Mixing of the photon with low-mass particles

Georg Raffelt
Astronomy Department, University of California, Berkeley, California 94720
and Institute for Geophysics and Planetary Physics, Lawrence Livermore National Laboratory, Livermore, California 94550

Leo Stodolsky
Max-Planck-Institut für Physik und Astrophysik, Postfach 401212, 8000 München 40, Federal Republic of Germany
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Photons can mix with low-mass bosons in the presence of external electromagnetic fields if these particles—not necessarily of spin 1—couple by a two-photon vertex. Important examples are the hypothetical axion (spin 0) and graviton (spin 2). We develop a formalism which is adapted to study the evolution of a photon (axion, graviton) beam in the presence of external fields. We apply our results to discuss the possibility of detecting axions by a measurement of the magnetically induced birefringence of the vacuum. We also discuss photon-axion (graviton) transitions in pulsar magnetic fields. The QED-induced nonlinearity of Maxwell's equations causes magnetic birefringence effects which are much stronger than the axion-induced effects in the range of axion parameters allowed by astrophysical constraints. Also, this QED effect induces an index of refraction for photons in vacuum which is so large near pulsars that photon-axion (graviton) transitions are strongly suppressed. However, this QED effect can be canceled by plasma refractive effects, leading to degeneracy between photons and axions so that resonant transitions can occur in analogy with the Mikheyev-Smirnov-Wolfenstein effect. The adiabatic condition can be met only in spatially extended systems, possibly in the magnetosphere of magnetic white dwarfs. Our conclusions differ substantially from several recent discussions of various aspects of these mixing phenomena.

I. INTRODUCTION

A particle, if it has a two-photon vertex, may be created by a photon entering an external electromagnetic field. Furthermore it may be that this particle is very light or has zero mass, leading to a near-degeneracy with the photon. In this case we expect a "mixing" phenomenon between the photon and the particle, where a coherent superposition of the two arises, as is familiar from the famous K^0-meson system.[1] Since the external field is present, angular momentum for the beam need not be conserved and the superposition can contain components with various spins and polarizations. This is in sharp contrast to the K^0 system or with neutrino-flavor-mixing effects[2] where only states of equal spin and polarization mix. In particular, photon mixing with spin-0 or spin-2 particles is possible. In the first case there is the much discussed axion[3,4] which is supposed to couple to two photons in analogy with the neutral pion. Although many astrophysical[5] and cosmological[6] arguments have severely constrained the interaction parameters and mass of the axion and similar hypothetical particles, their possible existence or nonexistence remains an open question with far-reaching consequences for particle physics, astrophysics, and cosmology. For spin 2 there is the graviton, which also—as is shown "experimentally" by the bending of light by the sun—must have a two-photon vertex. The spin 1 case is more subtle since the Landau-Yang theorem[7] prohibits the coupling of two *real* photons to a spin-1 state. What remains unobtainable, of course, is mixing with nonintegral spin particles. Strong, large-scale magnetic fields exist in the laboratory as well as in the astrophysical context, so we will concentrate our attention on external *magnetic* fields.

Since we are dealing with cases of near degeneracy, one has to consider otherwise small effects which can be important in lifting the degeneracy and so cause a severe alteration of the problem. Even in vacuum, there will be an effective index of refraction for the photon due to QED effects[8–10] which will play an important role, particularly in the strong magnetic fields near pulsars. The neglect of this point, we believe, has lead to errors in a number of papers.[11,12] We shall take account of this point by using the Euler-Heisenberg effective Lagrangian,[13] which is the lowest-order expression of the nonlinearity of Maxwell's equations in vacuum. Its effects are characterized by the parameter

$$\xi=(\alpha/45\pi)(B_e/B_{\rm crit})^2 \ ,$$

where[14]

$$B_{\rm crit}=m_e^2/e\approx 4.41\times 10^{13}\ {\rm G}$$

is the critical field strength. This parameter ξ will always be small, although not necessarily small compared to our other small parameters, the axion or graviton couplings to the photon.

We shall discuss three types of problems, principally: production of axions and gravitons by photons (or vice versa) in strong fields[11,12,15,16] and indirect effects as in sensitive laboratory tests[17,18] of photon propagation in strong magnetic fields. In addition there is the conversion of axions into electromagnetic power in a resonant cavity, as suggested by Sikivie[19] in his original paper. He suggested that this method can be used to detect the hypothetical galactic axion flux that would exist if axions were the dark matter of the Universe. This method has been further discussed[20] and a detector of this sort has produced first negative results,[21] at a level, however, not yet significant compared to the theoretical prediction. Our approach is not well suited to treat this case and we have nothing to add to this discussion.

Our method is very well adapted, however, to cope with the conversion of beams. An example is the possibility of detecting the solar axion flux.[19,22] Also, axions might be detectable by "shining light through walls," i.e., by propagating a laser beam through a magnetic field, blocking it halfway by a "wall" and measuring the reemerging photons due to the axion component which penetrates the obstacle.[15] Since the quoted discussions are adequate treatments of these questions, we will apply our mixing formalism in some detail only to those cases where the existing discussions, we believe, contain some errors: The problem of axion-induced birefringence of the vacuum,[17] and axion-photon transitions in pulsar magnetic fields.[11,12] Other points we shall touch on are the use of periodic external fields to enhance the birefringence effect and the influence of matter on these effects, including the possibility of resonant oscillation effects due to cancellations between the plasma and vacuum refractive effects.

We begin our discussion in Sec. II with the derivation of a suitable wave equation to describe the axion-(graviton-)photon system. The main ingredient will be a matrix of the refractive index which, in the presence of an external magnetic field, is nondiagonal in the particle states. In Sec. III we discuss the evolution of a photon-axion beam in the presence of various external field configurations. In Sec. IV we apply these results to a discussion of experimental possibilities to detect axionlike particles by measurements of magnetically induced birefringence. Section V is devoted to axion-photon and photon-graviton transitions in stellar magnetic fields. In Sec. VI we summarize our conclusions.

II. EQUATIONS OF MOTION IN THE PRESENCE OF EXTERNAL FIELDS

A. The axion-photon system

We begin our discussion with a derivation of the equations of motion for the axion-photon system where the term "axion" stands generically for any light *pseudoscalar* particle. We shall see later that a very similar equation is found for the graviton-photon system so that the following discussion is generic to the whole class of problems that we wish to address. A suitable Lagrangian density is given by

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(\partial_\mu a\,\partial^\mu a - m_a^2 a^2) + \frac{1}{4M}F_{\mu\nu}\tilde{F}^{\mu\nu}a + \frac{\alpha^2}{90 m_e^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \tfrac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right] , \quad (1)$$

where a is the axion field, m_a its mass, $F_{\mu\nu}$ the electromagnetic field tensor, and $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ its dual. The third term describes the *CP*-conserving interaction between the pseudoscalar and the electromagnetic field where the energy scale M is a phenomenological parameter to characterize the interaction strength. The most recent discussion of the evolution of red giants in connection with observational data on the number of "clump" giants in open clusters gives the rather firm bound[23] $M > 10^{10}$ GeV, independently of specific model assumptions concerning the axion or any similar particle. We anticipate that in the photon-graviton system M will be replaced by essentially the Planck mass $M_{\rm Pl}$. The last term in Eq. (1) is the Euler-Heisenberg effective Lagrangian[13] arising from the vacuum polarizability [Fig. 1(a)]. It describes photon-photon interactions in the limit where the photon frequencies are small in comparison with the electron mass m_e and all field strengths are weak in comparison with the critical field strengths.

We stress that Eq. (1) is written in terms of natural, rationalized electromagnetic units[24] (natural Lorentz-Heaviside units) where $\hbar = c = 1$ and the fine-structure constant is given as $\alpha = e^2/4\pi \approx \frac{1}{137}$. These units are commonly employed in field theory and have been used

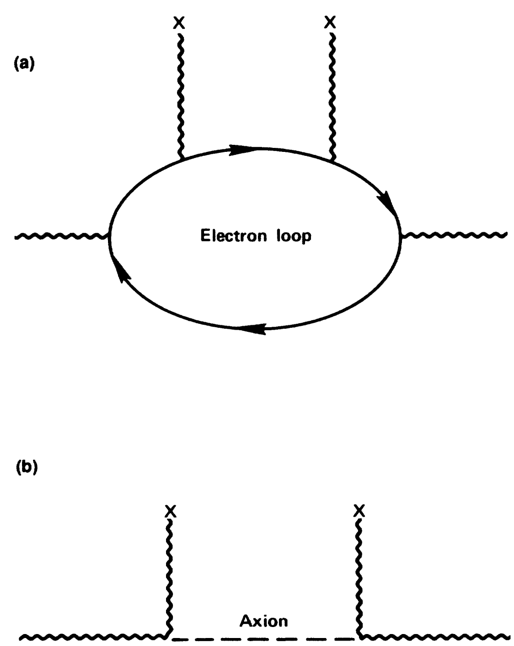

FIG. 1. (a) Feynman diagram for the refractive index of a photon propagating in an external magnetic field. The sources for this field are denoted by crosses (×). (b) Feynman diagram as in (a) for the axion contribution. Both contributions lead to magnetically induced birefringence effects of the vacuum. In (b) the axion can be replaced by a graviton or any other particle with a two-photon vertex.

in the axion literature. In the literature concerning the Euler-Heisenberg Lagrangian and its applications,[9,10,13,25] unrationalized (Gaussian) units have been used where $\alpha = e^2 \approx \frac{1}{137}$. Therefore care must be taken when comparing our results with those in this literature. In Ref. 17, both systems of units have been used simultaneously, leading to numerical errors when results written in the two different systems are compared.

Since it is central to the following, we begin with a discussion of the role of the external magnetic field $\mathbf{B}_e$ in giving a refractive index to the photon. This problem has been studied by a number of authors,[9,10] we follow the results given by Adler.[10] It is found that the two states of linear polarization parallel ($\|$) and perpendicular ($\perp$) to the external field are the eigenstates of propagation and have the following (different) refractive indices:

$$n_\perp = 1 + \tfrac{4}{2}\xi \sin^2\Theta, \quad n_\| = 1 + \tfrac{7}{2}\xi \sin^2\Theta \ , \tag{2}$$

where

$$\xi = (\alpha/45\pi)(B_e/B_{\rm crit})^2 \tag{3}$$

and the critical field strength is[14] $B_{\rm crit} = m_e^2/e$. In contrast to Adler,[10] we follow the usual convention and take the photon polarization vectors to represent the direction of the *electric* field of a plane wave. The angle Θ is the angle between the external field direction and the photon momentum, $\cos\Theta = \hat{\mathbf{B}}_e \cdot \hat{\mathbf{k}}$. Furthermore, although the $\|$ and $\perp$ states correspond roughly to the usual linear polarization states in vacuum, the E and B fields are not necessarily transverse to the direction of propagation, e.g., in general $\nabla\cdot\mathbf{E} \neq 0$. This nontransversality is proportional to ξ and we shall neglect it in our mixing couplings since it always appears multiplying another small factor.

In order to have mixing we need a transverse external field component for the following reason. The conversion from a free photon to a spin-0 axion or a spin-2 graviton involves a change in the azimuthal (J_z) quantum number of angular momentum. The photon has $J_z = \pm 1$, and the axion or graviton have $J_z = 0$ or $J_z = \pm 2$, respectively. A longitudinal field, i.e., one which gives the problem an azimuthal symmetry, however, cannot induce a change in J_z and so will give us no transitions.

In principle, the mixing problem is multichannel, involving the two polarization states of the photon with the axion or with the two states of the graviton, yielding a 3 or 4 channel problem. It may be seen, however, that *CP* simplifies the problem, at least in the absence of matter, so that only one of the photon polarization states couples to one of the other states, giving in effect a two-channel mixing problem. The argument is as follows. Let the parity operation be defined as a reflection $x_\perp \to -x_\perp$ in the plane containing the external magnetic field $\mathbf{B}_e$ and the beam direction $\mathbf{k}$. This operation induces $E_\perp(x_\perp, x_\|) \to -E_\perp(-x_\perp, x_\|)$ and $E_\|(x_\perp, x_\|) \to E_\|(-x_\perp, x_\|)$, where the component $\perp$ of a vector is the component perpendicular to the plane of reflection while the $\|$ component stands for the two components in this plane. The magnetic fields transform as $B_\perp(x_\perp, x_\|) \to B_\perp(-x_\perp, x_\|)$ and $B_\|(x_\perp, x_\|) \to -B_\|(-x_\perp, x_\|)$ and $a(x_\perp, x_\|) \to -a(-x_\perp, x_\|)$ for the pseudoscalar axion field. In particular, the external field $\mathbf{B}_e$ will reverse sign. The interactions under consideration are *P* and *C* invariant, so we apply the *C* operation to reverse $\mathbf{B}_e$ again. If $\mathbf{B}_e$ is sufficiently slowly varying with $x_\perp$ the Lagrangian Eq. (1) is invariant under *P* and *CP*, and since for these conditions $\mathbf{B}_e$ is invariant under *CP*, our wave field solutions in the presence of $\mathbf{B}_e$ can be classified into eigenstates of this *CP* operation. The plane-wave photon states $\perp$ and $\|$ are even and odd, respectively (this remains true when the small longitudinal components are taken into account[10]), while the plane-wave axion states are odd. Thus only the $\|$ photon state mixes with the axion.

We then find the following stationary wave equation for particles propagating along the z axis:

$$\left[\omega^2 + \partial_z^2 + \begin{bmatrix} Q_\perp & 0 & 0 \\ 0 & Q_\| & B_t\omega/M \\ 0 & B_t\omega/M & -m_a^2 \end{bmatrix}\right] \begin{bmatrix} A_\perp \\ A_\| \\ a \end{bmatrix} = 0 \ , \tag{4}$$

where B_t is the transverse part of $\mathbf{B}_e$ and is, in general, a slowly varying function of the space coordinates. In vacuum, $Q_j = 2\omega^2(n_j - 1)$ where the refractive indices n_j are given by Eq. (2), and these quantities also vary in space. $A_\perp$, $A_\|$, and a are the amplitudes of the $\perp$ and $\|$ photon states and the axion, respectively.

In general we may assume that the variation of the magnetic field in space occurs on much larger scales than the photon or axion wavelength. Then we use the expansion $\omega^2 + \partial_z^2 = (\omega + i\partial_z)(\omega - i\partial_z) = (\omega + k)(\omega - i\partial_z)$ for propagation in the positive z direction. The dispersion relation can be expressed as $k = n\omega$ with the refractive index n, and since in our case always $|n - 1| \ll 1$ we may approximate $\omega + k = 2\omega$. Therefore in all practical cases it is convenient to use the linearized form of the wave equation,

$$\left[\omega + \begin{bmatrix} \Delta_\perp & 0 & 0 \\ 0 & \Delta_\| & \Delta_M \\ 0 & \Delta_M & \Delta_a \end{bmatrix} - i\partial_z\right] \begin{bmatrix} A_\perp \\ A_\| \\ a \end{bmatrix} = 0 \ , \tag{5}$$

where

$$\Delta_\perp = \tfrac{4}{2}\omega\xi \sin^2\Theta, \quad \Delta_\| = \tfrac{7}{2}\omega\xi \sin^2\Theta, \quad \Delta_a = -m_a^2/2\omega \ . \tag{6}$$

These quantities are the momentum differences of the respective modes compared to photons of the same energy in field-free vacuum, $\Delta_j = k_j - \omega$. Using refractive indices, $k_j = n_j\omega$ so that $n_j = 1 + \Delta_j/\omega$. The axions are assumed to be relativistic, $m_a \ll \omega$. The off-diagonal component is

$$\Delta_M = (B_e/2M)\sin\Theta \ . \tag{7}$$

The inverse quantities Δ_j^{-1} are the natural length scales of the problem and have, as will become clear shortly, a very graphic interpretation as oscillation lengths.

We stress that $\Delta_a < 0$, while, in vacuum, $\Delta_{\perp,\|} > 0$ so that the presence of the magnetic field enhances the splitting between axions and photons. Therefore, in vacuum, "level crossing" where the axions and photons become degenerate cannot occur. The positive sign of $\Delta_{\perp,\|}$ can be physically understood if one recalls that the relevant photon frequencies are below the energy of the intermediate e^+e^- state whence the frequencies are below the relevant resonance, like light in ordinary materials. For massless particles (gravitons) and for propagation along the field lines, $\Delta_{\perp,\parallel}=0$, and degeneracy occurs even in vacuum, but then also $\Delta_M=0$ and no mixing effects arise.

In practice, a perfect vacuum does not exist either in the laboratory or in the astrophysical context. Therefore the $\Delta_{\perp,\parallel}$ terms in Eq. (5) should be extended to represent the *total* refractive indices. If a separation of the two contributions is necessary we shall use the notation $\Delta^{\rm vac}$ for the vacuum contribution Eq. (6) and $\Delta^{\rm gas}$ for the gas contribution so that $n=1+(\Delta^{\rm vac}+\Delta^{\rm gas})/\omega$. The $\Delta^{\rm gas}$ are functions of density, temperature, chemical composition of the gas, and magnetic field strength. We emphasize that $\Delta_\perp^{\rm gas}\neq\Delta_\parallel^{\rm gas}$, the difference giving rise to the Cotton-Mouton effect, i.e., the birefringence of gases and liquids in the presence of transverse magnetic fields. This effect is usually accounted for by the relationship[26]

$$(\Delta_\parallel^{\rm gas}-\Delta_\perp^{\rm gas})/\omega=(n_\parallel^{\rm gas}-n_\perp^{\rm gas})_{\rm CM}=C\lambda B_e^2\ , \tag{8}$$

where C is the Cotton-Mouton constant and λ the wavelength of the light. We note that, in the presence of longitudinal magnetic field components, there are also off-diagonal gas contributions which couple the $\parallel$ and $\perp$ modes (Faraday effect). We finally note that the case of a scalar instead of a pseudoscalar particle is fully analogous with the exchange of the roles of the $\parallel$ and $\perp$ modes.

B. Gravitons

Whatever the ultimate fate of a quantum theory of gravity, it seems safe to assume that at low energy compared to the Planck mass $M_{\rm Pl}\approx10^{19}$ GeV and in weak gravitational fields, there is a field quantum, the graviton. It couples as a massless spin-2 particle to the energy-momentum tensor, in particular to that of the electromagnetic field. There is therefore a two-photon vertex of known structure by which transverse gravitons could be produced by photons in a magnetic field. It is an intriguing thought that this might be possible in the strong magnetic fields of pulsars, although we will find that, as for axions, these effects will be very small.

The two states of the graviton can be represented[27] by transverse, symmetric, traceless tensors ϵ^+ and $\epsilon^\times$ where, for graviton propagation in the z direction, $\epsilon_{xx}^+=-\epsilon_{yy}^+=1$, $\epsilon_{xy}^\times=\epsilon_{yx}^\times=1$, and all other components are zero. The space parts of the electromagnetic energy-momentum tensor are $T_{ij}=E_iE_j+B_iB_j-\frac{1}{2}\delta_{ij}(E^2+B^2)$ so that the coupling is of the form $\epsilon_{ij}(E_iE_j+B_iB_j)$ where a summation over repeated indices is implied. The relevant coupling constant is the inverse Planck mass. Given an external magnetic field $\mathbf{B}_e$, only its transverse part B_t couples, and taking B_t in the x direction, B_x of a photon couples to ϵ^+, while B_y couples to $\epsilon^\times$. Therefore ϵ^+ couples to the $\perp$ photon mode, while $\epsilon^\times$ couples to the $\parallel$ mode. This result can also be derived with our above *CP* argument because the tensor components ϵ_{xx} and ϵ_{yy} are even under the *CP* operation defined above, while ϵ_{xy} and ϵ_{yx} are odd so that ϵ^+ is even and $\epsilon^\times$ is odd under this *CP* operation. Therefore we find a linearized wave equation of the form

$$\left[\omega+\begin{pmatrix}\Delta_\perp & \Delta_M & 0 & 0\\ \Delta_M & 0 & 0 & 0\\ 0 & 0 & \Delta_\parallel & \Delta_M\\ 0 & 0 & \Delta_M & 0\end{pmatrix}-i\partial_z\right]\begin{pmatrix}A_\perp\\ G_+\\ A_\parallel\\ G_\times\end{pmatrix}=0\ , \tag{9}$$

where G_+ and $G_\times$ are the amplitudes of the states ϵ^+ and $\epsilon^\times$, respectively, and, aside from factors of order unity,

$$\Delta_M\approx(B_e/M_{\rm Pl})\sin\Theta\ . \tag{10}$$

Therefore this problem is fully analogous to the axion-photon system which we shall use as our generic case. The main difference is that the graviton is strictly massless so that the diagonal entries for G_+ and $G_\times$ are zero.

III. EVOLUTION OF A PHOTON BEAM IN MAGNETIC FIELDS

We now turn to a general discussion of the evolution of a light beam in a magnetic field region. We will focus on our generic case, the mixing of photons with light pseudoscalar particles to which we refer as "axions." In other words, we will explore several important specific solutions to the wave equation Eq. (4). The interpretation of these solutions in the context of specific experimental or astrophysical scenarios is left to Secs. IV and V.

A. General solution in a homogeneous magnetic field

If the external field $\mathbf{B}_e$ is homogeneous the first equation in (5) may be easily Fourier transformed in order to obtain the dispersion relation $k=n_\perp\omega$. The lower part of Eq. (5) can be diagonalized by a rotation to primed fields:

$$\begin{pmatrix}A_\parallel'\\ a'\end{pmatrix}=\begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}A_\parallel\\ a\end{pmatrix}\ . \tag{11}$$

The strength of the mixing is characterized by the ratio of the off-diagonal term in Eq. (5) to the *difference* of the diagonal terms:

$$\tfrac{1}{2}\tan2\vartheta=\Delta_M/(\Delta_\parallel-\Delta_a)\ . \tag{12}$$

We have $\frac{1}{2}\tan2\vartheta\approx\vartheta$ for $\vartheta\ll1$, the "weak mixing case." Turning to the dispersion relation for the diagonal fields $A_\parallel'$ and a' we find

$$\Delta'=\frac{\Delta_\parallel+\Delta_a}{2}\pm\frac{\Delta_\parallel-\Delta_a}{2\cos2\vartheta}\ , \tag{13}$$

where the plus sign refers to $\Delta_\parallel'$ and the minus sign to Δ_a'. The refractive indices of the mixed modes are given as $n'=1+\Delta'/\omega$.

Now consider a beam of frequency ω propagating in the z direction. Our discussion is simplified if we measure the phases of all modes relative to the unmixed $A_\parallel$ component, neglecting a common phase $e^{i(\omega t-\omega z-\Delta_\parallel z)}$. Then the component $A_\perp$ develops as

$$A_\perp(z)=e^{-i(\Delta_\perp-\Delta_\parallel)z}A_\perp(0)\ . \tag{14}$$

The phase in this solution represents the QED-induced and Cotton-Mouton contribution to magnetic birefringence. For the mixing components we find

$$\begin{pmatrix} A_\parallel(z) \\ a(z)\end{pmatrix}=\mathcal{M}(z)\begin{pmatrix} A_\parallel(0) \\ a(0)\end{pmatrix}, \tag{15}$$

where

$$\mathcal{M}(z)=\begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix} e^{-i(\Delta_\parallel'-\Delta_\parallel)z} & 0 \\ 0 & e^{-i(\Delta_a'-\Delta_\parallel)z}\end{pmatrix}\begin{pmatrix}\cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta\end{pmatrix}. \tag{16}$$

With Eqs. (6), (12), and (13) these equations completely solve the wave equation Eq. (5). These formulas apply for an arbitrary direction of $\mathbf{B}_e$ in the case of vacuum, or for a transverse $\mathbf{B}_e$ when matter is present. The general case of $\mathbf{B}_e$ with longitudinal components plus matter can be handled by diagonalizing the analogous 3×3 matrix.

B. Weak mixing case and applications

We now take the external homogeneous field to be sufficiently weak so that $\vartheta \ll 1$ and the weak mixing case applies. The mixing matrix Eq. (16) can be written as

$$\mathcal{M}(z)=\mathcal{M}_0(z)+\vartheta\mathcal{M}_1(z)+\vartheta^2\mathcal{M}_2(z)\ . \tag{17}$$

We must work to second order in the small mixing angle ϑ in order to exhibit the back reaction on the incoming channels as given by $\mathcal{M}_2$ [see also Fig. 1(b)]. We also introduce the notation

$$\Delta_{\rm osc}=\Delta_\parallel-\Delta_a \tag{18}$$

for the momentum difference between relativistic axions of energy ω and photons in the state $A_\parallel$ of the same energy. Furthermore, it is convenient to measure the beam path z in units of $\Delta_{\rm osc}^{-1}$, which is the only natural length scale of the problem:

$$\zeta=\Delta_{\rm osc}z\ . \tag{19}$$

Then we find for the expansion of the mixing matrix, where one must be careful to work to the relevant order in ϑ in the exponentials also,

$$\mathcal{M}_0(\zeta)\begin{pmatrix}1 & 0\\ 0 & e^{i\zeta}\end{pmatrix},\quad \mathcal{M}_1(\zeta)=(1-e^{i\zeta})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\mathcal{M}_2(\zeta)=\begin{pmatrix}-i\zeta-(1-e^{i\zeta}) & 0\\ 0 & (1-e^{i\zeta})+i\zeta e^{i\zeta}\end{pmatrix}. \tag{20}$$

This result can be used to discuss some applications of the mixing formalism.

1. Photon birefringence effects

We begin with laboratory experiments where small effects on the photon polarization state are investigated. We assume that some residual gas is present in the apparatus and the field $\mathbf{B}_e$ is transverse. In the absence of axions, the magnetically induced phase shift between the modes $A_\perp$ and $A_\parallel$ is simply given from Eq. (14) by $\phi=(\Delta_\parallel-\Delta_\perp)z=\phi_{\rm QED}+\phi_{\rm CM}$. The QED and the Cotton-Mouton contributions are, respectively,

$$\phi_{\rm QED}(z)=\frac{2\alpha^2B_e^2}{15m_e^4}\omega z,\quad \phi_{\rm CM}(z)=2\pi CB_e^2z\ . \tag{21}$$

We emphasize again that these results are expressed in natural Lorentz-Heaviside units.[24]

These phases represent the magnetically induced birefringence. The eigenmodes are $A_\parallel$ and $A_\perp$ whence this effect causes a linearly polarized beam to develop a small elliptical polarization component. The effect is biggest when the initial linear polarization is at an angle[28] $45°$ to $\mathbf{B}_e$. The resulting ellipticity (ratio of minor to major axis) is $|\phi_{\rm QED}+\phi_{\rm CM}|/2$, a possibly measurable quantity.[25] A *rotation* of the plane of polarization would be obtained if the eigenstates of refraction were the circularly polarized modes (helicity states) of the photons as is the case in optically active media or for photon propagation in a medium *along* the magnetic field lines (Faraday effect). In order to measure $\phi_{\rm QED}$ the apparatus must be sufficiently evacuated so that $\phi_{\rm CM}\ll\phi_{\rm QED}$ which, as we shall see, can be achieved.

In the presence of axions, there is an extra contribution ϕ_a from the imaginary part of the upper diagonal term in $\mathcal{M}(z)$. We find for the total phase $\phi=\phi_{\rm QED}+\phi_{\rm CM}+\phi_a$, where

$$\phi_a(z)=-\,{\rm Im}\mathcal{M}_{(11)}=\vartheta^2(\Delta_{\rm osc}z-\sin\Delta_{\rm osc}z)\ . \tag{22}$$

If this contribution were of similar order as the QED effect one could prove the existence of axionlike particles from a measurement of the magnetic birefringence of the vacuum as was first pointed out by Maiani, Petronzio, and Zavattini[17]—see also Ref. 18.

The term $\mathcal{M}_{(11)}$ also has a real part which deviates from unity by the amount

$$\varepsilon(z)=1-{\rm Re}\mathcal{M}_{(11)}=2\vartheta^2\sin^2(\Delta_{\rm osc}z/2)\ . \tag{23}$$

Assuming initially $a(0)=0$, the magnitude of $A_\parallel$ will be reduced by the factor $1-\varepsilon(z)$ due to the conversion of photons into axions. Therefore the plane of polarization will be *rotated* by an angle $\varepsilon(z)/2$, assuming initially $45°$ against $\mathbf{B}_e$. This effect could also be used to detect axions.[17] There are, then, two axion-induced effects, both of order ϑ^2, an extra phase shift for $A_\parallel$ and a reduction in its magnitude. We will study these effects in the context of a specific experimental proposal in Sec. IV.

2. *Photon-axion production*

The photon-axion or axion-photon transition amplitude is given by the off-diagonal terms in $\mathcal{M}$. The transition probability is

$$p(\gamma_\parallel \to a) = |\mathcal{M}_{(12)}|^2 = 4\vartheta^2 \sin^2(\Delta_{\rm osc} z/2) = 2\varepsilon(z) \ . \quad (24)$$

If the beam path is a multiple of $l_{\rm osc} = 2\pi/\Delta_{\rm osc}$, the transition rate is zero so that $l_{\rm osc}$ is to be interpreted as the axion-photon oscillation length.

These transitions can be used to possibly measure cosmological or astrophysical axion fluxes.[19,20,22] One could also produce axions and detect them by their reconversion into photons ("shining light through walls").[15,16,29] The quoted discussions offer an adequate treatment of these problems and we shall not consider them any further although we feel that our mixing formalism is a more elegant approach. In Sec. V, however, we will study the problem of axion-photon and photon-graviton transitions in pulsar magnetic fields because the existing discussions,[11,12] we believe, are erroneous due to their neglect of the $\Delta^{\rm vac}$ contribution to $\Delta_\parallel$, leading to errors in the calculation of ϑ and $l_{\rm osc}$ by many orders of magnitude.

C. Maximum mixing and "level crossing"

We also consider the case of maximum mixing, large ϑ, where $\mathcal{M}$ has a particularly simple form. In vacuum, axions and photons can never be degenerate as discussed above. In a medium, however, one might have $\Delta_\parallel < 0$ and specifically $\Delta_\parallel = \Delta_a$ so that maximum mixing with $\vartheta = 45°$ occurs. The momentum difference between the mixed eigenmode $A'_\parallel$ or a' with photons in field-free vacuum is, in this case, $\Delta' = \Delta_a \pm \Delta_M = \Delta_\parallel \pm \Delta_M$. Then we find for the mixing matrix Eq. (16) the result

$$\mathcal{M}_{\rm deg}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cos(\Delta_M z) - i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sin(\Delta_M z) \ . \quad (25)$$

We stress that the diagonal terms are purely real so that no axion-induced birefringence (or ellipticity) effects occur while the ε (or rotation) effect remains. The transition rate is

$$p(\gamma_\parallel \to a) = \sin^2(\Delta_M z) \quad (26)$$

and the oscillation length is

$$l_{\rm deg} = \pi/\Delta_M = 2\pi M/B_e \ , \quad (27)$$

assuming a purely transverse field. In this case a complete transition between photons and axions is possible.

In practice, it does not seem possible to precisely adjust the properties of the medium such that the degenerate case would occur. It is conceivable, however, that the beam passes through a region with a shallow gradient of the density of the medium so that initially $\Delta_\parallel < \Delta_a$ and at a later position $\Delta_\parallel > \Delta_a$, or vice versa. This would be the case, as we shall see later, for axions passing through the surface layers of a pulsar. Then at some intermediate point "level crossing" occurs. If the gradient is so shallow that this crossing occurs adiabatically, photons would completely oscillate into axions (gravitons) or vice versa. This is fully analogous to "resonant" neutrino oscillations that have been proposed to solve the solar-neutrino problem [Mikheyev-Smirnov-Wolfenstein (MSW) effect].[30]

The gradient is sufficiently shallow if the degeneracy is maintained for a distance larger than $l_{\rm deg}$. More precisely, we require that the rate of change of the mixing angle at resonance, $|d\vartheta/dz|$, be less than the oscillation wave number at resonance, $2\pi/l_{\rm deg} = 2\Delta_M$. This translates into the condition

$$\left| \frac{d}{dz}(\Delta_\parallel^{\rm gas} + \Delta_\parallel^{\rm vac}) \right| < 8\Delta_M^2 \ . \quad (28)$$

If at resonance $\Delta_\parallel^{\rm gas}(z)$ varies much faster than $\Delta_\parallel^{\rm vac}(z)$ this condition may be rewritten

$$\left| \frac{d}{dz} \ln N_e \right|^{-1} > |l_{\rm deg}/8\pi\vartheta^{\rm vac}| = (7\alpha/180\pi)(\omega M^2/B_{\rm crit}^2) \ , \quad (29)$$

where $\vartheta^{\rm vac} = \Delta_M/(\Delta_\parallel^{\rm vac} - \Delta_a)$ is the small mixing angle in the absence of the medium. We have assumed that $\Delta_\parallel^{\rm gas} \propto N_e$, the local electron density.

We stress that these conditions are necessary but not sufficient for resonant transitions to occur. The presence of the medium will cause the photon component of the beam to be scattered. Substantial oscillations occur only if the photon mean free path exceeds the oscillation length $l_{\rm deg}$.

D. Perturbative solution in inhomogeneous fields

Our previous formalism, although conceptually very useful, breaks down for inhomogeneous fields. Of practical interest are the spatially varying fields near pulsars or periodic field configurations in the laboratory. In this case one may consider a perturbative solution of the linearized wave equation Eq. (5). To this end we rewrite Eq. (5) as a "Schrödinger equation" with the z coordinate playing the role of time:

$$i\partial_z \mathbf{A} = (\mathcal{H}_0 + \mathcal{H}_1)\mathbf{A} \ , \quad (30)$$

where $\mathbf{A} = (A_\perp, A_\parallel, a)$ and the "Hamiltonian" 3×3 matrices are

$$\mathcal{H}_0(z) = \omega + \begin{bmatrix} \Delta_\perp(z) & 0 & 0 \\ 0 & \Delta_\parallel(z) & 0 \\ 0 & 0 & \Delta_a \end{bmatrix} ,$$
$$\mathcal{H}_1(z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_M(z) \\ 0 & \Delta_M(z) & 0 \end{bmatrix} . \quad (31)$$

For $M \to \infty$ we have $\Delta_M \to 0$ and this equation is solved exactly by $\mathbf{A}(z) = \mathcal{U}(z)\mathbf{A}(0)$ with

$$\mathcal{U}(z) = \exp\left[-i \int_0^z \mathcal{H}_0(z')dz' \right] .$$

This includes the QED-induced phase shift and the Cotton-Mouton effect as in the homogeneous case with the substitution $B_e^2 z \to \int_0^z B_e^2(z')dz'$.

We then use the "interaction representation" by means of the transformation $\mathbf{A}_{\rm int} = \mathcal{U}^\dagger \mathbf{A}$. In terms of these transformed fields our Schrödinger equation is $i\partial_z \mathbf{A}_{\rm int} = \mathcal{H}_{\rm int} \mathbf{A}_{\rm int}$ where $\mathcal{H}_{\rm int} = \mathcal{U}^\dagger \mathcal{H}_1 \mathcal{U}$. The complete solution is now obtained order by order from the usual iteration:

$$\mathbf{A}_{\rm int}^{n+1}(z) = -i \int_0^z dz' \mathcal{H}_{\rm int}(z') \mathbf{A}_{\rm int}^n(z') \ . \tag{32}$$

This iteration is started with the zeroth-order solution $\mathbf{A}_{\rm int}^0(z) = \mathbf{A}(0)$.

For the photon-axion transition rate we find, from the first-order solution,

$$p(\gamma_\parallel \to a) = \left| \int_0^z dz' \Delta_M(z') \times \exp\left[i\Delta_a z' - i \int_0^{z'} \Delta_\parallel(z'')dz'' \right] \right|^2 . \tag{33}$$

We stress that this result crucially depends on our "distorted wave function" approach where part of the perturbation, the terms $\Delta_\parallel$ and $\Delta_\perp$, have been absorbed by the unperturbed "Hamiltonian" $\mathcal{H}_0$. Had we considered these terms as part of the perturbation $\mathcal{H}_1$, a higher-order expansion would have been necessary to obtain this result. All previous discussions[11,12,19] have effectively used this latter approach *without* expansion to higher order because these authors were unaware of the existence of the term $\Delta_\parallel^{\rm vac}$. In laboratory fields and for the range of interesting axion parameters this omission is not important because the argument of the exponential is dominated by $\Delta_a z'$. In pulsar magnetic fields, however, $\Delta_a z'$ is entirely negligible and the previously derived results[11,12] require substantial corrections.

For the axion-induced phase shift we find, from a second-order expansion,

$$\phi_a(z) = \mathrm{Im}\left[\int_0^z dz' \int_0^{z'} dz'' \Delta_M(z')\Delta_M(z'') \times \exp\left[i\Delta_a(z'-z'') - i\int_{z''}^{z'} dz''' \Delta_\parallel(z''') \right] \right] . \tag{34}$$

In laboratory fields typically $|\Delta_\parallel| \ll |\Delta_a|$ so that the second term in the exponential is negligible.

For homogeneous fields we recover our previous results of the weak mixing case. Of course the maximum mixing case and level-crossing effects cannot be treated with these perturbative methods so that the earlier mixing formalism is a necessary tool to treat these cases.

E. Resonant effects in periodic fields

It is possible to consider providing the missing momentum transfer between the channels $A_\parallel$ and a from a spatially periodic B_e field. When the oscillation period matches $|\Delta_a| = m_a^2/2\omega$ there will be resonantlike effects in analogy with the Rabi resonance in spin resonance experiments. Therefore we consider a sinusoidal field configuration:

$$B_e(z) = B_0 \cos\Delta_0 z \ . \tag{35}$$

This field is assumed sufficiently weak that in Eq. (33) and (34) the argument of the exponential is dominated by $\Delta_a z$.

Then we find for the transition rate, keeping only the resonant term of $\cos\Delta_0 z = (e^{i\Delta_0 z} + e^{-i\Delta_0 z})/2$,

$$p(\gamma_\parallel \to a) = (B_0 z/2M)^2 g(\zeta) \ , \tag{36}$$

where $\zeta \equiv (\Delta_a - \Delta_0)z$ and

$$g(\zeta) = \zeta^{-2}\sin^2(\zeta/2) \ . \tag{37}$$

We note that for a constant field, $\Delta_0 = 0$, this result is identical to our previous result for homogeneous fields Eq. (24) aside from an extra factor $\frac{1}{4}$ which occurs because we have kept only the resonant contribution of the periodic field. On resonance $g(\zeta) = \frac{1}{4}$, see Fig. 2.

For the axion-induced phase shift we find, again keeping only resonant terms,

$$\phi_a(z) = (B_0 z/2M)^2 f(\zeta)/4 \ , \tag{38}$$

where

$$f(\zeta) = \zeta^{-1} - \zeta^{-2}\sin\zeta \ . \tag{39}$$

On resonance, $\phi_a = 0$ as in the degenerate case. Near resonance, however, there is a strongly enhanced effect, see Fig. 2.

All resonant effects occur for $\zeta \approx 1$ so that the width of the resonance is about $\delta\Delta_a \approx z^{-1}$. In other words, strongly enhanced effects occur near $m_a = \sqrt{2\omega\Delta_0}$ in a mass interval of width $\delta m_a \approx (\omega/m_a)z^{-1}$ where z is the

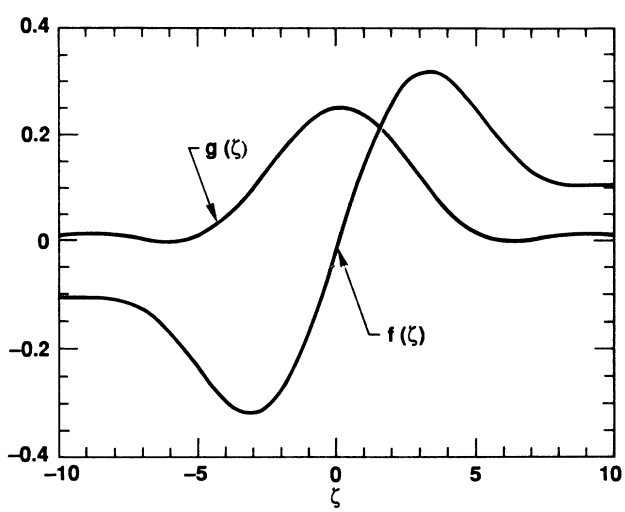

FIG. 2. Effects in periodic fields. The axion-photon transition rate is proportional to $g(\zeta)$ as defined in Eq. (37) while the axion-induced birefringence effect ϕ_a is proportional to $f(\zeta)$ as defined in Eq. (39).

length of the periodic field region. We stress that near resonance $\phi_a \propto z^2$ while $\phi_{\rm QED} \propto z$.

IV. BIREFRINGENCE EXPERIMENTS TO SEARCH FOR AXIONS

Conversion experiments to search for axions have been extensively studied in the literature,[15,16,19,20,22,29] the results agree with our Eq. (24) and we have nothing essential to add to these discussions. We merely remark that implicit assumptions concerning the neglect of vacuum and residual gas refractive effects have neither been stated nor justified in these papers, although the approximations made tend to be appropriate for the most interesting—but not the full—range of axion parameters. The applicability of various approximations becomes much more transparent, we believe, in our mixing formalism. The existing discussion[17] of possible measurements of the axion-induced birefringence of vacuum, however, needs some correction and does not include the possibility of periodic fields. Since these experiments are interesting even for the QED contribution alone, we now discuss the range of axion parameters to which such experiments might be sensitive.

A. Homogeneous fields

1. Approximations for laboratory conditions

Assuming transverse $\mathbf{B}_e$ we begin our discussion by an estimate of the relative importance of the Cotton-Mouton contribution. To this end we use the value of C for oxygen given in Ref. 26 and scale it down to a residual pressure of 10^{-11} Torr. We find $C \approx -5\times 10^{-30}$ G^{-2} cm$^{-1} = -2.6\times 10^{-31}$ eV^{-3}, whence for $\omega = 2.4$ eV we have $|\phi_{\rm CM}/\phi_{\rm QED}| \approx 7\times 10^{-3}$, slightly larger but in essential agreement with the corresponding ratio obtained in Ref. 25. It is certainly small enough to assure that the Cotton-Mouton contribution may be neglected subsequently for the phase shift between $A_\perp$ and $A_\parallel$.

In order to estimate the mixing angle we recall that, from Eq. (12), in the weak mixing case,

$$\vartheta = \frac{2\omega\Delta_M}{2\omega\Delta_\parallel^{\rm vac} + 2\omega\Delta_\parallel^{\rm gas} + m_a^2} . \tag{40}$$

A typical value for the frequency of the laser beam would be (Ref. 25) $\omega = 2.4$ eV while the magnetic field might be as strong as $B_e = 10^5$ G. With these values we find $2\omega\Delta_\parallel^{\rm vac} = 1.1\times 10^{-20}$ eV^2. A typical residual pressure in the experimental apparatus may be (Ref. 25) 10^{-11} Torr, and scaling the refractive index of air at 760 Torr, $n-1 \approx 3\times 10^{-4}$, to this low pressure we find $(n-1) \approx 10^{-17}$, corresponding to $2\omega\Delta^{\rm gas} \approx 10^{-16}$ eV^2. This is substantially larger than the $\Delta_\parallel^{\rm vac}$ contribution. Note that although the total refractive index of the residual gas is much larger than the vacuum contribution, the *difference* of the refractive indices between the $\parallel$ and $\perp$ modes is much larger for the vacuum term so that, indeed, $\phi_{\rm QED} \gg \phi_{\rm CM}$. We shall be interested only in axion masses far exceeding 10^{-8} eV so that we may approximate

$$\vartheta \approx B_e\omega / M m_a^2 . \tag{41}$$

The weak mixing approximation then translates, for the quoted values of ω and B_e, into the requirement $M \gg B_e\omega/m_a^2 = (4.7\times 10^{-6}$ GeV) $[(1$ eV$)/m_a]^2$. This condition is easily met for the range of interesting (M, m_a) values. For these conditions we may also approximate $\Delta_{\rm osc} = \Delta_\parallel - \Delta_a \approx -\Delta_a = m_a^2/2\omega$.

The authors of Ref. 17 also mentioned the case of massless "axions", i.e., true Goldstone bosons. In this case the dominant contribution to the denominator of Eq. (40) comes from the residual gas in the apparatus so that a discussion of this case is much more complicated. We have made no attempt to consider this case in any detail.

2. Single- vs multiple-beam-path experiments

In a practical experiment one would reflect the laser beam back and forth N times between two mirrors so as to accumulate a larger effect with a field region of given size. We assume that the distance between subsequent reflections is l so that the total beam path in the magnetic field is $L = Nl$. In the proposed experiment to measure the QED effect[25] the specifications are $l = 4m$, $N = 500$ and therefore $L = 2$ km. Obviously any physical mirror will be transparent to axions so that only the photon component of the beam can be reflected. Therefore each reflection effectively acts as a "filter" which sets the axion component of the beam back to zero. The compound effect is $\phi(L) = N\phi(l)$ where, in general, $N\phi(l) \neq \phi(Nl)$ because $\phi(z)$ is, in general, not linear in z. Therefore we find, for the QED and axion contributions,

$$\begin{aligned} \phi_{\rm QED}(L) &= \frac{2\alpha^2 B_e^2}{15 m_e^4}\omega L , \\ \phi_a(L) &= \vartheta^2[1-(\sin\Delta_{\rm osc} l)/\Delta_{\rm osc} l]\Delta_{\rm osc} L , \end{aligned} \tag{42}$$

and

$$\varepsilon(L) = 2\vartheta^2 N \sin^2(\Delta_{\rm osc} l/2) . \tag{43}$$

These results disagree with the corresponding conclusions of Ref. 17 where the fact has been ignored that a physical mirror will act as an "axion filter" and will, therefore, not preserve the composition of the beam. Aside from problems concerning the use of electromagnetic units, the results of Ref. 17 would be correct for a single-path experiment with $N = 1$ and $L = l$.

Considering now the case of very small axion masses, $\Delta_{\rm osc} l \ll 1$, these results may be expanded as

$$\phi_a(L) = N\frac{(B_e m_a)^2}{48\omega M^2} l^3, \quad \varepsilon(L) = N\frac{B_e^2}{8M^2} l^2 . \tag{44}$$

We believe[31] that this result replaces Eq. (16) of Ref. 17. [Note that in the second formula of their Eq. (16) apparently k^2 should read k or, in our notation, ω.] From this result it is clear that experiments looking for ϕ_a with massless particles that mix with the photon is inappropriate because, to lowest order, the effect vanishes for $m_a \to 0$.

3. *Range of measurable effects: large m_a*

In the case when the distance between two reflections is large compared with the oscillation length ($\Delta_{osc} l \gg 1$) we neglect $\sin\Delta_{osc} l$ against $\Delta_{osc} l$ and find

$$\phi_a(L) = (B_e / m_a M)^2 \omega L \ . \tag{45}$$

This case would apply to relatively large axion masses. Then it is easy to compare the QED and axion contributions. The requirement $\phi_a \geq \phi_{QED}$ translates into

$$m_a M \leq (15 m_e^4 / 2\alpha^2)^{1/2} \ . \tag{46}$$

So far our discussion has been completely independent of any specific assumptions concerning the relationship between m_a and M. However, for the actual axion models discussed in the literature, there is a definite relationship between these quantities:

$$M = R(0.7 \times 10^{10} \text{ GeV})[(1 \text{ eV})/m_a] \ , \tag{47}$$

where R is a numerical factor of order unity.[4] In these models, aside from model-dependent numerical factors $M \approx f_a/\alpha$ where f_a is the axionic decay constant. Then our result Eq. (46) is $R \lesssim 10^{-5}$ for a measurable effect to occur. Therefore R would have to take on a rather extreme value in these models for axions to be visible in this type of experiment. Even relaxing our requirement to $\phi_a \geq 0.1\phi_{QED}$ leaves us with the requirement $R \lesssim 10^{-4}$ which is more restrictive than the $R \lesssim 10^{-3}$ of Ref. 17. We mention in passing that neutral pions satisfy Eq. (47) with R of order unity because $m_a f_a \approx m_\pi f_\pi$, a formula which reflects the fact that axions have a nonzero mass due to their mixing with the π°, the mixing angle being approximately f_π / f_a. Pions, then, do not contribute to a measurable effect.

4. *Range of measurable effects: m_a arbitrary*

We now evaluate Eq. (42) in more detail without recourse to specific assumptions concerning the value of $\Delta_{osc} l$. For the ratio of the axion and QED effect we find

$$r_a \equiv \frac{\phi_a}{\phi_{QED}} = \frac{15 m_e^4}{4\alpha^2 M^2 m_a^2} \left| 1 - \frac{2\omega}{m_a^2 l} \sin\frac{m_a^2 l}{2\omega} \right| \ . \tag{48}$$

We note that this ratio depends critically on the length l of the optical resonator, but not on the absolute value of B_e or L, although these numbers are, of course, instrumental for the absolute magnitude of the effect.

Taking the specifications of Ref. 25 as an example ($B_e = 10^5$ G, $\omega = 2.4$ eV, $L = 2$ km, $l = 4$ m, and $N = 500$), we find $\phi_{QED} = 0.96 \times 10^{-11}$. With the optimistic assumption that ϕ can be measured with a 10% precision, axions can be seen if $r_a > 0.1$. In Fig. 3(a) we have shaded this area, i.e., where $\phi_a > 0.1\phi_{QED}$, or, given the above numbers, where $\phi_a \gtrsim 10^{-12}$. We also show the curve which would border this region if $l = 40$ m, keeping $L = 2$ km constant.

The region of (M, m_a) values to which this experiment is sensitive extends to largest M values for $m_a^2 = 2\pi\omega / l$ [the "nose" of the curves in Fig. 3(a)]. At this m_a value we find

$$l = (25.8 \text{ km}) r_a M_{10}^2 \omega_{eV} \ , \tag{49}$$

where $M_{10} = M/(10^{10}$ GeV) and $\omega_{eV} = \omega/(1$ eV). In other words, even if r_a can be measured to the 0.1 level, one needs a cavity length in the km range to improve on the astrophysical bound[23] of $M > 10^{10}$ GeV even in a narrow range of masses. It is remarkable, however, that it is quite feasible to "beat" the solar bound (Ref. 23) ($M > 4 \times 10^8$ GeV) since for $\omega = 2.4$ eV and $r_a = 0.1$ only $l \approx 10$ m is required.

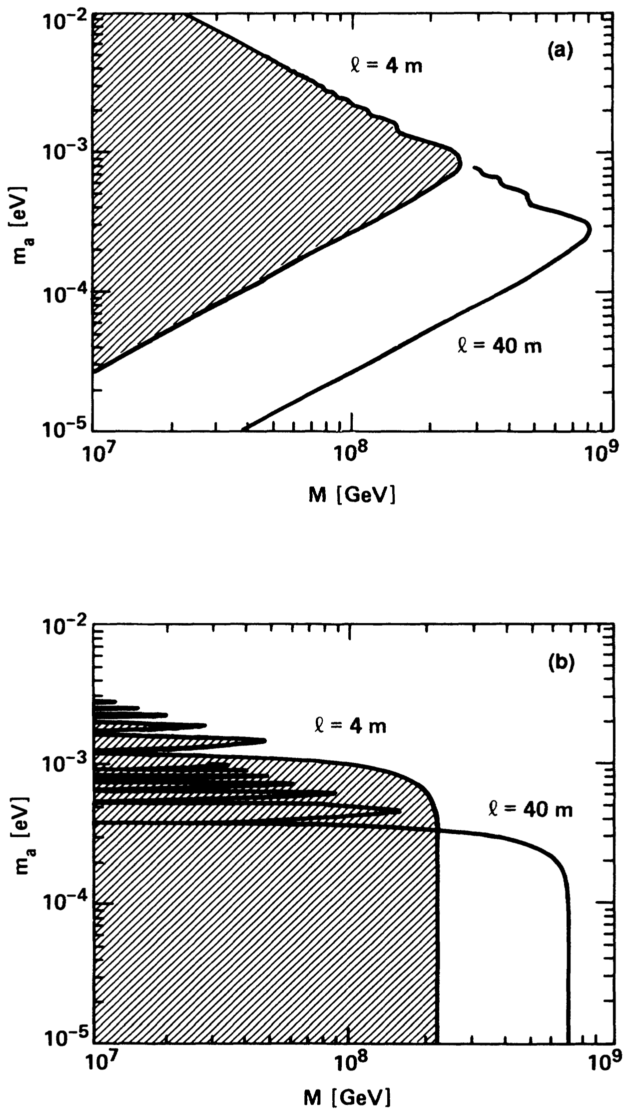

FIG. 3. (a) In the shaded area the phase shift ϕ_a would exceed 10% of ϕ_{QED} for a beam frequency $\omega = 2.4$ eV and a cavity length $l = 4$ m. The second curve gives the same result for $l = 40$ m. (b) In the shaded area the rotation of the plane of polarization would be measurable ($\varepsilon > 10^{-12}$). It is assumed that $\omega = 2.4$ eV, $B_e = 10^{12}$ G, $L = 2$ km, and a cavity length $l = 4$ m. The second curve gives the same result for $l = 40$ m while keeping L fixed at 2 km.

5. *Rotation of the plane of polarization*

For completeness, finally, we follow Ref. 17 and assume that the rotation of the plane of polarization of the laser beam was also measurable at the same level of precision as the ellipticity. The requirement that ε as given in Eq. (43) exceeds the value 10^{-12} for the above experimental data translates into the shaded area of Fig. 3(b) which indicates the range where a measurable effect may be hoped for. Again, our results differ from those of Ref. 17. They are similar, aside from problems concerning the use of electromagnetic units, for $l=L$. The oscillatory pattern in our result reflects the fact that for sufficiently large axion masses the oscillation length becomes shorter than the optical cavity and may then fit an integral fraction of l. We recall that a measurement of ε amounts to a measurement of the photon to axion transition rate and that the "shining light through walls" possibility appears to be a more practical approach to this problem.[15]

B. Birefringence in periodic fields

In Sec. III we have shown that the phase shift ϕ_a will be enhanced for a certain range of masses in a periodic, transverse magnetic field while $\phi_{\rm QED}$ remains essentially unchanged. Therefore the ratio $r_a=\phi_a/\phi_{\rm QED}$ is enhanced. One may again reflect the beam back and forth through the same magnetic field region (containing now a periodic field) in order to accumulate a larger total effect while r_a remains unaffected by this approach. We stress that the resonance does not become "sharper" if the beam is reflected more often, although the photons "see" more periods on their path. This is again due to the transparency of any mirror to axions.

For the ratio r_a of the axion and QED birefringence effect in a sinusoidal B_e field we find from Eq. (21) where we substitute $B_e^2 \to B_0^2/2$ and, from Eq. (38),

$$r_a=\frac{\phi_a}{\phi_{\rm QED}}=\frac{15m_e^4}{16\alpha^2M^2\omega}lf(\zeta)\ , \tag{50}$$

where, again, l is the length of the optical cavity, not the total beam path, and $\zeta=(\Delta_a-\Delta_0)l$. The extreme values for f are approximately ± 0.32 (see Fig. 2) so that we can estimate the necessary cavity length in order to obtain a measurable signal:

$$l=(5.1\ {\rm km})r_aM_{10}^2\omega_{\rm eV}\ . \tag{51}$$

Therefore one still needs a cavity length in the km range to improve on the astrophysical bound[23] of $M>10^{10}$ GeV even in a narrow range of masses. The periodic external field, therefore, does not improve on the maximum M value which is reachable for a given cavity length, it rather shifts the m_a value at which the maximum result is obtained.

In summary we find that the performance of a birefringence experiment to measure the QED effect would be sensitive to a large range of axion parameters. This range can be enhanced by a periodicity of the magnetic field. In view of practical limitations, however, it does not appear possible to probe a range of parameters not already excluded by astrophysical considerations.

V. AXION AND GRAVITON CONVERSION IN STELLAR MAGNETIC FIELDS

A. Axion conversion in pulsar magnetospheres

Axions could be abundantly produced in the interior of neutron stars from bremsstrahlung emission in nucleon-nucleon collisions.[32] This could, indeed, constitute the dominant cooling mechanism for these stars, a fact which may be used to set useful bounds on axion parameters.[33] Recently Morris[11] has proposed that this axion flux may be detectable through the secondary photons from axion conversion in the exterior magnetic field of these compact stars. The energy of these x rays would be characteristic of the *internal* temperature of the star (typically 50 keV) while the thermal x-ray emission from the surface would rather lie in the 1-keV range or below.

The magnetic field strength can be taken on the order[34] 4×10^{12} G, the axion energy as 10 keV, and for the cases considered by Morris[11] one would have M on the order $10^{12}-10^{13}$ GeV so that there is a strong axion flux, yet not so strong that the pulsar cools too fast to have any internal thermal energy left. Then the axion masses, given Eq. (47), are in the range $10^{-2}-10^{-3}$ eV. Therefore we obtain, in our mixing matrix Eq. (5),

$$|\Delta_a|\approx10^{-8}-10^{-10}\ {\rm eV} \tag{52}$$

while the off-diagonal term is

$$\Delta_M\approx10^{-10}-10^{-11}\ {\rm eV}\ . \tag{53}$$

If one sets $\Delta_\parallel=0$, the mixing angle would turn out somewhere in the range $\vartheta=\Delta_M/|\Delta_a|\approx10^{-1}-10^{-2}$, a formidable value which gives rise to the large conversion rate discussed by Morris.

To estimate the actual value of $\Delta_\parallel$ we begin with the vacuum contribution which, according to Eq. (6), turns out roughly

$$\Delta_\parallel^{\rm vac}\approx10^{-2}\ {\rm eV}\ . \tag{54}$$

This is much larger than $|\Delta_a|$ whence axions can be treated as effectively massless particles, $\Delta_a=0$. Concerning the contribution of free charges in the pulsar magnetosphere we emphasize that this environment cannot be a perfect vacuum.[35] Estimating the electron density we use the expression[35]

$$N_e=(7\times10^{-2}\ {\rm cm}^{-3})[B_z/(1\ {\rm G})][(1\ {\rm sec})/P]\ , \tag{55}$$

where P is the pulsar period and B_z is the magnetic field component along the rotation axis in the specific model of the aligned rotator for the pulsar magnetosphere that was used to derive this expression. With the expression $\omega_{\rm pl}^2=4\pi\alpha N_e/m_e$ for the plasma frequency we can estimate

$$\Delta_\parallel^{\rm gas}\approx-\omega_{\rm pl}^2/2\omega\approx-2\times10^{-14}\ {\rm eV}\ , \tag{56}$$

where $P=1$ sec and $B_z=4\times10^{12}$ G has been used. This is the smallest of all contributions to the mixing matrix.

Hence the mixing angle will now be on the order $\vartheta=\Delta_M/\Delta_\parallel^{\rm vac}\approx10^{-8}$, about six or seven orders of magnitude smaller than previously thought. Since $p(a\to\gamma)$

$\approx\vartheta^2$, the transition rate is down by 12–14 orders of magnitude. Although in a detailed calculation of the axion-photon conversion rates the inhomogeneity of the field must be properly included as in our perturbative solution Eq. (33), it is clear that the conversion is now dramatically suppressed due to the magnetically induced vacuum index of refraction. Given this suppression it is difficult to imagine the occurrence of *observable* effects.[36]

B. Photon-graviton conversion near pulsars

Although we can see from the preceding section that the possibility of substantial photon-graviton transitions in a stellar magnetic field will be very small it is very instructive to consider this case explicitly because gravitons are massless and the Planck scale is a known number so there are no free parameters in the problem besides the magnetic field. We may neglect all effects of the gas near the pulsar surface and find for the mixing angle

$$\vartheta\approx\frac{m_e^4}{\alpha^2 M_{\rm Pl} B_e \omega}\ , \tag{57}$$

where we have neglected factors of order unity. For $\omega=1$ keV, typical for blackbody photons at the pulsar surface, and for $B_e=10^{12}$ G this is $\vartheta\approx10^{-14}$. The oscillation length is approximately

$$l_{\rm osc}=2\pi/\Delta_\parallel\approx\frac{m_e^4}{\alpha^2 B_e^2\omega}\ . \tag{58}$$

For the present conditions this is $l_{\rm osc}\approx10^{-5}$ cm so that many oscillations take place before the photon passes even one scale length of the magnetosphere. Therefore $p(\gamma\to{\rm grav})\approx\vartheta^2\approx10^{-28}$.

Using smaller magnetic fields yields larger values for ϑ but also larger oscillation lengths. Taking magnetic white dwarfs with $B_e=10^8$ G as an example and for $\omega=10$ eV we find $\vartheta\approx10^{-8}$ and $l_{\rm osc}\approx1$ km, so that $p(\gamma\to{\rm grav})\approx\vartheta^2\approx10^{-16}$. For objects with still weaker magnetic fields, such as the Sun, finally the oscillation length would far exceed the length scale R of the system. Then one may estimate the transition probability as

$$p(\gamma\to{\rm grav})\approx4\vartheta^2\sin^2(\Delta_0R/2)\approx B_e^2R^2/M_{\rm Pl}^2\ . \tag{59}$$

Hence the transition rate decreases with decreasing field strength as naively expected. Of course, it is not obvious that the presence of gas can be neglected in these cases of very weak magnetic fields. We have considered, for example, photon-graviton mixing in the galactic magnetic field where the momentum difference between these channels is dominated by refractive effects of the interstellar gas. The transition rate is very small.

C. Level crossing near the pulsar surface

In the interior of neutron stars, axion-photon transitions are suppressed by plasma effects which give the photon an effective mass, not by the vacuum refractive index. Taking, for example, even a very low electron density such as 10^{25} cm^{-3}, the corresponding plasma frequency would be about $\omega_{\rm pl}^2\approx10^4$ eV^2 so that $\Delta^{\rm gas}\approx-\omega_{\rm pl}^2/2\omega$ is the dominant scale in the mixing matrix. The detailed dispersion relation in a strongly magnetized plasma would be much more complicated than is indicated by the simple $k^2=\omega^2-\omega_{\rm pl}^2$, because resonant transitions between the Landau levels of the electrons are possible. The magnetically induced vacuum index of refraction for photons is larger than unity, while plasma effects tend to reduce it to values less than unity. Therefore we have the situation that the photon refractive index in the *interior* of the neutron star is substantially *below unity*, in the *exterior* it is substantially *above unity* so that somewhere near the surface a crossover must occur, photons and axions or gravitons would be degenerate. Therefore one can imagine that adiabatic "level crossing" transitions could occur near the pulsar surface, as discussed in Sec. III C.

Near the pulsar surface, the electron density varies much faster than the magnetic field so that we may apply our criterion Eq. (29) for resonant effects to occur. We find, for the oscillation length from Eq. (27),

$$l_{\rm deg}=(0.64\ {\rm km})M_{10}B_{12}^{-1}\ , \tag{60}$$

where $M_{10}=M/10^{10}$ GeV and $B_{12}=B_e/10^{12}$ G. Furthermore,

$$\vartheta^{\rm vac}=\Delta_M/\Delta_\parallel^{\rm vac}=1.1\times10^{-5}M_{10}^{-1}B_{12}^{-1}\omega_{\rm keV}^{-1}\ , \tag{61}$$

where $\omega_{\rm keV}=\omega/{\rm keV}$ and axions have been treated as massless particles. Therefore,

$$|\,l_{\rm deg}/8\pi\vartheta^{\rm vac}\,|=(2.4\times10^3\ {\rm km})M_{10}^2\omega_{\rm keV} \tag{62}$$

independently of B_e and far beyond the scales over which the electron density varies. Therefore the adiabatic condition cannot be met for the axion and the situation is even worse for the graviton.

Most recently, Yoshimura[12] has independently pointed out the possibility of resonant level-crossing effects between axions and photons in the magnetosphere of a pulsar. Unfortunately, he has neglected the QED contribution to the photon dispersion relation. Therefore all of his numerical estimates are incorrect by many orders of magnitude. His scenario is that of a binary pulsar where a shallow gradient of matter is obtained from an accretion flow. We do not believe that resonant effects would occur in this scenario.

D. Level crossing in magnetic white dwarfs

It is not unthinkable, however, that resonant level-crossing effects occur in larger systems where the electron density changes on scales larger than that given by Eq. (62). An example would be magnetic white dwarfs. For $\omega\approx10$ eV, we find $l_{\rm deg}/\vartheta^{\rm vac}=600$ km $M_{10}^2\omega_{10}$, where $\omega_{10}=\omega/10$ eV. Since typical white dwarf radii are on the order of 10^4 km, a dilute atmosphere with a scale height in the 1000-km range appears possible. The magnetic field would be in the 10^8-G range, and we shall assume that axions have sufficiently small masses that the vacuum mixing angle is still given by Eq. (61). The photon dispersion relation is very complicated for the conditions at hand because atomic binding energies are also in the eV range. Furthermore, even for free electrons, the spacing of the Landau levels is $(e/m_e)B=(1.16\ \text{eV})\ B_8$ where $B_8=B/10^8$ G so that these magnetic resonances cannot be ignored for photons in the eV range.

We parametrize the effect of the gas as

$$\Delta_\parallel^{\text{gas}}=-(\omega_{\text{pl}}^2/2\omega)F(\omega,T,B)\ , \tag{63}$$

where T is the temperature of the magnetosphere of the star, possibly much larger than the surface temperature so that the medium could be fully ionized. The plasma frequency is given as $\omega_{\text{pl}}^2=4\pi\alpha N_e/m_e$ where N_e is the local electron density. Neglecting Δ_a, degeneracy occurs for $\Delta_\parallel^{\text{vac}}+\Delta_\parallel^{\text{gas}}=0$, i.e., for

$$N_e=\frac{14\alpha}{90\pi}\frac{B^2\omega^2}{m_e^3}F(\omega,T,B)^{-1}$$

$$=(1.34\times10^8\ \text{cm}^{-3})B_8^2\omega_{10}^2/F\ . \tag{64}$$

In other words, if $F\approx1$ the medium must be extremely dilute. Parametrizing the photon scattering cross section with the Thomson cross section as

$$\sigma=(8\pi\alpha^2/3m_e^2)G(\omega,T,B) \tag{65}$$

the mean free path is

$$(\sigma N_e)^{-1}=(1.12\times10^{13}\ \text{km})B_8^{-2}\omega_{\text{eV}}^{-2}F/G \tag{66}$$

far exceeding all relevant length scales. We conclude that for M near the astrophysical bound of 10^{10} GeV resonant oscillations of photons into (massless) axions may possibly occur in extended systems such as magnetic white dwarfs. In practice, it would be very difficult, of course, to find *observational* evidence for these oscillation effects.

VI. CONCLUSIONS

We have provided a detailed discussion of mixing phenomena between photons and light bosons of spin 0 or 2 in the presence of external magnetic fields. Our study is complementary to several recent discussions of axion-photon mixing phenomena.[11,12,15,16,19,20,29] We go beyond this previous work by including the spin-2 case (gravitons), and by developing a mixing formalism for these phenomena. We also give a perturbative approach for the resulting multicomponent wave equation, and by including the QED-induced nonlinearity of Maxwell's equations we take into account an important effect in the lifting of the near degeneracy of the mixing particles. We identify the range of parameters where vacuum and matter refractive effects are important for photon-axion or graviton transitions and for photon birefringence effects. In detail our conclusions are as follows.

(a) *Photon birefringence experiments.* Experiments to measure the QED-induced birefringence of the vacuum in an external magnetic field could exclude a large range of axion parameters. These, however, are already excluded by astrophysical bounds.[23] Even the use of periodic field configurations, while enhancing the effect for a certain range of axion masses, would not give a measurable effect. However, since this type of experiment is important in its own right it is worth noting that it would yield *independent* insight into the question of axion parameters as a by-product.

(b) *Photon-axion conversion experiments.* The most promising approach to measure this effect appears to be "shining light through walls" as described in Ref. 15. Optimistically, there appears to be a range of parameters not excluded by astrophysical evidence where a measurable effect is possible, although this parameter range is very small and does not overlap with the predictions of any existing axion model—see Ref. 15. Again, this type of experiment would yield independent evidence on the range of allowed axion parameters.

(c) *Photon-axion (-graviton) conversion near stars.* The transition rates appear to be always very small, mainly because the QED-induced photon refractive effects strongly suppress these effects. This QED effect could be canceled by plasma refractive effects, possibly leading to resonant oscillation phenomena. However, near pulsars the adiabatic condition cannot be met because the generic length scales of the system are far too short. In more extended systems such as magnetic white dwarfs, however, resonant transitions may possibly occur, although a fortuitous combination of axion and white-dwarf parameters is required.

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