Geometrical representation of neutrino oscillations in vacuum and matter

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A simple formula from which a geometrical picture of neutrino oscillations with two flavors may be constructed is derived from the equation of motion for the neutrinos. Applications of the picture to the nonadiabatic as well as adiabatic Mikheyev-Smirnov-Wolfenstein effects in the solar-neutrino problem are given. A generalization of the picture to the three-generation case is also briefly discussed.

The neutrino oscillation can be described as the rotation of a unit vector $\boldsymbol{\nu}_\alpha(t)$ representing a neutrino of flavor α around the mass eigenstate $\boldsymbol{\nu}_i(t)$. This is analogous to the precession of a magnetic dipole in a (steady and varying) external magnetic field. The projection of $\boldsymbol{\nu}_\alpha(t)$ on $\boldsymbol{\nu}_i(t)$ is related to the amplitude of finding $\nu_i(t)$ in the state $\nu_\alpha(t)$. Although the geometrical representation of neutrino oscillation is a convenient way to understand and analyze the oscillation problems, the representation itself is not unique. For example, Mikheyev and Smirnov[1] (MS) have used an orthogonal basis with the axes $\boldsymbol{\nu}_1$, Re$\boldsymbol{\nu}_2$, and Im$\boldsymbol{\nu}_2$ to describe the oscillation of two generations of neutrinos. In their picture, the flavor vector, e.g., $\boldsymbol{\nu}_e$ will rotate around $\boldsymbol{\nu}_1$ with an angle θ starting from the initial position $\boldsymbol{\nu}_e(0)=\text{Re}\boldsymbol{\nu}_e=(\cos\theta,\sin\theta,0)$. For oscillations in matter, known as the Mikheyev-Smirnov[2]-Wolfenstein[3] (MSW) effects, the mass eigenstates $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ in vacuum are replaced by the effective (in matter) mass eigenstates $\boldsymbol{\nu}_1^{(m)}$ and $\boldsymbol{\nu}_2^{(m)}$. Although this representation correctly describes the oscillations both in vacuum and matter, the choice of the orthogonal basis is somewhat arbitrary.

On the other hand, Messiah,[4] and Kim, Nussinov and Sze[5] have recently discussed the use of a geometrical picture in three-dimensional Euclidean flavor space deduced from the two-valued representation in flavor space of the rotation group. They have independently applied this picture to the case of adiabatic approximation of the MSW effects in the solar-neutrino problem.

In this paper we present a clear and simple derivation of the geometric picture discussed in Refs. 4 and 5 starting from the original equation of motion for two generations of neutrinos. Applications of the picture to nonadiabatic as well as adiabatic MSW effects in the solar-neutrino problem are given. A generalization to the three-generation case is also discussed.

We will start with the case of two neutrino flavors. Let $\psi=(\nu_e\ \ \nu_\mu)^T$ be a neutrino state expressed in the weak basis and normalized such that $\psi^\dagger\psi=1$. The equation of motion for neutrinos in matter is given by

$$i\frac{d}{dt}\begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix}=\frac{1}{4E}\begin{bmatrix}A-\Delta\cos\theta & \Delta\sin 2\theta\\ \Delta\sin 2\theta & -A+\Delta\cos 2\theta\end{bmatrix}\begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix}, \tag{1}$$

where $A=2\sqrt{2}G_F N_e E$ with G_F the Fermi constant, N_e the electron number density in matter, and E the neutrino energy. Also in Eq. (1), $\Delta=m_2^2-m_1^2$, the mass-squared difference. In a vacuum, we have $A=0$. In this case Eq. (1) is the equation of motion for the weak-eigenstate neutrinos in a vacuum.

We will rewrite Eq. (1) in the form

$$i\frac{d\psi}{dt}=-\frac{\boldsymbol{\sigma}}{2}\cdot\mathbf{B}\psi\ , \tag{2}$$

where

$$\mathbf{B}=(1/2E)[-\hat{\mathbf{x}}\Delta\sin 2\theta+\hat{\mathbf{z}}(\Delta\cos 2\theta-A)]\ , \tag{3}$$

when expressed in terms of the orthonormal unit vectors $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ which define a right-hand coordinate system, and the components of $\boldsymbol{\sigma}$ are the Pauli matrices. [We note that the coordinate system in Ref. 5 is a left-hand one which gives a positive x component for $\mathbf{B}$. It can be obtained by a similarity transform of Eq. (2) by σ_1 followed by an inversion.] The vector $\mathbf{B}$ is related to its vacuum value,

$$\mathbf{B}_0=(\Delta/2E)(-\hat{\mathbf{x}}\sin 2\theta+\hat{\mathbf{z}}\cos 2\theta)\ , \tag{4}$$

by $\mathbf{B}=\mathbf{B}_0-(A/2E)\hat{\mathbf{z}}$. The negative sign on the right-hand side of Eq. (2) is chosen such that $\mathbf{B}_0$ is almost parallel to $\hat{\mathbf{z}}$ for small θ values. Multiplying Eq. (2) by $\psi^\dagger$ on the right, multiplying its Hermitian conjugate by ψ on the left, and then taking their difference, we get

$$i\frac{d}{dt}(\psi\psi^\dagger)=\left[\psi\psi^\dagger,\frac{\boldsymbol{\sigma}}{2}\cdot\mathbf{B}\right]. \tag{5}$$

We can parametrize $\psi\psi^\dagger$ by a three-vector $\mathbf{m}$ defined by

$$\psi\psi^\dagger=\frac{\sigma_0}{2}+\frac{\boldsymbol{\sigma}}{2}\cdot\mathbf{m}\ , \tag{6}$$

where σ_0 is the 2×2 identity matrix. The coefficient of σ_0 is $\frac{1}{2}$ by our assumption that ψ has unit norm. This equation assigns a unique vector $\mathbf{m}$ to each given ψ. Conversely, when $\mathbf{m}$ is given, ψ will be determined up to an overall phase. It is readily seen that in this representation the pure electron-neutrino state corresponds to $\mathbf{m}=\hat{\mathbf{z}}$, while the pure muon neutrino state corresponds to $\mathbf{m}=-\hat{\mathbf{z}}$. With this parametrization Eq. (5) becomes

$$i\frac{\sigma}{2}\cdot\frac{d\mathbf{m}}{dt}=i\frac{\sigma}{2}\cdot\mathbf{m}\times\mathbf{B}\ . \tag{7}$$

Since the Pauli matrices are linearly independent of one another, we obtain, from Eq. (7),

$$\frac{d\mathbf{m}}{dt}=\mathbf{m}\times\mathbf{B}\ , \tag{8}$$

which is the equation of motion for a magnetic moment $\mathbf{m}$ with a gyromagnetic ratio $g=1$ precessing about an external magnetic field $\mathbf{B}$. This equation, together with $\mathbf{B}=\mathbf{B}_0-[A(t)/2E]\hat{\mathbf{z}}$, completely specifies the evolution of the system. The precession frequency is $\omega=-|\mathbf{B}|$; the negative sign means that $\mathbf{m}$ is precessing in the clockwise sense when viewed from the direction of $\mathbf{B}$. The probability of P_{ee} of a neutrino in the state ψ being detected as an electron neutrino at time t is given by $|\psi^\dagger\psi_{\nu_e}|^2=\mathrm{tr}(\psi\psi^\dagger\psi_{\nu_e}\psi_{\nu_e}^\dagger)$. Employing the parametrization given in Eq. (6) for ψ and ψ_{ν_e}, we see that this can be written as $\mathrm{tr}[\frac{1}{4}(1+\mathbf{m}\cdot\hat{\mathbf{z}})\sigma_0+\text{traceless parts}]$. Hence we get

$$P_{ee}=\tfrac{1}{2}(1+\mathbf{m}\cdot\hat{\mathbf{z}}) \tag{9}$$

Starting from the identity $|\psi^\dagger\psi|^2=1$ and proceeding in a similar way, we deduce that $\mathbf{m}^2=1$. That is, $\mathbf{m}$ is a magnetic dipole moment of unit strength. Therefore $\mathbf{m}\cdot\hat{\mathbf{z}}$ is simply the cosine of the angle between $\mathbf{m}$ and $\hat{\mathbf{z}}$. If we call this angle 2α, then from Eq. (9) we immediately get $P_{ee}=\cos^2\alpha$.

For oscillations in vacuum, the vector $\mathbf{B}$ is a constant vector equal to $\mathbf{B}_0$ since A is identically zero. To see that in this case Eq. (9) reduces to the usual formula in neutrino oscillations, let us pick another set of coordinate vectors $(\mathbf{i}_0,\mathbf{j}_0,\mathbf{k}_0)$ so that $\mathbf{k}_0$ is parallel to $\mathbf{B}_0$:

$$\begin{aligned}\hat{\mathbf{x}}&=\mathbf{i}_0\cos2\theta-\mathbf{k}_0\sin2\theta\ ,\\ \hat{\mathbf{y}}&=\mathbf{j}_0\ ,\\ \hat{\mathbf{z}}&=\mathbf{i}_0\sin2\theta+\mathbf{k}_0\cos2\theta\ .\end{aligned} \tag{10}$$

Here we want to solve Eq. (8) subject to the initial condition $\mathbf{m}(t=0)=\hat{\mathbf{z}}$. The solution can be written as

$$\mathbf{m}=\mathbf{k}_0\cos2\theta+\sin2\theta(\mathbf{i}_0\cos\omega_0t+\mathbf{j}_0\sin\omega_0t)\ , \tag{11}$$

where $\omega_0=-|\mathbf{B}_0|=-\Delta/2E$ is the vacuum value of ω. This equation shows that $\mathbf{m}$ describes the surface of a cone with axis $\mathbf{B}$ and an opening angle 2θ. Note that in our picture the angle between the axis $\mathbf{B}$ (which is in the direction of the mass eigenstate ν_1) and the $\mathbf{z}$ axis is 2θ instead of θ. This is due to the fact that Eq. (8) is expressed in the flavor O(3) space whereas Eq. (1) is in the flavor SU(2) space, i.e., a result of the well-known Caley-Klein parametrization.

Substituting the above into Eq. (9) and inserting the value of ω_0 we get $P_{ee}=1-\sin^22\theta\sin^2(\Delta t/4E)$ which is the familiar result for neutrino oscillations in vacuum. In the solar-neutrino problem, one is only interested in the time average of the probability given by $\langle P_{ee}(t)\rangle$.

Qualitatively the precession picture of the MSW effect can be understood as follows. Within the core of the Sun where the electron neutrinos are produced, the high electron density implies a large A value for neutrinos of high enough energies. This means that $\mathbf{B}$ will have a negative z component for neutrinos of energies higher than a certain critical value E_c. In fact, for small θ values, $\mathbf{B}$ is almost antiparallel to $\hat{\mathbf{z}}$ if $E>E_c$. Initially $\mathbf{m}$ is equal to $\hat{\mathbf{z}}$, corresponding to the fact that the neutrino is born an electron neutrino, and the opening angle of the precession cone is close to 180°. (We have to imagine a cone turned inside out here.) Alternatively m is precessing about $-\mathbf{B}$ in the counterclockwise sense with an opening angle close to zero; the point is that $\mathbf{m}$ gets anchored to the axis defined by $\mathbf{B}$ and will try to follow the shift of the axis if it can catch up. As the neutrino emerges out of the Sun, the A value decreases, causing $\mathbf{B}$ to migrate to the vacuum position $\mathbf{B}_0$. If the migration is slow enough, $\mathbf{m}$ will follow the motion and finally ends up precessing around $\mathbf{B}_0$, still with an opening angle close to 180°. It means that in the final configuration $\mathbf{m}$ is almost parallel to the $-\hat{\mathbf{z}}$ direction, and the corresponding neutrino state is dominated by the ν_μ component which escapes detection. This is the case of adiabatic transition. It should be contrasted with oscillation in a vacuum in which $\mathbf{m}$ always precess about $\mathbf{B}_0$ with a small opening angle.

It is easy to quantify the preceding argument including the case when the migration of $\mathbf{B}$ is fast, i.e., nonadiabatic processes, and get the expression for P_{ee} in terms of the relevant angle variables. Again we pick a set of unit vectors $(\mathbf{i},\mathbf{j},\mathbf{k})$ such that $\mathbf{k}$ is along the $\mathbf{B}$ direction. In this case, however, $\mathbf{B}$ is time dependent; thus $(\mathbf{i},\mathbf{j},\mathbf{k})$ determines a moving coordinate system. The transformation between $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ and $(\mathbf{i},\mathbf{j},\mathbf{k})$ can be obtained from Eq. (10) by replacing $(\mathbf{i}_0,\mathbf{j}_0,\mathbf{k}_0)$ and θ by $(\mathbf{i},\mathbf{j},\mathbf{k})$ and the effective mixing angle ϕ, respectively,

$$\begin{aligned}\hat{\mathbf{x}}&=\mathbf{i}\cos2\phi-\mathbf{k}\sin2\phi\ ,\\ \hat{\mathbf{y}}&=\mathbf{j}\ ,\\ \hat{\mathbf{z}}&=\mathbf{i}\sin2\phi+\mathbf{k}\cos2\phi\ .\end{aligned} \tag{12}$$

Since $\mathbf{B}$ always remain in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane, the problem is simply a two-dimensional one. For subsequent calculations the expression of $\mathbf{i}$ and $\mathbf{k}$ in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ as given by

$$\begin{aligned}\mathbf{i}&=\hat{\mathbf{x}}\cos2\phi+\hat{\mathbf{z}}\sin2\phi\ ,\\ \mathbf{k}&=-\hat{\mathbf{x}}\sin2\phi+\hat{\mathbf{z}}\cos2\phi\ ,\end{aligned} \tag{13}$$

will be useful. From these we readily obtained the expression for the time rate of change of $\mathbf{i}$ and $\mathbf{k}$:

$$\frac{d\mathbf{i}}{dt}=\omega_m\mathbf{k},\quad \frac{d\mathbf{k}}{dt}=-\omega_m\mathbf{i}\ , \tag{14}$$

where the migration frequency ω_m is defined to be $d(2\phi)/dt$.

The solution to Eq. (8) at any instant can be written in a form similar to Eq. (11) as

$$\mathbf{m}=\mathbf{k}\cos2\beta+\sin2\beta[\mathbf{i}\cos2\Phi(t)+\mathbf{j}\sin2\Phi(t)]\ , \tag{15}$$

where 2β is the opening angle of the cone swept out by $\mathbf{m}$ and

$$2\Phi(t)=\int_0^t \omega(\tau)d\tau+\cdots \ , \tag{16}$$

where the ellipsis denotes the parts of Φ due to the motion of the **i**-**j**-**k** coordinate system. In the vacuum case $2\Phi=\omega_0 t$ and $\beta\equiv\theta$, so we get back Eq. (11). For solar neutrinos, on the other hand, the effective mixing angle is initially equal to θ_m. Consequently the initial value of β is given by $\beta_i=\theta_m$. As **B** migrates towards $\mathbf{B}_0$, β will in general vary with time, eventually attaining some final value β' when **B** settles down to $\mathbf{B}_0$. Thereafter we have $\mathbf{m}=\mathbf{k}_0\cos2\beta'+\sin2\beta'(\mathbf{i}_0\cos2\omega_0 t+\mathbf{j}_0\sin2\omega_0 t)$. (An unimportant additive phase to $\omega_0 t$ has been dropped.) As we will see later, except for the adiabatic case, β' depends on the resonance time t_R and has to be averaged over. This averaging over the resonance time t_R, which is necessary because neutrinos with different energies correspond to **m** with different precession frequency and hence different t_R, is not to be confused with the averaging over the detection time t, which is always implicitly taken in solar-neutrino calculation. Noticing that the average of $\cos2\omega_0 t$ with respect to t is zero, the averaged value of $\mathbf{m}\cdot\hat{\mathbf{z}}$ can be obtained as $\cos2\theta\langle\cos2\beta'\rangle_{t_R}=\cos2\theta\cos2\beta_f$. Here the angular brackets with t_R as subscript denotes the averaging with respect to t_R. Moreover we have defined β_f, the effective final β value, by $\cos2\beta_f=\langle\cos2\beta'\rangle_{t_R}$. So finally we have

$$P_{ee}=\tfrac{1}{2}(1+\cos2\theta\cos2\beta_f) \ . \tag{17}$$

Thus the problem reduces to the calculation of β_f.

In the adiabatic case, the migration rate of **B** is always infinitesimally small compared with the precession frequency of **m**. This means that **m** can always catch up with the shift in **B**, and β stays equal to its initial value throughout. Therefore both β_f and β_i are equal to θ_m. Substituting this back to Eq. (17) we get

$$P_{ee}=(1+\cos2\theta\cos2\theta_m)/2 \ ,$$

which is the well-known result.

In general, the angle β_f will be different from β_i. We will define another parameter γ by writing $\cos2\beta_f\equiv\cos2\gamma\cos2\beta_i$. The significance of γ can be seen by substituting this expression for $\cos2\beta_f$ into Eq. (17) to get

$$P_{ee}=[1+(1-2\sin^2\gamma)\cos2\theta\cos2\theta_m]/2 \ .$$

Comparing with the formula given by Parke,[6] we obtain the expression for the Landau-Zener[7] probability P_x which describes the transition between the mass levels:

$$P_x=\sin^2\gamma \ . \tag{18}$$

We have just seen that the extreme adiabatic case corresponds to $\beta_f=\beta_i$, so γ and P_x both vanish. This is the case if $A(t)$ is a slowly varying function of t.

In the other extreme when the transition is instantaneous,[8] $A(t)$ will be a step function assuming its initial value until $t=t_R$ when it drops abruptly to zero. In this case **B** will retain its original position at an angle $\beta_i=\theta_m$ from the $\hat{\mathbf{z}}$ axis for $t<t_R$, flip to $\mathbf{B}_0$ at $t=t_R$, and stay there afterwards. Hence we have the picture of **m** originally precessing about **B** and described by the equation

$$\mathbf{m}=\mathbf{k}\cos2\beta_i+\sin2\beta_i(\mathbf{i}\cos2\omega t+\mathbf{j}\sin2\omega t)$$

for $t<t_R$. At time t_R, $(\mathbf{i},\mathbf{j},\mathbf{k})$ will switch to align with $(\mathbf{i}_0,\mathbf{j}_0,\mathbf{k}_0)$. Right after, the **m** "sees" the new magnetic field $\mathbf{B}_0$ and precesses about it instead. The angle between **m** and $\mathbf{B}_0$ appropriately averaged over as described below, will give $2\beta_f$. Recalling that $\mathbf{k}_0$ is the unit vector along $\mathbf{B}_0$, we can see that the cosine of the angle included by **m** and $\mathbf{B}_0$ is simply given by $\mathbf{m}\cdot\mathbf{k}_0$. The average of this cosine will give $\cos2\beta_f$, and its evaluation is similar to the calculation of the average of $\mathbf{m}\cdot\hat{\mathbf{z}}$ earlier. Namely, from $\mathbf{k}_0=\mathbf{i}\sin2(\theta_m-\theta)+\mathbf{k}\cos2(\theta_m-\theta)$ and $\langle\cos2\omega t_R\rangle_{t_R}=0$ we get $\cos2\beta_f=\langle\mathbf{m}\cdot\mathbf{k}_0\rangle_{t_R}=\cos2(\theta_m-\theta)\cos2\beta_i$. Comparing with the definition of γ we readily obtain $\gamma=\theta_m-\theta$ and $P_x=\sin^2(\theta_m-\theta)$. In almost all cases θ_m is practically equal to $\pi/2$ and consequently $P_x=\cos^2\theta$.

In the more general case which is neither of the two extreme, we can evaluate β_f by solving the differential equation for β, at least in principle. To this end we note that since $\cos2\beta$ is given by $\mathbf{m}\cdot\mathbf{k}$, to get an expression for $d\beta/dt$ is to start with the equality

$$\frac{d}{dt}\cos2\beta=\frac{d}{dt}(\mathbf{m}\cdot\mathbf{k}) \ . \tag{19}$$

The left-hand side of the above is simply equal to $-2\sin2\beta(d\beta/dt)$. Furthermore, from Eqs. (8) and (14) we have

$$\frac{d}{dt}(\mathbf{m}\cdot\mathbf{k})=(\mathbf{m}\times\mathbf{B})\cdot\mathbf{k}-\omega_m\mathbf{m}\cdot\mathbf{i} \ . \tag{20}$$

Since **k** is taken to be parallel to **B** by definition, the first term $(\mathbf{m}\times\mathbf{B})\cdot\mathbf{k}$ vanishes identically. Also we have $\mathbf{m}\cdot\mathbf{i}=\sin2\beta\cos2\Phi$ from Eq. (15). Substituting all these back into Eq. (19) and taking out the common factor $\sin2\beta$ on both sides we get

$$\frac{d\beta}{dt}=\frac{\omega_m}{2}\cos2\Phi \ , \tag{21}$$

which describe the evolution of β in t. However, the right-hand side of this equation involves Φ whose time dependence we are yet to find out.

The time derivative of Φ can be obtained in a procedure similar to that outlined above by starting with the equality [see Eq. (15)]

$$\frac{d}{dt}(\sin2\beta\cos2\Phi)=\frac{d}{dt}(\mathbf{m}\cdot\mathbf{i}) \ . \tag{22}$$

The expansion of the left-hand side of this equation is straightforward:

$$\frac{d}{dt}(\sin2\beta\cos2\Phi)=-2\sin2\beta\sin2\Phi\frac{d\Phi}{dt}+\cos2\beta\cos2\Phi\frac{d\beta}{dt} \ . \tag{23}$$

On the other hand, the right-hand side of Eq. (22) can be evaluated from Eqs. (8) and (14) as

$$\frac{d}{dt}(\mathbf{m}\cdot\mathbf{i})=(\mathbf{m}\times\mathbf{B})\cdot\mathbf{i}+\omega_m\mathbf{m}\cdot\mathbf{k}\ . \tag{24}$$

The second term is just $\omega_m\cos2\beta$. The first term can be reduced by recalling that $\mathbf{B}=-\omega\mathbf{k}$, from which we have $(\mathbf{m}\times\mathbf{B})\cdot\mathbf{i}=-\omega(\mathbf{m}\times\mathbf{k})\cdot\mathbf{j}$. This can be further simplified as $-\omega\mathbf{m}\cdot(\mathbf{k}\times\mathbf{i})=-\omega\mathbf{m}\cdot\mathbf{j}$. Since $\mathbf{m}\cdot\mathbf{j}$ is just $\sin2\beta\sin2\Phi$, the first term of Eq. (24) is equal to $-\omega\sin2\beta\sin2\Phi$. Substituting Eqs. (21), (23), and (24) back to Eq. (22) we get

$$-2\sin2\beta\sin2\Phi\frac{d\Phi}{dt}+\omega_m\cos2\beta\cos^22\Phi=\omega_m\cos2\beta-\omega\sin2\beta\sin2\Phi\ . \tag{25}$$

After further reduction we finally arrive at

$$\frac{d\Phi}{dt}=\frac{\omega}{2}-\frac{\omega_m}{2}\sin2\Phi\cot2\beta\ . \tag{26}$$

Equations (21) and (26) furnish a coupled set of equations from which β and Φ can be solved.

We will conclude with a brief description on the generalization of the above formalism to the case of three neutrino flavors.[9,10] In place of Eq. (2) we have the following:

$$i\frac{d\psi}{dt}=-\frac{\boldsymbol{\lambda}}{2}\cdot\mathbf{B}\psi\ . \tag{27}$$

Here $\boldsymbol{\lambda}$ has as components the Gell-Mann SU(3) matrices, $\mathbf{B}$ is a vector in an eight-dimensional space, and $\psi=(\nu_e\ \ \nu_\mu\ \ \nu_\tau)^T$ is the normalized neutrino wave function. Defining λ_0 to be $\sqrt{2/3}I_3$, where I_3 is the 3×3 identity matrix, Eq. (6) will be modified to

$$\psi\psi^\dagger=\frac{\lambda_0}{\sqrt{6}}+\frac{\boldsymbol{\lambda}}{2}\cdot\mathbf{m}\ . \tag{28}$$

It can be verified that we always have $|\mathbf{m}|=2/\sqrt{3}$. Let $(\mathbf{e}_i)$, with i ranging from 1 to 8, be a set of unit coordinate vectors corresponding to the λ_i. Furthermore denote the vectors corresponding to ν_e,ν_μ, and ν_τ states by $\mathbf{u}_e,\mathbf{u}_\mu$, and $\mathbf{u}_\tau$, respectively. Then we have

$$\begin{aligned}\mathbf{u}_e&=\mathbf{e}_3+\frac{1}{\sqrt{3}}\mathbf{e}_8\ ,\\ \mathbf{u}_\mu&=-\mathbf{e}_3+\frac{1}{\sqrt{3}}\mathbf{e}_8\ ,\\ \mathbf{u}_\tau&=-\frac{2}{\sqrt{3}}\mathbf{e}_8\ .\end{aligned} \tag{29}$$

That is, the three vectors $\mathbf{u}_e$, $\mathbf{u}_\mu$, and $\mathbf{u}_\tau$ span an equilateral triangle. Proceeding in similar fashion as in the two-generation case we get the following equation, which is identical in form to Eq. (8) except that here $\mathbf{m}$ and $\mathbf{B}$ are eight-vectors:

$$\frac{d\mathbf{m}}{dt}=\mathbf{m}\times\mathbf{B}\ . \tag{30}$$

The cross product is defined by $(\mathbf{m}\times\mathbf{B})_i=f_{ijk}\mathbf{m}_j\mathbf{B}_k$, f_{ijk} being the structure constants of the SU(3) algebra in the λ basis. Again $\mathbf{B}$ is related to the vacuum field $\mathbf{B}_0$ by $\mathbf{B}=\mathbf{B}_0-(A/2E)\mathbf{u}_e$. Note that since $\mathbf{e}_3\times\mathbf{e}_8=0$ there will be no precession of the vector $\mathbf{m}$ with $\mathbf{m}(t=0)=\mathbf{u}_e$ if $\mathbf{B}$ lies on the $\mathbf{e}_3\times\mathbf{e}_8$ plane, which corresponds to the case when the mass-square matrix is diagonal. The probability P_{ee} is given by

$$P_{ee}=\tfrac{1}{3}+\tfrac{1}{2}\mathbf{m}\cdot\mathbf{u}_e\ , \tag{31}$$

which is similar to the two-generation case, except that the constant term is $\frac{1}{3}$ instead of $\frac{1}{2}$. This means that in the case of "maximal mixing" when $\mathbf{m}$ does not have any preferred orientation with respect to $\mathbf{u}_e$, $\mathbf{u}_\mu$, and $\mathbf{u}_\tau$ so that $\mathbf{m}\cdot\mathbf{u}_e$ averages to zero, we will have $P_{ee}=\frac{1}{3}$ as expected.

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