Critical dimension of strings with an extrinsic curvature

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The conformal anomaly is calculated by using the path-integral method to determine the critical dimension for a string theory with an extrinsic curvature by appropriately defining the first-order form of this Lagrangian. The critical dimension, defined by the vanishing of the Liouville kinetic term, is found to be $D=26$, the same as for the ordinary bosonic string theory.

A new type of string theory has been recently proposed[1] whose actions depend on the extrinsic curvature of the world sheet. This model is proposed with the hope that the Nambu-Goto action with this term added can describe "smooth" strings for QCD hadrons. The reciprocal of the coefficient for this extrinsic-curvature term is found to be perturbatively asymptotically free, which realizes the "smooth" phase. The classical properties of this theory have been extensively studied[2] since its proposal, especially in regard to the mass spectrum, but the quantum aspects of this theory have not yet been well understood. One aspect of the quantum feature is the critical dimension: i.e., in which dimension does a consistent quantum theory exist? Can this be $D=4$ to describe QCD appropriately, or again $D=26$?

There are several methods to determine this critical dimension.[3–5] Here we will adopt the method taken by Fujikawa:[4] i.e., the path-integral method. In Ref. 4, Fujikawa fixed the gauge of the two-dimensional general coordinate invariance, quantized the Lagrangian for a bosonic string in the manner of Becchi, Rouet, and Stora[6] (BRS), fixed the functional measure so that there arises no BRS anomaly, and calculated the conformal anomaly. Vanishing of the conformal anomaly is necessary to make the conformal degree cease to be another dynamical degree of freedom, and determines the critical dimension. For strings with extrinsic curvature, it is not necessary for the conformal anomaly to vanish as it is already broken at the tree level. Nevertheless, it is of interest to investigate the coefficient of the Liouville mode for this case. In order to perform this procedure, we need to construct the first-order form including the extrinsic curvature so that we can quantize this Lagrangian following Ref. 6. Let us do this first.

The original form proposed in Ref. 1 has taken, as a first term, the Nambu-Goto area-law action in the second-order form:

$$I_0=-\int d^2\xi\sqrt{g}\;,\quad g_{ab}=\partial_a X^\mu \partial_b X_\mu\;, \tag{1}$$

whose first-order form is well known to be

$$I_0=-\tfrac{1}{2}\int d^2\xi\sqrt{g}\,g^{ab}\partial_a X^\mu\partial_b X_\mu\;. \tag{2}$$

That is, by taking the variation in g^{ab}, one can recover Eq. (1). The term involving the extrinsic curvature has the form (the other forms are equivalent to this up to a total divergence)

$$I_1=-\frac{a}{2}\int d^2\xi\sqrt{g}\,(\partial^2 X)^2\;, \tag{3}$$

where a is a constant and $\partial^2 X$ is defined to be

$$\partial^2 X^\mu=\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\,g^{ab}\partial_b X^\mu)\;. \tag{4}$$

The first-order form of the action is taken to be the sum of I_0 and I_1 in which both the metric and X^μ are treated as independent. In Polyakov's formulation the action is the sum of (3) and a term

$$I_3=\int \lambda^{ab}(\partial_a X^\mu\partial_b X_\mu - g_{ab})d^2\sigma\;, \tag{5}$$

where λ^{ab} is a constraint field that fixes the induced metric g_{ab} to be equal to $\partial_a X^\mu \partial_b X_\mu$. It is shown that if one considers λ^{ab} as a fixed background field of the form $\frac{1}{2}\sqrt{g}\,g^{ab}$, then our formalism is equivalent to Polyakov's. The effect of "fast" fluctuations of background fields is shown to renormalize the coupling a and thus does not affect our anomaly calculations.

The action is invariant under two-dimensional general coordinate transformations; hence, we can follow the quantization procedure employed in Ref. 4. That is, the gauge-fixing term for two-dimensional general coordinate invariance can be taken to be the same as in Ref. 4 and hence so can the Faddeev-Popov term:

$$L_{\rm GF}=B_0A^0+B_1A^1\;, \tag{6}$$

$$\begin{aligned} L_{\rm FP}=&\,i\bar{C}_0[(C^a\partial_a+\tfrac{1}{2}\partial_a C^a)A^0+(\partial_1C^0+\partial_0C^1)A^1\\ &\quad+(\partial_1C^0-\partial_0C^1)A^2]\\ &+i\bar{C}_1[(C^a\partial_a+\tfrac{1}{2}\partial_aC^a)A^1+(\partial_1C^0+\partial_0C^1)A^0\\ &\quad+(\partial_0C^0-\partial_1C^1)A^2]\;, \end{aligned} \tag{7}$$

where B_a are auxiliary fields, C^a and $\bar{C}_a$ are the Faddeev-Popov (FP) ghosts, which are denoted as η and ξ, respectively, in Ref. 4 and

$$\mathbf{g}_{ab}=g^{-1/4}g_{ab}=\begin{bmatrix} A^1+A^2 & A^0\\ A^0 & A^1-A^2\end{bmatrix}\;. \tag{8}$$

Here we have changed some notations so that there occurs no confusion with other papers. We work in Minkowski space while Fujikawa has worked in Euclidean space. The BRS transformations for fields do not change by adding Eq. (3) and are given in Ref. 4. The functional measure which is the origin of all anomalies, if there are any, is given by

$$d\mu = d\tilde{X}\, d\tilde{C}^{a} d\bar{C}_a dB_a dA^i \ . \tag{9}$$

Here the fields X and C^a are defined so that there is no BRS anomaly arising from the measure (9):

$$\tilde{X}^{\mu} = g^{1/4}X^{\mu}, \quad \tilde{C}^{a} = g^{1/2}C^{a} \ . \tag{10}$$

Now the partition function

$$Z = \int d\mu \exp\left[i \int d^2x (L_0 + L_{\rm GF} + L_{\rm FP}) \right] \tag{11}$$

is invariant under the BRS transformation due to the two-dimensional general coordinate transformation and is locally invariant under the Weyl (local scale) transformation at the tree level. To get the conformal anomaly we integrate over B_a, A^0, and A^1 to obtain the simplified Lagrangian as

$$Z = \int dA^2 d\tilde{X}\, d\tilde{C}^{a} d\bar{C}_a \exp\left[-\frac{i}{2} \int d^2x\, \tilde{X}_{\mu}(H_0 + aH_0^2)\tilde{X}^{\mu} + i \int d^2x\, \bar{C}\rho^{1/2} i\partial\!\!\!/(\rho^{-1}\tilde{C}) \right] , \tag{12}$$

where $i\partial\!\!\!/ = -i\sigma_2\partial_0 + \sigma_3\partial_1$ with the Pauli matrices σ_i and $\rho^{1/2} = g^{1/4} = A^2$. The original kinetic operator for the field X is simply H_0 and is given by

$$H_0 = -\rho^{-1/2}\partial^2\rho^{-1/2} \ , \tag{13}$$

which is a Hermitian operator and hence the total kinetic term $H_0 + aH_0^2$ is also Hermitian. Now we calculate the anomaly contribution from the field X, which is the only difference between Ref. 4 and our work. Following carefully the procedure taken in Ref. 4, the anomaly is expressed in the path-integral method and is obtained, after tedious but rather straightforward calculations, as

$$\begin{aligned} A^{\tilde{X}} &= \lim_{M\to\infty} \int \frac{d^2k}{(2\pi)^2} e^{-ikx} \exp[-(H_0 + aH_0^2)/M^2] e^{ikx} \\ &= \frac{1}{24\pi}\partial^2 \ln\rho + \frac{1}{4\pi} \lim_{M\to\infty} \left[M^2 \int_0^{\infty} dz \exp\left[-\frac{aM^2}{\rho} z^2 - z \right] \Big/ \rho \right] \\ &\quad + \frac{1}{4\pi} \lim_{M\to\infty} \frac{5}{aM^2} \left\{ 1 - (1 + 2aM^2) \int_0^{\infty} dz \exp\left[\left[-\frac{aM^2}{\rho} z^2 - z \right] \Big/ \rho \right] \right\} , \end{aligned} \tag{14}$$

where we have kept the coupling a and a cutoff M finite within the limit $M \to \infty$. The third term in (14) vanishes if we take the limit $M \to \infty$ irrespective of the value of a. However, the second term in (14) becomes different expressions depending on which limit is taken first. If one first lets $a \to 0$ within the limit $M \to \infty$, the second term becomes

$$\frac{1}{4\pi} M^2 \rho \ , \tag{15}$$

which is the same form as the one obtained in Ref. 4. On the other hand, if one first lets $M \to \infty$, keeping the coupling a finite, then it becomes

$$\lim_{M\to\infty} \frac{M\rho}{4\sqrt{2\pi a}} \ , \tag{16}$$

where one cannot take the limit $a \to 0$. The order of two limits, $M \to \infty$ and $a \to 0$, is not interchangeable. This means that the theory with $a = 0$ cannot be analytically connected to the one with a finite a. In any case, however, those two divergent expressions, (15) and (16), can be absorbed by the counterterm of the form $\mu^2\rho$ as in Ref. 4. Finally, the total conformal anomaly, together with contributions from the FP ghosts included, is given by

$$A_{\rm Weyl} = \frac{26 - D}{48\pi} \partial^2 \ln\rho \ , \tag{17}$$

which appears in the anomalous Ward-Takahashi identity

$$\int d\rho\, \rho \frac{\delta\tilde{S}}{\delta\rho} = \int d\rho \left[A_{\rm Weyl} + \int d\mu' \frac{a}{2} \int d^2\xi\, \rho^{-1} \partial^2(\tilde{X}/\sqrt{\rho}) \partial^2(\tilde{X}/\sqrt{\rho}) e^{iS} \Big/ \int d\mu' e^{iS} \right] , \tag{18}$$

where $\tilde{S}$ is the effective action of the field ρ, S is the total action appearing in Eq. (12), the $d\mu' = d\tilde{X}\, d\tilde{C}^{a} d\bar{C}_a$. The second term in Eq. (18) exists because of the explicit breaking term given by Eq. (3). If we require that the quantum correction to the Ward-Takahashi identity vanish, the critical dimension does not change since the first term in Eq. (15) is the same as the one obtained in Ref. 4 without the I_1 term, Eq. (3). It is again given by $D = 26$.

This result may be anticipated from the original form of the extrinsic curvature term, Eq. (3). When we substitute the conformal gauge $g_{\mu\nu}=\rho\eta_{\mu\nu}$ into Eq. (3), we notice that this term is directly proportional to ρ^{-1} without replacing X_μ with $\tilde{X}_\mu$ in Eq. (10). That is, this term already breaks the two-dimensional conformal invariance, which is implicitly shown by the dimensional parameter a in Eq. (3) and is explicitly shown in Eq. (18). An explicit breaking term may not contribute to the anomaly just as the fermion mass term does not contribute to the chiral anomaly.

Recently, Pisarski[7] has analyzed the perturbative stability of smooth strings. He has calculated the Liouville term directly by computing the path integral. Our procedure in this paper has been to calculate the contribution of the extrinsic curvature to the Weyl anomaly.

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