particles, as outlined in Appendix A. The final form of the qqq equation, Eq. (3.29), after the inclusion of one-gluon-exchange effects (summarized in Appendix B), represents an algebraic solution in the form $F_B(M)=N+3$, where F_B is a *known,* nonlinear, function of (M,N). Section IV gives a limited comparison with a representative collection of experimental data,[11] directly in terms of the above form of the solution (without attempting to invert it). The possibility of mixing between different qqq states arising from the vector nature of the confinement (as distinct from one-gluon-exchange effects[1]) is indicated in Appendix C through the structure of certain spin-dependent correction terms that appear in this model and illustrated for the case of $P'_{11}(1440)$ and $P''_{11}(1710)$ mixing. Apart from good agreement with the data for most individual cases, the comparison also shows strong support from the qqq data trends (up to $N=6$) for the asymptotic prediction $M\sim N^{2/3}$ (as with light $q\bar{q}$ systems[21]), a feature that bears directly on vector confinement for both systems.

II. COVARIANT qqq EQUATION IN NPA FORM

The BSE for a qqq system with pairwise qq interaction under vector ($\gamma_\mu^{(1)}\gamma_\mu^{(2)}$) confinement was first written down in I (Ref. 14) and reduced to a three-dimensional form in the instantaneous approximation (IA). After a Lorentz-invariant generalization for the harmonic kernel, Eq. (1.1), was achieved,[20] a corresponding derivation of a covariant three-dimensional form in the null-plane approximation (NPA) was given in II (Ref. 20) in close parallel to the $q\bar{q}$ treatment.[20] While referring the interested reader to the details in I and II, including most notations, we recapitulate for easy reference the main sequence of steps leading to the final form of the covariant NPA equation for qqq. The four-dimensional BSE for $m_1=m_2=m_3\equiv m_q$ reads after a Gordon reduction as[14]

$$\Delta_1\Delta_2\Delta_3\Psi(p_1p_2p_3)=\sum_{123}\left[-\frac{2}{3}\right]\frac{1}{i}\Delta_3(2\pi)^{-4}\int d^4q'_{12}\langle q_{12}\,|\,V\,|\,q'_{12}\rangle T_{12}\Psi(p'_1p'_2p_3)\ , \tag{2.1}$$

$$p_{1,2}=\tfrac{1}{2}P_{12}\pm q_{12},\quad p'_{1,2}=\tfrac{1}{2}P_{12}\pm q'_{12}\ , \tag{2.2}$$

$$p_3=P-p_1-p_2\equiv P-P_{12}\ , \tag{2.3}$$

$$T_{12}=(P^\mu_{12})^2-(q^\mu_{12}+q'^\mu_{12})^2-2i(\sigma_1^{\mu\nu}+\sigma_2^{\mu\nu})q^\mu_{12}q'^\mu_{12}+iP^\mu_{12}(\sigma_2^{\mu\nu}-\sigma_1^{\mu\nu})(q^\nu_{12}-q'^\nu_{12})+\sigma_1^{\mu\nu}\sigma_2^{\mu\lambda}(q^\nu_{12}-q'^\nu_{12})(q^\lambda_{12}-q'^\lambda_{12})\ , \tag{2.4}$$

$$\Delta_i=m_q^2+p_i^2\quad (i=1,2,3)\ . \tag{2.5}$$

For any four-vector A_μ, the null-plane components are

$$A_\pm\equiv A^\pm=A_0\pm A_z,\quad \mathbf{A}_\perp\equiv\mathbf{A}^\perp=(A_x,A_y)\ . \tag{2.6}$$

The NPA form ψ of the BS wave function Ψ is defined as[20]

$$\psi(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)=\int \tfrac{1}{2}dq^-_{12}\tfrac{1}{2}dp^-_3\,\Psi(p_1p_2p_3)\ , \tag{2.7}$$

where the cyclic symmetry of $dq^-_{ij}dp^-_k$ ensures the validity of (2.7) for all the three terms of (2.1). To specify T_{12} and $V=V_{12}$ under NPA conditions, we make the spectator assumption[14] for the "timelike" component of p_3^μ in NPA language:[20]

$$p_3^-=\tfrac{1}{3}P_-\ . \tag{2.8}$$

In the NPA limit, the relative momentum $q_-\equiv q^-_{12}$ becomes[20]

$$q_-=-q_+L_+^{-2},\quad L_+^{-2}\equiv(m_q^2+\mathbf{q}_\perp^2)/p_{1+}p_{2+}\ , \tag{2.9}$$

where the Lorentz factor L_+ reduces, on the mass shell of the subsystem (12), to[20]

$$L^+_{12}=P^+_{12}/M_{12},\quad M_{12}^2=P^+_{12}P^-_{12}-(\mathbf{P}^\perp_{12})^2\ . \tag{2.10}$$

A consistent use of (2.8) under $|q_+|\ll M_{12}$ permits a deduction[20] of a more symmetric form of L^+_{12}:

$$L_+\equiv L^+_{12}=L^+_{31}=L^+_{23}=P_+/M\ , \tag{2.11}$$

which is consistent with the IA result[14] ($P_+\approx M$) but is no longer restricted to a slowly moving baryon. As in the $q\bar{q}$ case,[20] the Lorentz factor $L_+=P_+/M$ allows one to define any three-vector $\mathbf{A}$ *covariantly* under NPA as $(\mathbf{A}_\perp,A_3)$, where $A_3=A_+M/P_+$, so that the various four-momenta appearing in V_{12}, Eq. (1.1), and T_{12}, Eq. (2.4), can be simplified with the identifications

$$(P^\mu_{12})^2=-M^2_{12}=\tfrac{4}{9}M^2-\mathbf{p}_3^2\ , \tag{2.12}$$

$$\begin{aligned}(q_\mu-q'_\mu)^2&=(\mathbf{q}-\mathbf{q}')^2\ ,\\ \sigma_{\mu\nu}A_\mu B_\nu&=\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{B}\ ,\end{aligned} \tag{2.13}$$

and so on. Thus on carrying out the limit $m\to 0$ after m differentiation in (1.1), the function $V=V_{12}$ reduces to the form

$$V_{12}\Longrightarrow\tilde{V}(\mathbf{q}_{12}-\mathbf{q}'_{12})=\tfrac{3}{4}\omega_{qq}^2(2\pi)^3\left[\nabla^2_{12}+\frac{C_0}{\omega_0^2}\right]\delta^3(\mathbf{q}_{12}-\mathbf{q}'_{12}) \tag{2.14}$$

after employing the replacement (1.3), where all three-vectors must be read in the sense $A_3=MA_+/P_+$ for the longitudinal component. The form (2.14) for V_{12} now permits a rapid simplification of Eq. (2.1) when the NPA ansatz (2.7) for the wave function is introduced, remembering that L-invariant element:

$$d^4q'=d^2q'_\perp dq'_+\tfrac{1}{2}dq'_-=P_+M^{-1}d^3q'\tfrac{1}{2}dq'_-\ . \tag{2.15}$$

Thus the integration on the right-hand side (RHS) over $d^3q'_{12}$ gives

$$\int d^3q'_{12}T_{12}V_{12}\psi'=-\tilde{D}_{12}\psi\ , \tag{2.16}$$

where $\tilde{D}_{12}$ is the differential operator

$$\tilde{D}_{12}=M^2_{12}\left[\nabla^2_{12}+\frac{C_0}{\omega_0^2}\right]+\hat{Q}_{12}-8\mathbf{J}_{12}\cdot\mathbf{S}_{12}+12-2i(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{P}_{12}\times\nabla_{12} \tag{2.17}$$

with

$$\hat{Q}_{12}=4\mathbf{q}^2_{12}\nabla^2_{12}+8\mathbf{q}_{12}\cdot\nabla_{12}+6 \tag{2.18}$$

and

$$\mathbf{P}_{12}=\mathbf{p}_1+\mathbf{p}_2=-\mathbf{p}_3\quad\text{(in the c.m. frame)}\ . \tag{2.19}$$

Finally, the integration over dq^-_{12} on the RHS of (2.1) yields the characteristic NPA denominator function D^+_{12} for the (12) pair,[20]

$$2\pi i(D^+_{12})^{-1}=\oint\tfrac{1}{2}dq^-_{12}\Delta_1^{-1}\Delta_2^{-1}\ , \tag{2.20}$$

leading to

$$D_{12}^{+}=2P_{12}^{+}(m_q^2+\mathbf{q}_{12}^2-\tfrac{1}{9}M^2+\tfrac{1}{4}\mathbf{p}_3^2+R_{12}) , \qquad (2.21)$$

where R_{12} is a small correction term specified later in Eq. (3.15). Collecting all these results gives rise to the NPA equation for qqq in the covariant form

$$\psi(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)=\sum_{123}\frac{L_{12}^{+}}{D_{12}^{+}}\tfrac{1}{2}\omega_{qq}^2\tilde{D}_{12}\psi(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) , \qquad (2.22)$$

where the Lorentz factors L_{ij}^{+} are given by (2.11).

III. SOLUTION OF THE qqq EQUATION (2.22)

Equation (2.22) represents the starting point of this investigation of qqq spectra, where the spring constant ω_{qq}^2 is defined by Eq. (1.2), and C_0 stands for the effect of the vacuum structure. These inputs have the *same* values as in the corresponding $q\bar{q}$ investigation,[21] viz.,

$$\omega_0=158 \text{ MeV},\quad C_0=0.296 ,\qquad m_q=270 \text{ MeV} , \qquad (3.1)$$

while the structure of α_s is also in close parallel to the $q\bar{q}$ pattern,[21] but commensurate with a three-body dynamics for confinement, viz.,

$$\alpha_s(3m_q)=\frac{12\pi}{33-2f}\left[\ln\frac{9m_q^2}{\Lambda^2}\right]^{-1} ,\quad \Lambda=250 \text{ MeV} . \qquad (3.2)$$

The strong-Coulomb term is treated perturbatively, again as in the $q\bar{q}$ case,[21] with $\alpha_s(M)$ substituting for $\alpha_s(3m_q)$. This term does not yet appear in the kernel of Eq. (2.22), but can be included in a simple way through an appropriate addition to the final results, Eq. (3.22), leading to Eq. (3.29); see I.

For a reduction of Eq. (2.22) in the overall c.m. frame ($P_+=M$) it is necessary to use the relative coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ defined by[14]

$$\sqrt{3}\boldsymbol{\xi}=\mathbf{p}_3-\mathbf{p}_2,\quad 3\boldsymbol{\eta}=-2\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3 . \qquad (3.3)$$

However, we shall now employ a considerably more refined technique than attempted earlier[14,17] by taking a more conscious account of S_3 symmetry in respect to the denominator functions D_{ij}^{+} as well as the operators $\hat{Q}_{ij}$ in Eqs. (2.18)–(2.21). The basic strategy lies first in expressing the D_{ij} functions in terms of an S_3-symmetric function D_s, *plus* a balance Δ_{ij} which can be taken perturbatively. (This is very closely related to the corresponding method of a "large-n expansion," $n=6$, in a hyperspherical coordinate basis,[23] though differing greatly in details.) Thus we have

$$D_{ij}^{-1}=D_s^{-1}+D_s^{-1}\Delta_{ij}D_s^{-1} , \qquad (3.4)$$

$$D_s=\frac{4M}{3}(\tfrac{1}{2}\xi^2+\tfrac{1}{2}\eta^2+m_q^2-\tfrac{1}{9}M^2+\langle R\rangle) , \qquad (3.5)$$

$$\Delta_{12,13}=\frac{4M}{3}\left[\frac{1}{8}(\xi^2-\eta^2)\pm\frac{\sqrt{3}}{8}2\boldsymbol{\xi}\cdot\boldsymbol{\eta}+\delta R_{12,13}\right] , \qquad (3.6)$$

$$\Delta_{23}=-\Delta_{12}-\Delta_{31} . \qquad (3.7)$$

Here $\langle R\rangle$ is the S_3-symmetrized form of the small R_{ij} term in (2.21), with a balance δR_{ij} which is neglected hereafter, while $\langle R\rangle$ is specified in Eq. (3.15).

When (3.4) is substituted in Eq. (2.22), one can identify two distinct parts, each separately S_3 symmetric, viz., (i) the *main part,* proportional to D_s^{-1}, and (ii) a correction term proportional to D_s^{-2}, when the three different pieces $\tilde{D}_{ij}$ on the RHS of (2.22) are brought together. The coefficients of D_s^{-1} sum up to

$$\tilde{D}_0\equiv\tfrac{8}{9}M^2(\nabla_\xi^2+\nabla_\eta^2)+\frac{C_0}{\omega_0^2}(\tfrac{4}{3}M^2+6\xi^2+6\eta^2)+\hat{Q}_B-8\mathbf{J}\cdot\mathbf{S}+18 \qquad (3.8)$$

in conformity with a very similar structure obtained in I, revealing an explicit $\mathbf{J}\cdot\mathbf{S}$ structure for the spin dependence. The purely momentum-dependent effects are contained in

$$\hat{Q}_B=\hat{Q}_\xi+\hat{Q}_\eta+\hat{Q}_{\xi\eta} , \qquad (3.9)$$

$$\hat{Q}_\xi=4\xi^2\partial_\xi^2+8\boldsymbol{\xi}\cdot\boldsymbol{\partial}_\xi+6 ,\qquad \hat{Q}_\eta=4\eta^2\partial_\eta^2+8\boldsymbol{\eta}\cdot\boldsymbol{\partial}_\eta+6 , \qquad (3.10)$$

$$\hat{Q}_{\xi\eta}=4\boldsymbol{\xi}\cdot\boldsymbol{\partial}_\xi+4\boldsymbol{\eta}\cdot\boldsymbol{\partial}_\eta+8\boldsymbol{\xi}\cdot\boldsymbol{\eta}\boldsymbol{\partial}_\xi\cdot\boldsymbol{\partial}_\eta+6 . \qquad (3.11)$$

The Δ_{ij} terms in (3.4) similarly add up to another operator $D_s^{-2}\Delta H$, where

$$\tfrac{3}{4}M^{-1}\Delta H=-\tfrac{1}{36}M^2\hat{K}^{(2)}+\tfrac{1}{16}\hat{Q}_B^{(2)}+\frac{1}{16}\frac{C_0}{\omega_0^2}S^{(2)}+\text{SDT} , \qquad (3.12)$$

$$\hat{K}^{(2)}=4(\xi^2-\eta^2)(\partial_\xi^2-\partial_\eta^2)+16\boldsymbol{\xi}\cdot\boldsymbol{\eta}\boldsymbol{\partial}_\xi\cdot\boldsymbol{\partial}_\eta , \qquad (3.13)$$

$$\tfrac{1}{16}\hat{Q}_B^{(2)}=-\tfrac{3}{2}(\xi^2-\eta^2)(\boldsymbol{\xi}\cdot\boldsymbol{\partial}_\xi-\boldsymbol{\eta}\cdot\boldsymbol{\partial}_\eta)-3\boldsymbol{\xi}\cdot\boldsymbol{\eta}(\boldsymbol{\eta}\cdot\boldsymbol{\partial}_\xi+\boldsymbol{\xi}\cdot\boldsymbol{\partial}_\eta)+(\xi^2-3\eta^2)\boldsymbol{\xi}\cdot\boldsymbol{\eta}\boldsymbol{\partial}_\xi\cdot\boldsymbol{\partial}_\eta-(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2(3\partial_\eta^2+\partial_\xi^2)+\tfrac{1}{2}(\xi^2-\eta^2)[\eta^2\partial_\eta^2+(\eta^2-2\xi^2)\partial_\xi^2] , \qquad (3.14)$$

$$\tfrac{1}{16}S^{(2)}=-\tfrac{3}{4}(\xi^2-\eta^2)^2-\tfrac{3}{4}(2\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2\equiv 8M^2\langle R\rangle . \qquad (3.15)$$

Equation (3.15) now formally specifies $\langle R\rangle$ of (3.5). Finally, the spin-dependent correction terms, SDT, are listed in Appendix C. The "master" equation (2.22) now reads

$$D_s\psi=\tfrac{1}{2}\omega_{qq}^2(\tilde{D}_0+D_s^{-1}\Delta H)\psi . \qquad (3.16)$$

It may be checked that each of the operators making up ΔH is separately S_3 invariant. These quantities represent important corrections, but as they stand they offer little hope of solution for Eq. (3.16). To this end we adopt a strategy similar to, but more refined than, that employed in I, viz., to express the major effects of ΔH in terms of the principal quantum number N in a six-dimensional HO basis, after noting that the main terms of D_s and $\tilde{D}_0$ are indeed quadratic in $(\boldsymbol{\xi},\boldsymbol{\eta})$ and their derivatives. Such a reduction requires an extensive use of the techniques of SO(2,1) algebra,[24–26] in conjunction with S_3 symmetry, involving successively quadratic, biquadratic, and sextic combinations of the HO operators $\mathbf{a}_\xi$, $\mathbf{a}_\eta$, and their Hermitian conjugates. These are briefly outlined in Appendix A, yielding the following approximate eigenvalues for the different operators of ΔH in a six-dimensional HO representation characterized by the total quantum number $N=N_\xi+N_\eta$:

$$Q_N=\tfrac{3}{2}\sigma_N-2(N+3)^2-18-2\tau_N\ , \tag{3.17}$$

$$K_N^{(2)}=2\sigma_N-4\tau_N\ , \tag{3.18}$$

$$S_N^{(2)}=-6\beta^4(\sigma_N+2\tau_N)=128M^2\langle R\rangle_N\ , \tag{3.19}$$

$$\beta^{-2}Q_N^{(2)}=2(N+3)(\sigma_N+6\tau_N+24)-\tfrac{4}{3}N\sigma_N\ . \tag{3.20}$$

Here

$$\sigma_N=2(N+3)^2-8u(u+1),\quad \tau_N=\tfrac{4}{3}N+\tfrac{1}{6}\sigma_N-2\ ,$$
$$u(u+1)=\tfrac{3}{4}\ \text{(even } N),\ \ 2\ \text{(odd } N)\ . \tag{3.21}$$

β is a dimensional quantity,[14] Eq. (3.24), which governs the momentum scales of (ξ,η) in accordance with the HO wave function ψ, Eq. (3.26).

A more compact treatment than that given in Appendix A is possible with a complex[27] representation of S_3 symmetry, which was recently employed for some $q^2\bar{q}^2$ systems,[28] but the meaning of the total quantum number $N\ (=N_\xi+N_\eta)$ is less transparent in terms of its complex constituents (N_z,N_z^*) than in terms of the real (N_ξ,N_η) representation, and, therefore, is not yet suitable for contact with data for excited baryon states (which are traditionally classified in the real representation).

With the substitutions (3.17)–(3.21) for the various operators, anticipating that N retains its "diagonal" significance for the *reduced* Eq. (3.16), these correction terms may be treated as constants in the N representation, after a similar substitution has been made for the factor D_s^{-1} multiplying ΔH; see below in Eq. (3.27). Under these conditions Eq. (3.16) takes the form of a standard six-dimensional HO whose solution may be expressed as[14]

$$F_{\mathrm{HO}}+\delta F_{\mathrm{HO}}=N_\xi+N_\eta+3=N+3\ , \tag{3.22}$$

$$\beta^2\gamma_B^2F_{\mathrm{HO}}=\tfrac{1}{9}M^2-m_q^2-\langle R\rangle_N +\tfrac{1}{2}\omega_{qq}^2[MC_0\omega_0^{-2}+\tfrac{3}{4}M^{-1}(Q_N-8\mathbf{J}\cdot\mathbf{S}+18)]\ , \tag{3.23}$$

$$\gamma_B^2=1-\tfrac{9}{2}\omega_{qq}^2C_0\omega_0^{-2},\quad \beta^4=\tfrac{2}{3}M\omega_{qq}^2\gamma_B^{-2}\ , \tag{3.24}$$

$$16\beta^2\gamma_B^2\delta F_{\mathrm{HO}}=\tfrac{1}{2}\omega_{qq}^2\left\langle\frac{1}{D_s}\right\rangle\left[Q_N^{(2)}-\tfrac{4}{9}M^2K_N^{(2)}+\frac{C_0}{\omega_0^2}S_N^{(2)}\right]\ . \tag{3.25}$$

The physical significance of β^2 is most succinctly seen from the form of the ground-state wave function

$$\psi_0=\exp[-\tfrac{1}{2}\beta^{-2}(\boldsymbol{\xi}^2+\boldsymbol{\eta}^2)]\ . \tag{3.26}$$

Finally the value of $\langle D_s^{-1}\rangle$ in the correction term δF is governed by the following considerations. First we note that this term is in general quite small ($\lesssim 10\%$) and its relative effect decreases rapidly with M. Now, for large M, the operator D_s^{-1} is far away from its singularities and it should be reasonable, as a first approximation, to replace its variable part $\boldsymbol{\xi}^2+\boldsymbol{\eta}^2$ by $\langle\boldsymbol{\xi}^2+\boldsymbol{\eta}^2\rangle=\beta^2(N+3)$, leading to the form $\langle D_s\rangle^{-1}$. A more accurate formula, which holds all the way down to $N=1$, is given by

$$\langle D_s^{-1}\rangle=-\langle D_0\rangle^{-1}\exp[\tfrac{2}{3}M\beta^2(N+3)\langle D_0\rangle^{-1}]\ , \tag{3.27}$$

where

$$\langle D_0\rangle=\tfrac{4}{3}M(\tfrac{1}{9}M^2-m_q^2-\langle R\rangle_N)\ . \tag{3.28}$$

As for the $N=0$ cases (N,Δ), the smaller values of $\langle D_0\rangle$ tend to overestimate the exponential factor, necessitating more careful considerations, which effectively amounts to its replacement by its zeroth-order and first-order terms in its expansion for N and Δ cases, respectively. This completes the solution of (3.16) in the form (3.22).

For contact with the data on baryon spectra, Eq. (3.22) must be augmented by the effect of one-gluon exchange, including the Fermi-Breit term.[3] As explained in the original formulation,[14] and substantiated for in the $q\bar{q}$ systems with the present refinements,[21] it is adequate to consider this effect perturbatively for *light* (*uds*) hadrons only. The procedure, which has been described in I, consists of adding this extra contribution to the LHS of (3.22) which would now read

$$F_B(M)\equiv F_{\mathrm{HO}}+\delta F_{\mathrm{HO}}+F_{\mathrm{SC}}+\delta F_{\mathrm{FB}}=N+3\ , \tag{3.29}$$

where the two extra terms represent the strong Coulomb and Fermi-Breit contributions, respectively, in the *same* relative normalization as defines the principal term F_{HO}. This gives, for the Coulomb term in coordinate space,

$$F_{\mathrm{SC}}=\beta^{-2}\gamma_B^{-2}\sum_{123}\tfrac{1}{2}M^{-1}\alpha_{\mathrm{SC}}\langle M_{12}^2r_{12}^{-1}\rangle\ , \tag{3.30}$$

where M_{12}^2 is given by (2.12) as an *operator* in coordinate space, and α_{SC} is the strong-Coulomb coupling constant given by (3.2), but with $9mq^2\to M^2$. As to the shorter-range Fermi-Breit corrections, their *full* effect had been considered in I and found to be small. However, there are certain formal differences between the complete expressions[14] based on our Gordon reduction method and the traditional structure[3] based on reduction in terms of large and small components of the wave function. Because of their intrinsic smallness we have considered here only the terms $\sim\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j$ which agree exactly for both methods of reduction. These give

$$\delta F_{\mathrm{FB}}=-\beta^{-2}\gamma_B^{-2}M^{-1}\alpha_{\mathrm{SC}}\sum_{123}\left\langle\frac{4\pi}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\delta^3(\mathbf{r}_{12})\right\rangle\ . \tag{3.31}$$

Evaluation of (3.30) and (3.31) must be made in the coordinate-space representations of the qqq wave functions whose momentum-space form is illustrated in Eq. (3.26) for the ground state $(N=0)$. The higher (N,L) wave functions, including complete normalizations for arbitrary excitations, are described in an earlier paper,[17] the results of which are used in Appendix B for a short listing of Coulomb contributions from the relevant types (**56**,**70**) of SU(6)×O(3) states needed for the present analysis. The FB term (3.31), in particular, can be compactly represented as

$$\delta F_{\rm FB} = -\tfrac{1}{2}\beta^{-2}\gamma_B^{-2}M^{-1}\alpha_{\rm SC}\left[\frac{3\beta^2}{4\pi}\right]^{3/2}[4S(S+1)-9]W_{NL} , \quad (3.32)$$

where $S=\frac{1}{2}$ or $\frac{3}{2}$ for d or q states,[12] respectively, and W_{NL} is a geometrical weight factor depending on the (N,L) values of the state only. It is given by formula (B15) of Appendix B.

IV. RESULTS AND DISCUSSION

As noted at the outset our object in this paper is not so much to make a detailed comparison with the data as to provide a more consistent relativistic framework for an *integrated* view of both $q\bar{q}$ and qqq systems, with common values employed for the basic constants (ω_0, C_0, m_q). To that end we shall consider a sufficiently representative sample of baryon states, which should provide a fair number of "check points" to warrant meaningful conclusions about the theoretical predictions vis-à-vis the experimental trends, without going into too many fine-grained details. These presumably require an elaborate mixing program for states, as has been successfully carried out in the past,[1] using the dynamical mechanism of one-gluon exchange. We have little to add to *this* aspect of the problem beyond the assertion that the same facility formally exists within our BS framework, and indeed was found to be quite important for heavy quarkonia[21] where the *mixing* of several radial states via the Coloumb term was crucial[21] for a successful fit to these types of data.[11] Apart from the Coulomb term, the present formalism also facilitates mixing between states due to several types of spin-dependent corrections arising from the *vector confinement* (as distinct from the short-range effects of one-gluon exchange). These are listed as the SDT's of ΔH, in Appendix C. However, in keeping with the basic objective of not putting too much emphasis on details in this paper, we have not made any elaborate use of these terms, except to illustrate their possibilities with the help of one example: the mixing of $P'_{11}(1440)$ and $P''_{11}(1710)$ as members of $(\mathbf{56},0_2^+)$ and $(\mathbf{70},0^+)$ supermultiplets, respectively (see below for results).

Before presenting the numerical results it is useful to make some general comments on the specific role of *vector* confinement, apart from the crucial one of providing the same sign for qq and $q\bar{q}$ interactions[8] (unlike scalar confinement). First, the asymptotic behavior of M with respect to N is easily deduced, after necessary substitutions in Eq. (3.23), to be $M \sim N^{2/3}$, a result which is reminiscent of a linear potential operative within a nonrelativistic (NR) (Schrödinger) framework,[29] even though we have employed a harmonic kernel (within a BS framework). This would seem to suggest that, to the extent that the present HO formalism fits the data for $q\bar{q}$ systems[21] as well as for qqq systems (as we see below), a corresponding BS treatment with an effectively *linear* kernel for such light-quark systems would give too-narrow spacings between successive N-excitations and, therefore, is in disagreement with observations, as was indeed found sometime ago.[30] Further, within the same BS framework, a *scalar* confinement with a harmonic BS kernel can be shown to give an asymptotic behavior[31] $M \sim N^{2/5}$ and a still smaller power with a linear kernel. The same results are of course true of $q\bar{q}$ systems as well.[14,21] Second, vector confinement produces some characteristic momentum and spin-dependent terms, as may be seen from Eq. (3.23) in the form of Q_N and $2\mathbf{J}\cdot\mathbf{S}$, with additional diagonal corrections arising from $\delta F_{\rm HO}$, Eq. (3.25). [Spin-dependent corrections (nondiagonal) contributing to mixing between states of different N, L, S values are listed in Appendix C.]

Table I depicts the results for the mass spectra of a representative cross section of baryon states (nonstrange), and may be regarded as the qqq analog of the corresponding results[21] for a wider list of meson states, within a common QCD-oriented framework with identical parameters (3.1). To bring out the role of the Q_N, $2\mathbf{J}\cdot\mathbf{S}$, etc., terms more naturally, especially for higher-N states, it is useful to employ the same artifice as in earlier publications,[14,17] viz., to list the F_B values of the LHS of Eq. (3.29) for the *experimental* masses[11] of the baryons concerned and check against their "theoretical" values $N+3$ on the RHS. And since the vacuum structure is now hopefully simulated by the "known" constant C_0 (determined from $q\bar{q}$ spectra[21]), a comparison of the two columns will offer a direct test of whether, and to what extent, the zero-point-energy shortfalls[14,17] for qqq states are filled in this (new) form of the theory. (They just get filled for $q\bar{q}$ systems.[21])

Table I does indeed show that the large ZPE shortfall of as much as two units (which had plagued the earlier formulation[14]) is almost completely filled up, as seen from a comparison of the theoretically expected values $N+3$ of $F_B(M)$. Considering the fact that there are no adjustable parameters, this feature must be regarded as a nontrivial test of the relativistic three-body equation (3.16) which, despite its superficial similarity to a six-dimensional HO form, goes far beyond the Schrödinger description, in view of the rich (M,N) dependence of the various terms. These features are somewhat akin to those of the Todorov equation[4] or allied formulations,[6] but differ in theoretical assumptions as well as formulation details.

Next, the *unit-step* variations of $F(M)$ with N that are revealed through this comparison suggest that $F(M)$ is almost an $SU(6)\times SU(3)_{\rm HO}$-invariant quantity, depending only on the total quantum number N, as expected from the theory. The relatively small scatters that are visible in the $N=2$ region are mainly from states which are most likely to be affected by mixing between "like" states in **56** and **70**, as is known to be important from earlier studies,[1] and facilitated by the "SDT's" of ΔH in this paper. For the relatively unmixed states we do find that the scatter is indeed small, thus collectively reveal-

TABLE I. Baryon mass spectra: test of $F_B(M)=N+3$ against data (Ref. 8).

State	SU(6): $NJLS$	$F_{HO}+\delta F_{HO}$	$F_{SC}+\delta F_{FB}$	Total F_B	Expected $(N+3)$
$N(938)$	(**56**): $0_{1/2}0_{1/2}$	1.769	1.203	2.972	3
$\Delta(1236)$	(**56**): $0_{3/2}0_{3/2}$	1.920	0.899	2.819	3
$D_{13}(1520)$	(**70**): $1_{3/2}1_{1/2}$	2.968	1.011	3.979	4
$D_{15}(1675)$	(**70**): $1_{5/2}1_{3/2}$	2.865	0.933	3.798	4
$P'_{11}(1440)$	(**56,70**): $2_{1/2}0_{1/2}$	3.247	1.123	4.370[a]	5
$P''_{33}(1600)$	(**56,70**): $2_{3/2}0_{3/2}$	3.276	0.872	4.148[a]	5
$F_{15}(1680)$	(**56**): $2_{5/2}2_{1/2}$	3.876	1.136	5.012	5
$P''_{11}(1710)$	(**70**): $2_{1/2}0_{1/2}$	4.126	0.895	5.021[a]	5
$P''_{13}(1720)$	(**70**): $2_{3/2}0_{3/2}$	3.714	0.744	4.488[a]	5
$P''_{33}(1920)$	(**56,70**): $2_{3/2}0_{3/2}$	4.452	0.834	5.286[a]	5
$F_{35}(1905)$	(**56,70**): $2_{5/2}2_{3/2}$	4.462	0.817	5.279[a]	5
$F_{37}(1950)$	(**56**): $2_{7/2}2_{3/2}$	4.199	0.829	5.028	5
$F_{17}(1990)$	(**70**): $2_{7/2}2_{3/2}$	4.362	0.878	5.240	5
$G_{17}(2190)$	(**70**); $3_{7/2}3_{1/2}$	5.513	0.855	6.368	6
$G_{19}(2250)$	(**70**): $3_{9/2}3_{3/2}$	5.168	0.831	5.999	6
$H_{19}(2250)$	(**56**): $4_{9/2}4_{1/2}$	5.883	0.763	6.646	7
$H_{3,11}(2420)$	(**56**): $4_{11/2}4_{3/2}$	6.075	0.753	6.828	7
$I_{1,11}(2600)$	(**70**): $5_{11/2}5_{1/2}$	7.308	0.771	8.079	8
$K_{3,15}(2950)$	(**56**): $6_{15/2}6_{3/2}$	8.134	0.733	8.846	9

[a]Possibly mixtures of **70,56** states.

ing the role of the "diagonal" correction terms Q_N, $2\mathbf{J}\cdot\mathbf{S}$ and the pieces of ΔH. This is particularly manifest when one compares the pairs

$$N,\Delta,\ \ D_{13},D_{15},\ \ F_{15},F_{37},\ \ H_{19},H_{3,11}\ , \tag{4.1}$$

the near equality of the F_B values for these pair implying that their huge (mass)^2 differences are actually "understood" in this model.

To illustrate the possibilities of **56,70** mixing within this model, Appendix C also sketches a calculation of this effect, employing the $L=0$ term of the SDT, for one of the "bad" pairs in Table I, viz., $P'_{11}(1440)$ and $P''_{11}(1710)$. The resulting corrections δF to their unmixed F_B values are shown in (C9), the inclusion of which leads to the following corrected values ($\bar{F}$) for these states:

$$\bar{F}(P'_{11})=5.045,\ \ \bar{F}(P''_{11})=4.955\ , \tag{4.2}$$

in excellent agreement with the "expected" value of 5.00.

For a more direct comparison of the mass predictions with the data, and also to test the sensitivity of the function $F_B(M)$ to the actual mass M, we have also provided a second table (Table II) depicting the predicted masses through a *numerical* inversion of the equation $F(M)=N+3$ for the "appropriate" values of N. This has been rapidly facilitated by the observation that the M dependence of $F(M)$ is of the form

$$a_0M^{3/2}+a_1M^{-3/2}+b_0M^{1/2}+b_1M^{-1/2}\ ,$$

where (a_0,a_1) pertain to the confinement interaction and (b_0,b_1) to the one-gluon-exchange effect (Coulomb + Fermi-Breit). The comparison between the theoretical and experimental masses does indeed show a good overlap for most cases, except for the cases labeled with (a) or (b) which have already been recognized at the $F_B(M)$ level, Table I, to be affected by "mixing" within N-super multiplets. The sensitivity of $F(M)$ to M is reflected in the comparison of the columns $\delta M=M(\text{th})-M(\text{expt})$ vs $\Delta F(M)=F(M)-N-3$, deduced from Table I. This comparison suggests that the $F(M)$ representation is

TABLE II. Theoretical values of the masses (in MeV units) obtained from $F_B(M)=N+3$ [Eq. (3.29)] compared with the experimental data (Ref. 8) for the various baryon states. $\delta M=M(\text{th})-M(\text{expt})$; $\Delta F(M)=F(M)-N-3$ is deduced from Table I.

State	M(expt)	M(th)	δM	$\Delta F(M)$
N	938	944	+6	−0.028
Δ_{33}	1236	1264	+28	−0.181
D_{13}	1520	1524	+4	−0.021
D_{15}	1675	1707	+32	−0.202
P'_{11}	1440	1551[a]	+110	−0.630
		1432[b]	−8	+0.045
P''_{33}	1600	1737[a]	+137	−0.852
F_{15}	1680	1678	−2	+0.012
P''_{11}	1710	1706[a]	−4	+0.021
		1718[b]	+8	−0.045
P''_{13}	1720	1804[a]	+84	−0.512
P''_{33}	1920	1872[a]	−48	+0.286
F_{35}	1905	1858[a]	−47	+0.279
F_{37}	1950	1945	−5	+0.028
F_{17}	1990	1950	−40	+0.240
G_{17}	2190	2128	−62	+0.368
G_{19}	2250	2250	+0.2	−0.001
H_{19}	2250	2304	+54	−0.354
$H_{3,11}$	2420	2447	+27	−0.172
$I_{1,11}$	2600	2587	−13	+0.079
$K_{3,15}$	2950	2973	+23	−0.154

[a]Values obtained *without* taking mixing into account (see text).
[b]Values obtained *after* taking mixing into account (see text) by using the corrected values ($\bar{F}$) of $F_B(M)$ from Eq. (4.2).

conservative enough to "magnify," if anything, the actual extent of the difference between the experimental and predicted masses. Further, for the $P_{11}(1440)$ and $P'_{11}(1710)$ states, their mixing has already been found to result in a dramatic improvement in their $F(M)$ values, as evidenced by Eq. (4.2). The same is reflected in the actual mass predictions as well as δM in Table II, before (a) and after (b) their mixing is taken into account.

For a more global view of these mass patterns, manifesting through their F_B values, we have plotted them in Fig. 1 as functions of N. The straight line with *unit slope* is seen to pass through most of these points with very little scatter, all the way up to $N=6$, thus suggesting strongly that the asymptotic prediction $M \sim N^{2/3}$ is rather well satisfied by the data. This may be regarded as an observational test of the vector confinement which predicts this feature within the BS framework. A similar formulation with unequal-mass kinematics and corresponding results on (Λ,Σ) states are under preparation.

We conclude with a few remarks on the two unrelated issues of (i) Gordon reduction prior to the NPA and (ii) vector versus scalar confinement. First, Gordon reduction, which makes sense only on the mass shell, seems to be a rather natural step in the present context of NPA in which the mass shell condition essentially *defines* the component p_- in terms p_+ and $\mathbf{p}_\perp$. And the extent of simplification achieved through this device with respect to the traditional Salpeter-type reduction in terms of $(\pm\pm\pm)$ components of the qqq wave function may be gauged by a comparison of Eq. (2.22) with a recent derivation by Kopaleishvili *et al.*[32] of a *coupled set of equations (three pages)* connecting these various components in the traditional (Salpeter-type) approach. The second point concerns the perspectives on the question of vector vs scalar confinment. Since the very concept is phenomenological, in the absence of a formal solution to the QCD Lagrangian problem, neither vector nor scalar confinement can be the whole story anyway, as has been recognized earlier by other authors[33] as well. Further, as explained in Ref. 14 of Ref. 21, the fine-structure splittings in ^3P_J states of $C\bar{C}$ are only sensitive to the higher-order (α_s^2) corrections to the short-range one-gluon-exchange term, but *not* much to the structure of the long-range confinement term. On the other hand, the spectra of $q\bar{q}$ states[21] as well as of qqq states found here seem to favor the asymptotic variation $M \sim N^{2/3}$ (vector) to $M \sim N^{2/5}$ (scalar), within the BS dynamical framework. At a more fundamental level, only a vector confinement (*not* scalar) seems to offer the possibility of a *common sign* for the long-range $q\bar{q}$ and qq interactions, thus justifying a common parametrization for them, which represents a major theme of this investigation aiming to unify the spectra of $q\bar{q}$ and qqq states. Other tests will be clearly desirable.

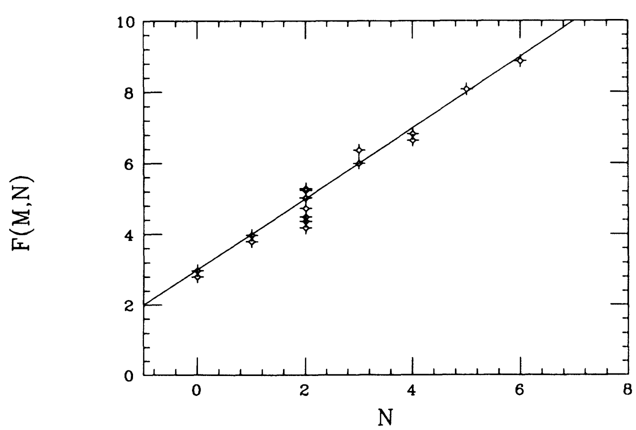

FIG. 1. Plot of the function $F_B(M)$ as a function of the total HO quantum number N, using experimental values of the baryon masses M. The expected line is $F_B(M)=N+3$. For a definition of $F_B(M)=F(M,N)$ see text.

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APPENDIX A

We briefly outline here a practical method based on an interplay of SO(2,1) algebra and S_3 symmetry for the eigenvalues of certain ξ- and η-dependent operators $\hat{Q}_B$ listed in Sec. III. Define two sets of HO operators[14,26] $a_{\xi i}$, $a_{\eta i}$, and $a^\dagger_{\xi i}$, $a^\dagger_{\eta i}$ through

$$\sqrt{2}\xi_i\beta^{-1}=a_{\xi i}+a^\dagger_{\xi i}, \quad \sqrt{2}\beta\partial_{\xi i}=a_{\xi i}-a^\dagger_{\xi i} , \tag{A1}$$

with an identical set for η. Similarly define the following scalar operators *quadratic* in the a_i's through

$$N_\xi=a^\dagger_{\xi i}a_{\xi i}, \quad N_\eta=a^\dagger_{\eta i}a_{\eta i} , \tag{A2}$$

$$A_\xi=a_{\xi i}a_{\xi i}, \quad A^\dagger_\xi=a^\dagger_{\xi i}a^\dagger_{\xi i} , \tag{A3}$$

$$A_\eta=a_{\eta i}a_{\eta i}, \quad A^\dagger_\eta=a^\dagger_{\eta i}a^\dagger_{\eta i} , \tag{A4}$$

$$B=a_{\xi i}a_{\eta i}, \quad B^\dagger=a^\dagger_{\xi i}\cdot a^\dagger_{\eta i} , \tag{A5}$$

$$C=C^\dagger=a_{\xi i}a^\dagger_{\eta i}+a_{\eta i}a^\dagger_{\xi i} . \tag{A6}$$

The basic commutation relations for the a_i's,

$$[a_{\xi i},a^\dagger_{\xi j}]=\delta_{ij}=[a_{\eta i},a^\dagger_{\eta j}] , \tag{A7}$$

with all other pairs commuting, lead to the following results for the various quadratic operators:

$$[A_\xi,N_\xi]=2A_\xi, \quad [A^\dagger_\xi,N_\xi]=-2A^\dagger_\xi , \tag{A8}$$

$$[A_\eta,N_\eta]=2A_\eta, \quad [A^\dagger_\eta,N_\eta]=-2A^\dagger_\eta , \tag{A9}$$

while pure ξ and pure η operators commute.

Now the two sets $(A_\xi,A^\dagger_\xi,N_\xi)$ and $(A_\eta,A^\dagger_\eta,N_\eta)$ represent the generators of two independent SO(2,1) algebras, any one of which is exactly the type that proved adequate for the derivation of the eigenvalues[14,26] of the operator $\hat{Q}_q$ appearing in the corresponding $q\bar{q}$ problem. In the present qqq problem, on the other hand, these two sets are also accompanied by "mixed" terms

B, $B^\dagger$, C thus considerably enlarging the algebra. For a systematic treatment consistent with S_3 symmetry, it is convenient to introduce the following combinations which have the desired permutation symmetries for a three-body system:

$$N \equiv N^s = N_\xi + N_\eta , \quad N^a = a^\dagger_{\xi i} a_{\eta i} - a^\dagger_{\eta i} a_{\xi i} , \tag{A10}$$

$$C \equiv N' = a^\dagger_{\xi i} a_{\eta i} + a^\dagger_{\eta i} a_{\xi i} , \quad N'' = N_\xi - N_\eta , \tag{A11a}$$

$$2B \equiv A' = 2 a_{\xi i} a_{\eta i} , \quad A'' = A_\xi - A_\eta , \tag{A11b}$$

$$A_s = A_\xi + A_\eta , \quad A_s^\dagger = A_\xi^\dagger + A_\eta^\dagger . \tag{A12}$$

Using the commutation relations (A8) and (A9) one obtains the following sets of commutators:

$$[A', N] = 2A' , \quad [A'^\dagger, N] = -2A'^\dagger , \tag{A13a}$$

$$[A'', N] = 2A'' , \quad [A''^\dagger, N] = -2A''^\dagger , \tag{A13b}$$

$$[N', N] = [N'', N] = [N^a, N] = 0 , \tag{A14}$$

$$\begin{aligned} &[A_s, N] = [A'', N''] = [A', N'] = 2A_s , \\ &[A_s^\dagger, N] = [A''^\dagger, N''] = [A'^\dagger, N'] = -2A_s^\dagger , \end{aligned} \tag{A15}$$

$$[A_s, A_s^\dagger] = [A'', A''^\dagger] = [A', A'^\dagger] = 4N + 12 , \tag{A16}$$

$$\begin{aligned} &[A', A''] = [A_s, N_a] = [A_s^\dagger, N_a] = 0 , \\ &[A', A''^\dagger] = -4N_a , \\ &[N', N_a] = 2N'' , \quad [N'', N_a] = -2N' , \\ &[A', N_a] = 2A'' , \\ &[A'', N_a] = -2A' , \quad [A'^\dagger, N_a] = 2A''^\dagger , \\ &[A''^\dagger, N_a] = -2A'^\dagger . \end{aligned} \tag{A17}$$

One thus finds several distinct sets of *coupled* SO(2,1) algebras, closing on an algebra as big as SO(m,n) where $m+n=10$, thus greatly reducing the practical value of this method, but for the observation that the operator chiefly responsible for the couplings is N_a, a *totally antisymmetric* object.

In SU(6) parlance, this operator makes it first nontrivial appearance (with a nonzero eigenvalues) for a **20** state whose totally antisymmetric spatial structure does not easily allow it to mix with the **56** and **70** states, as a consequence of which its identity is still not experimentally established. Taking advantage of this circumstance it is perhaps not unreasonable to drop the operator N_a from the list (A10)–(A17), which vastly simplifies the resulting algebra to an almost decoupled set of distinct SO(2,1) algebras. Before writing down the Casimir operators it is useful to express the operators (3.9)–(3.15) of the text in terms of the above quantities so as to exhibit their S_3 symmetry structure more explicitly. These give

$$\begin{aligned} \hat{Q}_B = {} & \tfrac{1}{2}[A_s + A_s^\dagger]^2 + \tfrac{1}{2}[A' + A'^\dagger]^2 + \tfrac{1}{2}[A'' + A''^\dagger]^2 \\ & - 2N'^2 - 2N''^2 - 2(N+3)^2 - 18 , \end{aligned} \tag{A18}$$

$$\begin{aligned} \hat{K}^{(2)} = {} & [A' + A'^\dagger]^2 + [A'' + A''^\dagger]^2 \\ & - 4(N'^2 + N''^2) + 8(A_s^\dagger - A_s) , \end{aligned} \tag{A19}$$

$$\begin{aligned} \beta^{-4} S^{(2)} = {} & -3(A' + A'^\dagger + 2N')^2 \\ & - 3(A'' + A''^\dagger + 2N'')^2 , \end{aligned} \tag{A20}$$

and a similar but more unwieldly expression for $\hat{Q}^{(2)}$, which we omit for brevity.

These operators still involve nondiagonal terms, but which, as in the $q\bar{q}$ case,[26] connect states differing by as much as $\Delta N = 4$, so that ignoring their effects is not likely to introduce any serious error. The diagonal terms are then almost expressible in terms of the following three SO(2,1) sets:

$$\tfrac{1}{2}(A', A'^\dagger, N+3), \quad \tfrac{1}{2}(A'', A''^\dagger, N+3) ,$$
$$\tfrac{1}{2}(A_s, A_s^\dagger, N+3) ,$$

all of which have identical Casimir operators, $U(U+1)$, with a rising spectrum[17]

$$\tfrac{1}{2}(N+3) \equiv -u + k , \quad k = 0, 1, 2, \ldots \tag{A21}$$

leading to the two classes of u values defined in Eq. (3.21) of the text. The only additional combination occurring in (A18)–(A20) which is not directly amenable to these quantum numbers is $N'^2 + N''^2$, which requires a more careful treatment involving the use of tensor operators $(A_{ij}, A_{ij}^\dagger) \ldots$, with $A_s = A_{ii}$, leading to the identification

$$N'^2 + N''^2 = A_{ij}^\dagger A_{ij} + 2N . \tag{A22}$$

Since, on the other hand, tensor operators are beyond the jurisdiction of our algebra (A10)–(A17), we must make the replacement $A_{ij}^\dagger A_{ij} \to \frac{1}{3} A_s^\dagger A_s$ to obtain (without serious error) the approximate result

$$N'^2 + N''^2 \approx \tau_N = \tfrac{4}{3}N - 2 + \tfrac{1}{3}(N+3)^2 - \tfrac{4}{3}u(u+1) \tag{A23}$$

in terms of the Casimir operator of the simpler SO(2,1). The resulting expressions are as listed in Eqs. (3.17)–(3.21) of the text.

APPENDIX B

We collect here the results of perturbative calculations of the strong Coulomb effects in coordinate space. Using the vectors $\mathbf{u}$, $\mathbf{v}$, which are conjugate to $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ ($\mathbf{u} = i\partial_\xi$, etc.) the quantities $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ are given by

$$\mathbf{x}_1 \equiv \mathbf{X} - \tfrac{2}{3}\mathbf{v} , \quad \mathbf{x}_{2,3} \equiv \mathbf{X} + \tfrac{1}{3}\mathbf{v} \mp (\tfrac{1}{3})^{1/2}\mathbf{u} . \tag{B1}$$

It is necessary to consider L excitations in a general way, but radial excitations of $n=1$ are adequate for our purposes. The possible states are of the types[1,17,12]

$$(\mathbf{56}, 2l^+); \quad (\mathbf{70}, 2l+1^-) ,$$
$$(\mathbf{70}, 2l+2), \quad l = 0, 1, 2, \ldots ,$$

of which only their spatial parts in their *maximally stretched* forms are needed, since the Coulomb interaction is spin independent. The **56** states are fully symmetric:[17]

$$\psi^s_{2l}(\mathbf{u},\mathbf{v})=N^s_{2l}(u_+^2+v_+^2)^l\left[\frac{\beta^2}{\pi}\right]^{3/2}\beta^{2l}\times\exp[-\tfrac{1}{2}\beta^2(\mathbf{u}^2+\mathbf{v}^2)]\ , \tag{B2}$$

$$N^s_{2l}=(l!)^{-1}\ . \tag{B3}$$

The $(\mathbf{70},2l+1)$ states ψ', ψ'' are of the form

$$\psi'_{2l+1};\psi''_{2l+1}=\tilde N_{2l+1}(\beta u_+;\beta v_+)\psi^s_{2l}/N^s_{2l}\ , \tag{B4}$$

$$\tilde N_{2l+1}=N^s_{2l}[2/(l+1)]^{1/2}\ . \tag{B5}$$

Finally the $(\mathbf{70},2l+2)$ states are of the form

$$\psi'_{2l+2};\psi''_{2l+2}=\tilde N_{2l+2}\beta^2(2u_+v_+;u_+^2-v_+^2)\psi^s_{2l}/N^s_{2l}\ , \tag{B6}$$

$$\tilde N_{2l+2}=N^s_{2l}[2/(l+1)(l+2)]^{1/2}\ . \tag{B7}$$

The radially excited functions are relevant for $P'_{11}(1440)$, $P'_{33}(1600)$, and possibly higher ones.[12,1] Their general construction has been described in Ref. 17 but here we need only the case of $n=1$, for which the normalized wave function is

$$\psi^s_{n=1}=(3\pi^3)^{-1/2}[\beta^2(\mathbf{u}^2+\mathbf{v}^2)-3]\exp[-\tfrac{1}{2}\beta^2(\mathbf{u}^2+\mathbf{v}^2)]\ . \tag{B8}$$

Analogous to $n=1$ of **56** states, there are possible $(\mathbf{70},0^+)$ states P''_{11}, P''_{13}, etc.,[17] whose corresponding wave functions are

$$(\psi';\psi'')_{n=1}=(3\pi^3)^{-1/2}\beta^2[2\mathbf{u}\cdot\mathbf{v};\mathbf{u}^2-\mathbf{v}^2]\times\exp[-\tfrac{1}{2}\beta^2(\mathbf{u}^2+\mathbf{v}^2)]\ . \tag{B9}$$

Calculation of the Coulomb contributions, Eq. (3.30), for these different cases is a straightforward, though lengthy, procedure. The results are expressible in the general form

$$F_{\rm SC}(L)=\tfrac{3}{4}\sqrt{3}\alpha_s M\beta^{-1}\gamma_B^{-2}N_L^2 2^{-2l}\times\sum_{r=0}^{l}\begin{bmatrix} l\\ r\end{bmatrix}^2(2r!)f_{lr}[\tfrac{4}{9}-\beta^2M^{-2}(2r+\tfrac{3}{2})]\times\Gamma^2(2l-2r+1)\Gamma^{-1}(2l-2r+\tfrac{3}{2})\ , \tag{B10}$$

where $f_{lr}=1$ for $(\mathbf{56},2l^+)$ states and

$$(\mathbf{70},2l+1):\ f^{\rm odd}_{lr}=\tfrac{1}{4}[2r+1+(2l-2r+1)^2\times(2l-2r+\tfrac{3}{2})^{-1}]\ , \tag{B11}$$

$$(\mathbf{70},2l+2):\ f^{\rm even}_{lr}=\tfrac{1}{8}(2r+1)(2r+2)+\frac{1}{8}\frac{(2l-2r+1)^2(2l-2r+2)^2}{(2l-2r+\frac{3}{2})(2l-2r+\frac{5}{2})}+\frac{1}{2}\frac{(2r+1)(2l-2r+1)^2}{2l-2r+\frac{3}{2}}\ . \tag{B12}$$

N_L^2 is the normalizer, listed above, appropriate to the state under consideration. For the $n=1$ cases we have

$$F_{\rm SC}(\mathbf{56};n=1)=\frac{1}{2}\left[\frac{3}{\pi}\right]^{1/2}M\alpha_{\rm SC}\beta^{-1}\gamma_B^{-2}(\tfrac{11}{9}-\tfrac{45}{8}\beta^2M^{-2})\ , \tag{B13}$$

$$F_{\rm SC}(\mathbf{70};0^+)=\frac{1}{2}\left[\frac{3}{\pi}\right]^{1/2}M\alpha_{\rm SC}\beta^{-1}\gamma_B^{-2}(\tfrac{19}{18}-\tfrac{109}{16}\beta^2M^{-2})\ . \tag{B14}$$

Finally the weight factor W_{NL}, Eq. (3.32), for orbitally excited **56**, **70** states in the L convention defined above, is given by the simple formula

$$W_{NL}=(2l-1)!!/(2^l l!)\ . \tag{B15}$$

In addition, for $(\mathbf{56},n=1)$ and $(\mathbf{70},0^+)$ states this quantity is $\frac{5}{4}$ and $\frac{5}{8}$, respectively.

APPENDIX C

We list here the spin-dependent terms (SDT) appearing in Eq. (3.12) of the text.

$$\mathrm{SDT}\equiv-\tfrac{3}{2}(\xi^2-\eta^2)\Sigma''_t-3\xi\cdot\eta\Sigma'_t \tag{C1}$$

$$+\tfrac{1}{2}\mathbf{\Sigma}\cdot[(\xi^2-\eta^2)\mathbf{L}''+2\xi\cdot\eta\mathbf{L}'] \tag{C2}$$

$$+\tfrac{3}{4}\mathbf{L}\cdot[(\xi^2-\eta^2)\mathbf{\Sigma}''+2\xi\cdot\eta\mathbf{\Sigma}'] \tag{C3}$$

$$+\tfrac{3}{4}\mathbf{L}_a\cdot[(\xi^2-\eta^2)\mathbf{\Sigma}'-2\xi\cdot\eta\mathbf{\Sigma}''] \tag{C4}$$

$$-\tfrac{3}{4}(\xi^2-\eta^2)[\mathbf{L}'\cdot\mathbf{\Sigma}'-\mathbf{L}''\cdot\mathbf{\Sigma}'']$$

$$-\tfrac{3}{2}\xi\cdot\eta[\mathbf{L}'\cdot\mathbf{\Sigma}''+\mathbf{L}''\cdot\mathbf{\Sigma}']\ . \tag{C5}$$

Here

$$i\mathbf{L}';i\mathbf{L}_a\equiv\xi\times\partial_\eta\pm\eta\times\partial_\xi\ ,$$

$$i\mathbf{L};i\mathbf{L}''\equiv\xi\times\partial_\xi\pm\eta\times\partial_\eta\ ,$$

$$\mathbf{\Sigma}\equiv\sigma_1+\sigma_2+\sigma_3\ ,$$

$$\sqrt{3}\mathbf{\Sigma}'\equiv\sigma_3-\sigma_2\ ,$$

$$3\mathbf{\Sigma}''\equiv-2\sigma_1+\sigma_2+\sigma_3\ ,$$

$$\sqrt{3}\Sigma'_t=\sigma_1\cdot(\sigma_2-\sigma_3)\ ,$$

$$3\Sigma''_t=-2\sigma_2\cdot\sigma_3+\sigma_1\cdot\sigma_2+\sigma_1\cdot\sigma_3\ .$$

The S_3 symmetry of SDT is explicit. As an illustration of the use of these terms, we consider the effect of the $L=0$ term (C1) which mixes the **56** radial state $P'_{11}(1440)$ and the $(\mathbf{70},0^+)$ state $P''_{11}(1710)$. Using the reduction method of Appendix A, and dropping the nondiagonal terms in N, we have

$$2\xi\cdot\eta\beta^{-2};(\xi^2-\eta^2)\beta^{-2}\Longrightarrow\tfrac{1}{2}[N';N''] ,$$

so that (C6)

$$(\text{SDT})_{L=0}=-\tfrac{3}{4}\beta^2[N'\Sigma_t'+N''\Sigma_t''] .$$

Inserting this term in (3.16) modifies it to two coupled equations involving the **56** radial and $(\mathbf{70},0^+)$ states whose SU(6)-cum-spatial structures (see Appendix B) are

$$\sqrt{2}\Psi_{56}=\psi^s(x'\phi'+x''\phi'') ,$$
$$2\Psi_{70}=(\psi'x''+\psi''x')\phi'+(\psi'x'-\psi''x'')\phi'' , \tag{C7}$$

where (χ,ϕ) are the notations for spin and isospin functions.[17] Elimination of the spin and isospin structures is straightforward after the operation of (C6) on (C7). The result is a set of three coupled equations in (ψ^s,ψ',ψ'') which are *self-consistent* with the identification

$$\psi'=N'\phi_s,\quad \psi''=N''\phi_s , \tag{C8}$$

ϕ_s being a single S_3-symmetric function associated with the $(\mathbf{70},0^+)$ state. Following the reduction technique of Sec. III, the two coupled equations in (ψ_s,ϕ_s) become

$$(N+3-F_B^{\text{I}})\psi^s+\alpha(N'^2+N''^2)\phi_s=0 ,$$
$$4\alpha\psi^s+(N+3-F_B^{\text{II}})\phi_s=0 ,$$

where

$$\alpha=\frac{3}{4\gamma_B^2}\tfrac{1}{2}\omega_{q\bar{q}}^2\langle -D_s^{-1}\rangle ;$$

see Eq. (3.27).

Using the result of (A23), and writing $F_B=F_B^0+\delta F$, where F_B^0 is the *unmixed* value of F_B for the state concerned, one immediately finds the mixing corrections δF to the respective states from a quadratic equation in x. The results for the states $P_{11}'(1440)$ and $P_{11}''(1710)$ are

$$\delta F(P_{11}')=+0.675,\quad \delta F(P_{11}'')=-0.066 . \tag{C9}$$

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