Gupta-Radford-Repko potential and the properties of the Υ system

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Because of recent improvements in the quality of the P-state splittings of the Υ system, we decided to reexamine some of the basic assumptions underlying an important recent calculation of the properties of the system by Gupta, Radford, and Repko (GRR). In particular, we want to examine the strategy chosen to determine the renormalization scale parameter. It is also of interest to know if one can develop a method for including Lüscher or Lüscher-type corrections to the form of the confinement potential that is consistent with some of the accomplishments of perturbation-theory potentials and the data. To this end, we have developed an abbreviated form of the GRR potential, which gives a fit to the measured energies that is somewhat better than the original GRR potential. By introducing a join radius, where the running coupling constant stops running, we are able to include a Coulomb term in the confinement potential. The amended Lüscher coefficient is determined from a continuity condition at the join radius. Its value is found to be near 0.55, which is consistent with a recent calculation of Olsson and Suchyta.

I. INTRODUCTION

In 1981 Eichten and Feinberg[1] (EF) took an important step forward in formulating the implications of QCD for the form of the spin-dependent parts of the heavy–quark-antiquark ($Q\bar{Q}$) potential; they were able to relate the spin-dependent parts of the potential to correlation functions of the color-electric and -magnetic fields. Assuming that the long-distance behavior of the static potential is uniquely determined by the longitudinal component of the color-electric field, a viewpoint often called electric confinement, they argued that the magnetic correlations were short range. Thus, one might expect the one-gluon-exchange (OGE) contribution to suffice for the spin-spin and tensor portions of the potential. Because of the presence of the classical spin-orbit term and the Thomas precession term, the derivatives of the static potential play a role in the spin-orbit force. However, the sign of the contribution of the long-range potential to the spin-orbit force obtained by EF was in error, as originally pointed out by Buchmüller.[2] Buchmüller's essential physical observation was that in the center-of-mass frame magnetic flux contributions to the confining force will also be important because of the velocities associated with rotations of the quark-antiquark pair. These arguments were given a convincing foundation by Gromes.[3,4]

Moxhay and Rosner[5,6] (MR) adopted much the same viewpoint at EF for their calculations of the energies, leptonic widths, and dipole transition rates of charmonium and the Υ system. They were able to include some important relativistic corrections to the spin-independent potentials and decided to use Richardson's form[7] for the static potential instead of the linear plus Coulomb potential. McClary and Byers[8] (MB) explored more of the consequences of relativistic effects and chose a scalar potential for the confining potential. The sign of the spin-orbit contribution from the scalar potential is the same as that resulting from Buchmüller's rotating flux-tube model and Gromes's analysis.[3,4]

Gupta, Radford, and Repko[9] (GRR) have applied perturbation theory to both the spin-independent and the spin-dependent parts of the $Q\bar{Q}$ potential. They include the effects of OGE as well as all one-loop corrections and thus obtain a potential correct to the second order, which is expected to have increased validity over OGE at small distances. They introduce a renormalization scale parameter whose magnitude is chosen in order to minimize the effects of higher-order renormalization-group improvements of the potential. Their treatment of the confining potential is equivalent to the effects of a scalar exchange. In at least two recent reviews,[10,11] their results for the fine-structure splittings of the Υ system have been cited as being superior in giving the magnitudes of the spin-orbit and tensor matrix elements. Their calculations of the $E1$ transition rates are a problem, however, disagreeing with the charmonium measurements by a factor of 2 or 3 (Refs. 12 and 13).

Recently the experimental results for the photon energies emitted in transitions among the χ_b and χ_b' states[11,14,15] have been improved, and thus the fine-structure splittings are now known to a precision of about 1 MeV (Ref. 16). Further, the character of the observed splittings has changed somewhat, away from a more or less equal-spacing scheme.[17] These new experimental developments immediately raise the question of whether the GRR calculation will be as successful in dealing with the new splittings as in dealing with the older ones. Another exciting question, posed by Pantaleone, Tye, and Ng,[18] is whether the new data will support a nonperturbative contribution to the spin-orbit potential.

The question of the form of the $Q\bar{Q}$ potential at large distances, where confinement effects must come into play, has long been a subject of intense theoretical interest. Most of the recent calculations of heavy-quarkonium

properties have been based on the linear potential. However, recent Monte Carlo lattice gauge (pure gauge)[19] calculations by Otto and Stack have shown that the confining potential should also include a Coulomb term: namely,

$$V_c(r)=Ar-\alpha/r \ , \tag{1}$$

where A is the string tension and $\alpha=0.25\pm0.02$. The parameter α is not the same as the strong-interaction coupling constant α_S.

The flux-tube (string) model affords another opportunity to derive a potential of the form of Eq. (1). Indeed, Lüscher[20] was able to derive a value of $\pi/12=0.26$ for α by considering transverse oscillations of the flux tube.[21,22] In spite of the agreement between the lattice gauge and the string calculations, the magnitude of the coefficient α that should be used in heavy-quarkonium calculations is an open question because one would expect substantial modifications from vacuum-polarization loops. Olsson and Suchyta[23] used the model of Poggio and Schnitzer[24] to show that vacuum polarization may effectively double the coefficient α.

Below we report two calculations of the leptonic widths and energies of the Υ system. The first is an abbreviated GRR calculation where we keep the α_S^2 corrections to the static energy but ignore these corrections for the spin-dependent parts of the potential. Originally, our motivation for this calculation was to warm up for the second calculation. However, as our experience with it grew we realized that there were several important reasons for doing the calculation. Among these were the following.

(1) To have an independent means of checking parts of GRR's calculation. Their results are based upon a variational approach, but ours are based on the numerical solution of the Schrödinger equation.

(2) To compare our results with GRR to see how much additional accuracy results from including the α_S^2 terms.

(3) To examine the strategy used by GRR to determine the renormalization scale parameter.

Our second calculation is undertaken to incorporate some of the latest thinking about the form of the confinement potential into a model that is based on the abbreviated GRR potential. We divide the space between the two quarks into a small-r region where the results of perturbation theory are expected to be valid and a large-r, or asymptotic, region where the $Q\bar{Q}$ potential is assumed to be adequately represented by the confinement potential. These two regions are separated by the join radius R_J. We examine the consequences of choosing successively smaller values for the join radius. With this procedure for setting up the potential it is straightforward to extract values of the asymptotic Coulomb coefficient α near 0.5 or 0.6, close to the value suggested by Olsson and Suchyta. With our procedure it would be impossible to extract a value near the Lüscher value of 0.2 or 0.3 for α without violent disagreement with experiment. In summary, our motivation for doing the second calculation is the following: (1) to see if reasonable values of the join radius lead to reasonable values for the asymptotic Coulomb coefficient α; (2) to see if a more complicated form for the confinement potential opens the prospect of any improvement of GRR-based results with experiment.

II. THE ABBREVIATED GRR POTENTIAL

Our calculations are based on a Hamiltonian that includes a relativistic form for the kinetic energy operator K, a confining potential V_C, a short-range spin-independent potential $V_{\rm SI}$, and a short-range spin-dependent potential $V_{\rm SD}$, that is,

$$H=K+V_C+V_{\rm SI}+V_{\rm SD} \ . \tag{2}$$

Since the Υ system is believed to be primarily a nonrelativistic system, it suffices to retain only the first three terms in the expansion of the kinetic energy operator: namely,

$$K=2m+p^2/m-p^4/4m^3+\cdots \ , \tag{3}$$

where m is the mass of the bottom quark and p is its momentum in the center-of-momentum frame.

We follow the lead of GRR (Refs. 9, 12, and 13) in choosing the form of $V_{\rm SI}$ and $V_{\rm SD}$ but do not include all of their α_S^2 corrections. Our choice for the $V_{\rm SI}$ may be written as a sum

$$V_{\rm SI}=V_{\rm NR}+V'_{\rm SI} \ , \tag{4}$$

where the nonrelativistic potential includes the most important α_S^2 corrections

$$V_{\rm NR}=-\frac{4\alpha_S}{3r}\left[1-\frac{3\alpha_S}{2\pi}+\frac{\alpha_S}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E]\right] . \tag{5}$$

In Eq. (5), the quantity n_f denotes the number of quark degrees of freedom which we take to be 4, $\gamma_E=0.5772$, and μ denotes the renormalization scale parameter. Its relationship to the QCD scale parameter Λ depends upon how one handles certain higher-order vacuum-polarization effects. The usual expression is based on a geometric series of iterated one-loop vacuum-polarization bubbles, i.e.,

$$\Lambda=\mu\exp\left[\frac{-6\pi}{(33-2n_f)\alpha_S}\right] . \tag{6}$$

One means of viewing the complications of the second-order terms in Eq. (5) is to think of them as giving an explicit expression for the running coupling constant $\alpha_S(r)$. At small values of r, its effect is to soften the coupling constant since the expression in large parentheses is negative there.

The term $V'_{\rm SI}$ of Eq. (4) includes some important relativistic corrections to the potential as well as a contact term, that is,

$$V'_{\rm SI}=V_{\rm NR}(p^2/m^2)-\frac{14\alpha_S^2}{9mr^2}+\frac{8\pi\alpha_S}{3m^2}\delta(\mathbf{r}) \ . \tag{7}$$

In the contact term of Eq. (7) we have chosen to keep only the lowest-order term since it is the most important contribution to this rather small term.

For the spin-dependent potential we retain only the leading contributions, that is,

$$V_{\rm SD}=\frac{32\pi\alpha_S}{9m^2}\mathbf{s}_1\cdot\mathbf{s}_2\delta(\mathbf{r})+\frac{2\alpha_S}{m^2}\frac{\mathbf{L}\cdot\mathbf{S}}{r^3}+\frac{4\alpha_S}{m^2}\frac{\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\hat{\mathbf{r}}-\frac{1}{3}\mathbf{s}_1\cdot\mathbf{s}_2}{r^3}\ . \tag{8}$$

Since all of the terms of Eq. (8) are just reductions of OGE terms, the spin-spin and tensor parts of Eq. (8) are the same as their counterparts in EF's works. We refer to the potential of Eq. (5), (7), and (8), as our abbreviated GRR potential.

In this section and the next, our confining potential has the form

$$V_c(r)=Ar+C-\frac{A}{2m^2}\frac{\mathbf{L}\cdot\mathbf{S}}{r}\ , \tag{9}$$

where the sign of the spin-orbit term is taken from the work of Buchmüller.[2] The spin-orbit contribution to Eq. (9) is the same as that resulting from a scalar exchange. Its physical origin may be traced to the Thomas precession term. Below we examine the extent to which the measured P-state splittings support the presence of this term. In Eq. (9) the quantity C denotes an overall constant. Because of the minus sign in front of the spin-orbit term in Eq. (9), one often speaks of "scalar" confinement although there is no fundamental scalar present in QCD.

Our basic approach for obtaining the eigenvalues and eigenfunctions of major energy levels is based upon a numerical solution to the Schrödinger equation and the use of perturbation theory to calculate the spin-dependent splittings and some small spin-independent terms. Thus, the unperturbed eigenvalue equation may be written

$$H_0\,|\,\psi_0(nL)\rangle=\epsilon_0\,|\,\psi_0(nL)\rangle\ , \tag{10}$$

where n is the radial quantum number and L is the orbital angular momentum quantum number and

$$H_0=\frac{p^2}{m}+Ar-\frac{4}{3}\frac{\alpha_S}{r}\left[1-\frac{3\alpha_S}{2\pi}+\frac{\alpha_S}{6\pi}(33-2n_f)\times[\ln(\mu r)+\gamma_E]\right]\ . \tag{11}$$

The relationship between the total energy of the system and the eigenvalue ϵ is

$$E(nLJ)=2m+C+\epsilon(nLJ)\ , \tag{12}$$

where J denotes the total angular momentum. The perturbing potential includes a kinetic energy contribution, the spin-independent terms of Eq. (7), and the entire spin-dependent potential of Eq. (8), that is,

$$H'=K'+V'_{\rm SI}+V_{\rm SD}\ . \tag{13}$$

Thus, the expression for the total energy of the system may be written in the form

$$E(nLJ)=E_0(nL)+\langle JMLSn\,|\,H'\,|\,JMLSn\rangle\ , \tag{14}$$

where the expectation value includes all of the improvements of first-order perturbation theory.

We adopt the following strategy to determine numerical values for our parameters. Since the calculated energies are not overly sensitive to the value of m (provided that it is not allowed to vary too far from about half the Υ mass), we simply use the GRR value of 4.78 GeV. GRR did not treat μ as a free parameter. It was chosen in such a manner that it minimized the effects of higher-order renormalization-group improvements of their perturbative potential. For a preliminary orientation we have adopted their value ($\mu=3.75$ GeV). We determine the string constant A and the strong coupling constant α_S by fitting the differences of the centers of gravity of $1P$ and $2P$ states with the ^3S energy of the lowest state at 9.460 GeV. In summary, our initial parameters for the abbreviated GRR calculation are

$$A=0.188\ \text{GeV}^2,\quad \alpha_S=0.269\ ,\qquad m=4.78\ \text{GeV},\quad \mu=3.75\ \text{GeV}\ . \tag{15}$$

Our values for A and α_S differ by less than 10% from GRR's. Below we shall study the effects of allowing μ to vary.

III. NUMERICAL RESULTS FOR THE ABBREVIATED GRR CALCULATION

After separating the angular part of the wave function of Eq. (10), the radial equation takes the form

$$\frac{d^2u}{dr^2}=\left[mV(r)-m\epsilon+\frac{L(L+1)}{r^2}\right]u\ , \tag{16}$$

where $V=V_{\rm NR}+Ar$, which includes the most important part of the confining potential and the running coupling constant potential of Eq. (5). Equation (16) is the general form

$$\frac{d^2u}{dr^2}=F(r)u(r)\ , \tag{17}$$

often encountered in numerical work in physics.[25,26] Thus, solutions to Eq. (16) may be generated by applying the Fox-Goodwin difference formula,[27,28] that is,

$$u(r+\Delta)\left[1-\frac{\Delta^2}{12}F(r+\Delta)\right]=2\left[1+\tfrac{5}{12}F(r)\right]u(r)-\left[1-\frac{\Delta^2}{12}F(r-\Delta)\right]u(r-\Delta)+\cdots\ . \tag{18}$$

Because of the symmetry of Eq. (18) under the interchange $\Delta \rightarrow -\Delta$ and the fact that the coefficients have been chosen to eliminate fourth-order terms, corrections to Eq. (18) are of order Δ^6. For our numerical work we choose $\Delta = 0.01-0.02$ GeV^{-1} and thus errors resulting from the use of Eq. (18) should be completely negligible, provided that we stay away from the singular points of Eq. (16) where numerical instabilities associated with admixtures of irregular solutions are a problem. Equation (18) may be used either for forward integration or for backward integration. The interval for the numerical work was taken to be [0, 20.0 GeV^{-1}].

Equation (16) has singular points at $r=0$ and at $r=\infty$, and it is important to respect the behavior of the solutions near these points in devising a strategy to solve for the eigenfunctions and eigenvalues. In order to minimize the problems associated with the appearance of irregular solutions, we used a method that involves both integrating in and integrating out. Equation (18) is used as a forward difference formula to integrate from the origin to an intermediate value of r, which we call the integration continuity radius (ICR). Then Eq. (18) is used as a backward difference formula to integrate from large r down to ICR. The requirement of a continuous logarithmic derivative at ICR, namely,

$$\left.\frac{du_1(r)}{dr}\Big/u_1(r)\right|_{r=R_{\rm ICR}} = \left.\frac{du_2(r)}{dr}\Big/u_2(r)\right|_{r=R_{\rm ICR}}, \tag{19}$$

may be used to determine the eigenvalue. To implement the search for the eigenvalues, it is helpful to think of the mismatch of the two logarithmic derivatives as a function of the trial energy ϵ. Then one can view the search for eigenvalues as the problem of finding the zeros of the mismatch function. Since the denominators of Eq. (19) have zeros at the nodes, it is important to keep the value of $R_{\rm ICR}$ away from the nodes. The effect of choosing $R_{\rm ICR}$ near one of the nodes is to make the range of initial values of ϵ that converge to the eigenvalue small. In practice, we found that choosing $R_{\rm ICR}=2.0$ GeV^{-1} or 3.0 GeV^{-1} would always suffice to find the eigenvalue.

When the potential $V_{\rm NR}$ is set equal to zero, the eigenvalue problem of Eq. (16) for S states reduces to the well-known linear potential problem. Its solution may be expressed in terms of Airy functions,[29–31] that is,

$$u(r)=N_1 \mathrm{Ai}[(mA)^{1/3}(r-\epsilon/A)], \tag{20}$$

where N_1 is a normalization constant. In this case, the eigenvalues are related to their zeros of the Airy function by the expression

$$\epsilon_n = x_n (A^2/m)^{1/3}, \tag{21}$$

where $x_n = 2.34$, 4.09, etc. The analytical expressions of Eqs. (20) and (21) provide an opportunity to check over eigenfunctions and eigenvalues for S states. We found that our results for these eigenvalues were accurate to four or five decimal places and thus our eigenvalues for S states should be accurate to better than 0.1 MeV, which is more than adequate for comparing theory and experiment. When $L \neq 0$, our prototype problem does not have an analytic solution. In this case we compared with the results for the lower P, D, and F states given in Table I of Gunion and Willey[32] and obtained satisfactory agreement. Of course, we also carried out some of the standard numerical checks by changing the step width and by choosing different values for the integration continuity radius. All of these checks gave a consistent picture of an accuracy of about 0.2 MeV for our numerical results.

Because of the δ function in Eq. (8), the spin-spin interaction contributes only to the energy shifts of the S states. The relevant matrix elements are thus related to the values of the wave function at the origin and are given by

$$\langle nS \mid V_{\rm SD} \mid nS \rangle = \frac{8\alpha_S}{9m^2} \mid R(0) \mid^2 \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}, \tag{22}$$

TABLE I. Energies of the Υ system for different renormalization scale parameters (MeV).

State	$\mu=3.75$ $A=0.188$ $\alpha_S=0.269$	$\mu=3.00$ $A=0.181$ $\alpha_S=0.292$	$\mu=2.50$ $A=0.180$ $\alpha_S=0.312$	$\mu=2.00$ $A=0.170$ $\alpha_S=0.350$	Expt.
$1\,^3S_1(\Upsilon)$	9460	9460	9460	9460	9460[a]
$2\,^3S_1(\Upsilon)$	10010	10011	10015	10018	10023
$3\,^3S_1(\Upsilon)$	10357	10355	10361	10360	10356
$1\,^3P_2(\chi)$	9907	9909	9910	9911	9913
$1\,^3P_1$	9895	9895	9894	9893	9892
$1\,^3P_0$	9875	9873	9870	9866	9860
$1\,^1P_1$	9900	9900	9900	9900	
$2\,^3P_2(\chi')$	10266	10266	10270	10269	10269[b]
$2\,^3P_1$	10255	10254	10256	10254	10256
$2\,^3P_0$	10239	10236	10236	10232	10231
$2\,^1P_1$	10260	10259	10261	10260	
$\sum \delta^2$ (MeV^2)	514	377	228	87	

[a]PDB (Ref. 16).
[b]CUSB Collaboration (Ref. 14).

TABLE II. Energies of the Υ system (MeV).

State	MR(83)	GRR(86)	GRR(82)	FUL(87)	Expt.
$1\,^3S_1(\Upsilon)$	9460	9460	9462	9460	9460[a]
$1\,^1S_0(\eta_b)$	9403	9416	9427	9417	
$2\,^3S_1$	10020	10013	10013	10018	10023
$2\,^1S_0$		9987	9994	9992	
$3\,^3S_1$	10350	10356	10355	10360	10356
$3\,^1S_0$		10336	10339	10339	
$1\,^3P_2(\chi)$	9914	9910	9910	9911	9913
$1\,^3P_1$	9903	9894	9893	9893	9892
$1\,^3P_0$	9877	9869	9868	9866	9860
$1\,^1P_1$	9906	9901	9900	9900	
$2\,^3P_2(\chi')$	10264	10268	10266	10269	10269[b]
$2\,^3P_1$	10256	10254	10252	10254	10256
$2\,^3P_0$	10237	10233	10232	10232	10231
$2\,^1P_1$	10257	10260	10258	10260	
$\sum\delta^2$ (MeV^2)	517	203	205	87	

[a]PDB (Ref. 16).
[b]CUSB Collaboration (Ref. 14).

where $R(r)=u(r)/r$ and the column vector notation is used to combine the shifts for the triplet (top) and singlet (bottom) states. Hence, in our treatment the triplet-singlet splitting is directly proportional to the leptonic width[33]

$$\Gamma_{ee}=\frac{4\alpha^2 e_Q^2}{M^2(Q\bar{Q})}\,|R(0)|^2\,, \tag{23}$$

where α is the fine-structure constant, e_Q denotes the charge of the quark, and M is the mass of the Υ system.

The results of allowing μ to vary through a succession of smaller values are shown in Table I, where the effects of scalar confinement are also included. In each case the values of A and α_S are obtained by fitting the $1P$-$1S$ and $2P$-$1S$ splittings. As a measure of the quality of the overall fit to the experimental data we compute the sum of the squares of the differences of the theoretical and experimental results. These are listed in the last row of Table I. From the values of $\sum\delta^2$ there, it is apparent that decreasing μ from the value used by GRR leads to a uniform improvement in agreement between theory and experiment. The best fit to the data occurs when $\mu=2.0$ GeV. Using this value and $\alpha_S=0.350$ we find $\Lambda=232$ MeV for the QCD scale parameter of Eq. (6). It is interesting to note that our value for Λ differs only slightly from the value of 227 MeV obtained with the parameter set of Eq. (15). Similar calculations with the other values of μ and α_S in Table I support a value of Λ near 230 MeV. Thus, the effect of allowing μ to vary is compensated for by a variation of α_S which keeps the value of the QCD scale parameter Λ constant, and the results of Table I support the existence of a single QCD scale parameter.

It is conceivable that the fit to the energies may become even better if μ is allowed to become smaller. However, we did not pursue this calculation further because the larger values of α_S associated with smaller values of μ give rise to repulsive effects of the potential of Eq. (5) near the origin. These are already beginning to appear in the wave function near the origin when $\mu=2.00$ GeV. Such effects probably arise because the form of the term in the large parentheses of Eq. (5) overcompensates for the effects of asymptotic freedom when α_S is large.

Our results for the Υ energies when $\mu=2.00$ GeV are compared with the calculated results of GRR and MR in

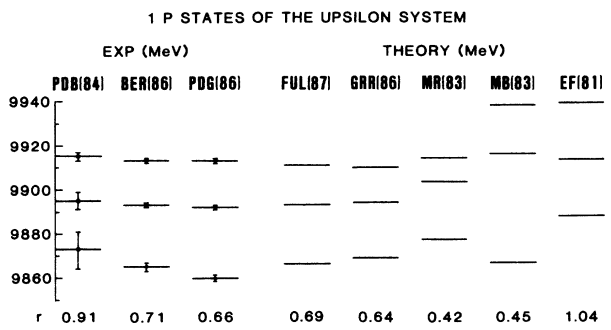

FIG. 1. Recent history of experiments and calculations for the $1P$ states of the Υ system.

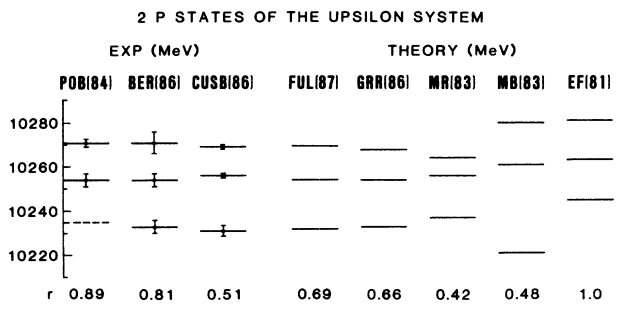

FIG. 2. Recent history of experiments and calculations for the $2P$ states of the Υ system.

TABLE III. Leptonic widths of low-lying S states of the Υ system (keV).

State	MR(83)	GRR(86)	GRR(82)	FUL(87)	Expt.
$\Upsilon(1S)$		1.64	1.29	0.85	1.22±0.05
		1.11[a]			
$\Upsilon(2S)$	$(2S/1S)=0.41$	0.84	0.62	0.44	0.54±0.03
		0.57[a]			
$\Upsilon(3S)$	$(3S/1S)=0.29$	0.61	0.46	0.33	0.40±0.03
		0.41[a]			

[a]Includes radiative correction.

Table II and Figs. 1 and 2. Roughly speaking, from the viewpoint of $\sum \delta^2$, the GRR results are about a factor of 2 better than the MR results and our results are about a factor of 2 better than the GRR results. Results from the calculations of MB and EF are also included in the figures. Our agreement with all the S- and P-state energies below the threshold for $B\bar{B}$ production is very good.

We are not as successful in the calculation of the leptonic widths, which are displayed in Table III. The cause of the discrepancy can be traced to the repulsive effects of the potential of Eq. (5). These effects tend to suppress the value of the wave function at the origin and thus lead to a reduced value for the leptonic width. Using the parameter set of Eq. (15) we were able to reproduce the GRR (82) values of the leptonic width to within 5–7%, an important confirmation of the technical correctness of their calculation and ours. In order to eliminate this artificial behavior near the origin, we would require an improved parametrization of the potential.

Since our spin-spin term contains a δ function, it does not contribute to nonzero angular momentum states. Thus the perturbation theory splittings of the P and D states may be computed with the expression[34–36]

$$\langle nL \mid V'_{\rm SD} \mid nL \rangle \equiv \begin{bmatrix} \Delta E(\text{trip};L+1) \\ \Delta E(\text{trip};L) \\ \Delta E(\text{trip};L-1) \\ \Delta E(\text{sing};L) \end{bmatrix}$$

$$= \left[\frac{2\alpha_S}{m^2} \langle nL \mid r^{-3} \mid nL \rangle - \frac{A}{2m^2} \langle nL \mid r^{-1} \mid nL \rangle \right] \begin{bmatrix} L \\ -1 \\ -L-1 \\ 0 \end{bmatrix}$$

$$+ \frac{4\alpha_S}{3m^2} \langle nL \mid r^{-3} \mid nL \rangle \begin{bmatrix} \dfrac{-L}{2(2L+3)} \\ \frac{1}{2} \\ \dfrac{-(L+1)}{2(2L-1)} \\ 0 \end{bmatrix}, \tag{24}$$

where the prime in the original matrix element denotes that we have included the spin-orbit part of the confining potential. In analyzing the spacings of the splittings of Eq. (24) it is often convenient to use the value of the ratio r, which is defined by

$$r = \frac{M(\text{trip};L+1) - M(\text{trip};L)}{M(\text{trip};L) - M(\text{trip};L-1)} . \tag{25}$$

If there is no tensor force then $r=2$ for all P states. Also, if there is no confinement contribution to Eq. (24), then $r=0.80$, another universal value for all P states. As a general rule, the "scalar" confining potential will make r smaller than 0.80 because of the negative sign in Eq. (24).

In their important work, Pantaleone, Tye, and Ng[18] have adopted the viewpoint that no nonperturbative contribution to the spin-dependent potential is necessary. In the context of our calculation, this means that the A-dependent term in Eq. (24) is not present, and thus the P-state splittings are given by OGE alone. We pursue this question in Table IV where the results of omitting the A-dependent term are compared with those obtained by including it. In each case, the $\sum \delta^2$ computed with OGE is smaller than when the "scalar" confinement contribution is included in Eq. (24). However, it would be premature to conclude that the data clearly supports OGE over OGE plus "scalar" confinement because the r values ob-

tained with "scalar" confinement (0.7) are closer to experiment than the values (0.8) obtained with OGE alone. In order to obtain the answer to this exciting question it will probably be important to improve experimental precision and to sort out the contributions of higher-order (α_S^2) terms to the spin-dependent potential.

Another interesting question about the P states that may soon be subject to experimental scrutiny is the location of the singlet P states. Because of our assumption that the spin-spin part of Eq. (8) contains a δ function the location of all singlet P states in the tables is the center of gravity of the triplet P states. However, if the spin-spin potential has a finite range, then the singlet P would be shifted away from the center of gravity, as emphasized in the recent study of Gupta and Ram.[37]

IV. CONFINING POTENTIAL WITH ATTRACTIVE COULOMB TERM

Using the model of Poggio and Schnitzer,[24] Olsson and Suchyta[23] were able to investigate the effects of vacuum-polarization loops on the quenched interquark confining potential. After carefully defining the confining potential in the purely gluonic sector, where no fermion loops are allowed, they were able to sum the effects of the finite chain of vacuum-polarization bubbles and obtain the expression

$$V_c(q)=V_G(q)[1-q^4\Pi_1(q)V_G(q)]^{-1} \tag{26}$$

for the renormalized confining potential V_c in momentum space. In Eq. (26) the quantity V_G denotes the interquark potential that includes (only) gluonic interactions (and self-interactions) to all orders. The quantity $q^4\Pi_1$ is the gluonic self-energy that results from inserting the one-fermion vacuum-polarization bubble. This form for the self-energy is obtained after the renormalization constants are determined to vanish. Because Π_1 satisfies a dispersion relation one can obtain the leading coefficients in the small-q expansion,

$$\Pi_1(q)\rightarrow D_0-D_1q^2 , \tag{27}$$

from the imaginary part of the self-energy correction. Using the Poggio-Schnitzer model, one may determine these coefficients in terms of the light-quark mass m_q, that is,

$$D_0=\frac{9}{80\pi^2m_q^2},\quad D_1=\frac{D_0}{14m_q^2} . \tag{28}$$

The result of transforming all equations to configuration space is to give the following form for the confining potential in the large-r region:

$$V_C(r)=Ar-(\alpha+\kappa)/r+\cdots , \tag{29}$$

where A is related to the gluonic sector string constant a by

$$A=\frac{a}{1+8\pi aD_0} \tag{30}$$

and

$$\kappa=16\pi D_1A^2=\frac{9A^2}{70\pi m_q^4}=0.20 , \tag{31}$$

if m_q is chosen to be 0.300 GeV (Refs. 34 and 38). Thus, the result of Olsson and Suchyta for the effects of vacuum polarization is almost to double the coefficient of the Coulomb term.

Because the light-quark mass that appears in Eq. (31) is not very well determined from experiments, it is highly desirable to have a means of determining the coefficient of the Coulomb term of Eq. (29) that is more closely connected with experiment. Now we develop a means of determining this coefficient that is based on a continuity argument.

One expects the perturbation approach of GRR to be valid at small values of r. Among the reasons for this is that the running coupling constant in the large parentheses of Eq. (5) serves to soften the effects of the OGE potential in this region which is consistent with the requirement of asymptotic freedom. However, at larger values of r the effect of the running coupling constant is to enhance OGE. On the other hand, the coefficients in Eq. (29) are supposed to be constants. Thus, the running coupling constant should stop running at some distance, which we call the join radius R_J. Equating the confining potential of Eq. (29) to the interior potential [from Eq. (5) and the linear part of Eq. (9)] leads to an expression for the coefficients of the Coulomb term in the asymptotic realm, that is,

TABLE IV. P-state splittings of the Υ system for different renormalization scale parameters.

State	$\mu=3.75$	$\mu=3.00$	$\mu=2.50$	$\mu=2.00$	Expt.
$1\,^3P_2(\chi)$	9910	9911	9912	9914	9913[a]
$1\,^3P_1$	9892	9892	9891	9891	9892
$1\,^3P_0$	9870	9868	9866	9862	9860
$2\,^3P_2(\chi')$	10269	10268	10271	10271	10269[b]
$2\,^3P_1$	10254	10252	10254	10252	10256
$2\,^3P_0$	10235	10232	10233	10229	10231
$\sum\delta^2$ (MeV^2)	129	86	50	30	
$\sum\delta^2$ (MeV^2)[c]	344	232	139	46	

[a]PDB (Ref. 16).
[b]CUSB Collaboration (Ref. 14).
[c]From Table I.

$$\alpha_{AL}=\frac{4}{3}\alpha_S\left[1-\frac{3\alpha_S}{2\pi}+\frac{\alpha_S}{6\pi}(33-2n_f)[\ln(\mu R_J)+\gamma_E]\right]. \tag{32}$$

We refer to this coefficient as the amended Lüscher coefficient. Our procedure for determining the amended Lüscher coefficient is depicted in Fig. 3 where the join radius is 2.0 GeV^{-1}. From Fig. 3, it is apparent that extrapolating the GRR potential into the large-r region yields a potential that differs somewhat from the purely linear form. Adding the amended Lüscher term has the effect of allowing us to introduce a potential that joins to V_{GRR} at R_J and that approaches the purely linear form at large r.

It is interesting to note that the potential of Eq. (29) has the same form as the EF potential[1] and thus the amended Lüscher coefficient should be compared with the EF coefficient K. Of course, the difference is that the EF potential is assumed to be valid for all values of r and hence does not behave as one would expect from asymptotic freedom. Thus, in some sense, our introduction of the join radius allows a synthesis of the GRR and EF potentials, in a manner that incorporates the strengths of both.

Our results for the Υ energies as the join radius is decreased from 2.0 GeV^{-1} to 0.75 GeV^{-1} are listed in Table V. In each case the parameters A and α_S, which are listed in Table VI, are determined from the energy differences of the centers of gravity of the $1\,^3P$ and $2\,^3P$ states from the $1\,^3S$ state. Since we are primarily interested in exploring the qualitative features associated with the introduction of the join radius, we use the GRR values of m and μ. As R_J varies from 2.0 to 0.75 GeV^{-1}, the parameters A and α_S follow approximate linear trends. The uniform decrease in the value of α_S is of interest because our values of α_S necessary to fit the data in Tables I and II are substantially higher than those derived from the two-gluon decay rate of triplet[10,39] P states (0.17) or the three-gluon decay rate of triplet[11,39] S states (0.16). Of course, some of this difference can be accounted for by recalling that the renormalization method of GRR is different from the modified minimal subtraction scheme underlying the decay rate calculation. The connection between the two coupling constants is given by

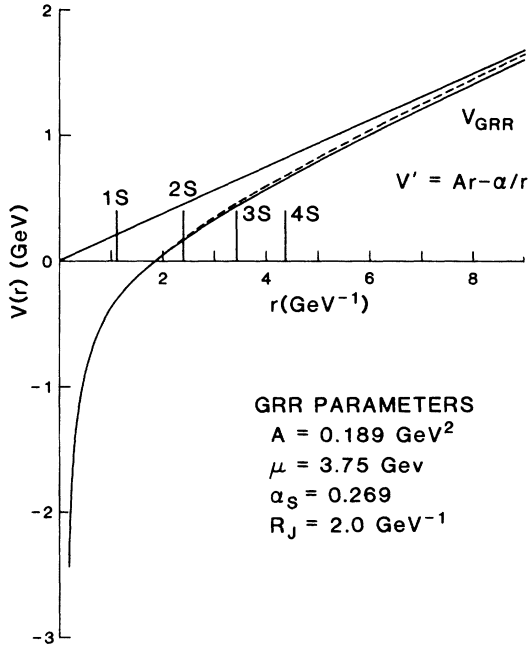

FIG. 3. The GRR potential and the confining potential with the amended Lüscher term (dashed line).

$$(\alpha_S)_{\text{GRR}}=\bar{\alpha}_S\left[1+\frac{\bar{\alpha}_S}{4\pi}(6+\tfrac{31}{3}-\tfrac{10}{9}n_f)\right]. \tag{33}$$

Using Eq. (33) to solve for $\bar{\alpha}_S$ yields $\bar{\alpha}_S=0.28$ if 0.35 is used as input for $(\alpha_S)_{\text{GRR}}$. Values for the amended Lüscher coefficients are calculated [Eq. (32)] once α_S is determined.

The average values and the squares of the wave functions listed in Table VI are all very close, with typical variations on the order of one or two precent. This lack of variation is perhaps surprising. It does suggest that the leptonic widths are not very sensitive to the value chosen for R_J.

The results of Tables V and VI suggest two criteria for determining the join radius R_J. The sum of the squares of the differences between calculated and experimental energies, that is $\sum\delta^2$, starts to increase strongly as R_J decreases below 1.50 GeV^{-1}. Because the difference between $\sum\delta^2$ at $R_J=2.0$ GeV^{-1} and $R_J=1.50$ GeV^{-1} is

TABLE V. Energies of the Υ system for different potential join radii (MeV).

State	$R_J=2.0$ GeV^{-1}	$R_J=1.5$ GeV^{-1}	$R_J=1.0$ GeV^{-1}	$R_J=0.75$ GeV^{-1}	Expt.
$1\,^3S_1$	9460	9460	9460	9460	9460[a]
$2\,^3S_1$	10012	10017	10011	10004	10023
$3\,^3S_1$	10353	10364	10361	10354	10356
$4\,^3S_1$	10625	10641	10645	10639	
$1\,^3D_{\text{cog}}$	10168	10172	10165	10156	
$\sum\delta^2$ (MeV^2)	157	100	169	365	

[a]PDB (Ref. 16).

TABLE VI. Selected Υ system properties/parameters for different potential join radii.

Property /parameter	$R_J=2.0$ GeV^{-1}	$R_J=1.5$ GeV^{-1}	$R_J=1.0$ GeV^{-1}	$R_J=0.75$ GeV^{-1}
A (GeV^2)	0.172	0.184	0.196	0.201
α_S	0.273	0.259	0.244	0.240
α_{AL}	0.658	0.576	0.487	0.448
$\langle r\rangle_{1S}$ (GeV^{-1})	1.11	1.11	1.11	1.11
$\langle r\rangle_{3S}$ (GeV^{-1})	3.51	3.47	3.45	3.44
$\lvert R_{1S}(0)\rvert^2$ (GeV^3)	4.36	4.28	4.26	4.34
C (MeV)	110	44	−35	−75

rather small, it is not clear if there is a local minimum near 1.5 GeV^{-1} or if the data are insensitive to R_J as long as $R_J \geq 1.5$ GeV^{-1}. Another criterion that one might use is to require the value of the overall constant C to vanish, thus eliminating one of the parameters. This occurs near $R_J=1.25$ GeV^{-1}, where $C=7.0$ MeV. Both of these results are consistent with a value of R_J between 1.50 and 1.25 GeV^{-1}.

The results listed in Table V show that the triplet D-state energies vary by 10–15 MeV as R_J varies. These differences are large enough to be observable and would furnish a good means of examining the properties of the confining potential. Thus the experimental determination of the D states should be given a high priority. One would expect the D states to be a better means for studying the confining potential than the lower angular momentum states because the larger angular barrier is more effective at preventing the wave function from penetrating the region where short-range effects are important.

V. CONCLUSIONS

Our abbreviated GRR calculation described in Secs. II and III generally supports the values that GRR obtained in their variational calculations. However, we found that their procedure for determining the renormalization scale parameter μ did not seem to be the optimum one. Introducing a little more freedom into their model by allowing μ to vary allowed us to obtain substantially better fits in the context of our abbreviated GRR model. It is especially noteworthy that all of our different values for the renormalization scale parameter μ lead to a value of about 230 MeV for Λ, the QCD scale parameter.

By introducing a potential join radius we were able to incorporate some of the modern thinking about the role of quantum fluctuations of the flux tube into the functional form for our confining potential. The join radius is defined as the distance at which the running coupling constant stops running. Thus, it separates the region where one might expect perturbation theory to have some validity from the region where one needs a phenomonological potential, such as the EF potential. The coefficient of the EF term in the confining potential, which we call the amended Lüscher coefficient, is determined from a continuity condition at the join radius. From the viewpoint of agreement of calculated energies and experimental energies values near $R_J=1.50$ GeV^{-1} are preferred. The trends of some of the parameters suggest that a join radius in the interval 1.25 to 1.50 GeV^{-1} would be reasonable. Such a value for the join radius leads to an amended Lüscher parameter $\alpha_{AL}\approx 0.55$. This value agrees with the recent result of Olsson and Suchyta,[23] who considered vacuum-polarization corrections to Lüscher's calculation. Our procedure for determining α_{AL} would not allow values as small as those originally determined by Lüscher.

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