Light-front Hamiltonian approach to relativistic two- and three-body bound-state problems in $1+1$ dimensions

A. Harindranath and J. P. Vary
W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125 and Physics Department, Iowa State University, Ames, Iowa 50011
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We utilize the Hamiltonian approach to the light-front formulation of quantum field theory to study two- and three-body relativistic bound-state problems in a truncated Fock-space basis in $1+1$ dimensions. The problem is numerically solved by diagonalizing the invariant-mass operator in the truncated basis. We present results for binding energies, valence wave functions, and the momentum distribution functions. We discuss the advantages of the present technique over the usual integral-equation approach.

I. INTRODUCTION

Presently a great deal of effort is being devoted to the study of relativistic bound states. So far very little progress has been made in solving the problems exactly. In this regard the Bethe-Salpeter equation[1] (more precisely, the ladder approximation to the Bethe-Salpeter equation) has received much attention. In this work we study the two- and three-body bound-state problems in the Hamiltonian formulation of light-front field theory. We restrict our discussions to the theory in $1+1$ dimensions and we employ a discretized version of the theory.[2,3] We construct the light-cone ladder approximation by a truncation of the Fock-space basis. The problem is solved exactly in the truncated basis by diagonalizing the invariant-mass operator. Binding energies and wave functions are calculated as well as the valence and the nonvalence contributions to the momentum distribution functions. We compare and contrast the present approach with the usual integral-equation approach.

For simplicity we study the self-interacting scalar model. More specifically the interaction Lagrangian density is chosen to be $(\lambda/3!)\phi^3$. It is well known that the resulting Hamiltonian is unbounded and hence unsuitable as a model for realistic interactions.[4] On the other hand, the perturbation theory is well defined for this interaction.[5] By solving the theory exactly, i.e., by diagonalizing the invariant-mass operator in the "full" Fock space, we readily identify the pathology since the mass operator gives negative eigenvalues once the coupling is raised above a certain value. On the other hand, when treated within a truncated Fock-space basis, this interaction does not reveal any pathology and readily produces two- and three-body bound states in the weak and moderately strong-coupling regions.

The present work is motivated by the current interest in relativistic field-theory problems. We are interested in building our intuition of the relationships between interactions and the resulting spectra and wave functions.

This paper is organized as follows. In Sec. II we review the light-front quantization of scalar field models. Discretization is reviewed in Sec. III. The pathology of the model when solved in the full Fock-space basis is discussed in Sec. IV. In Sec. V we present and discuss the results for the two- and three-body states in the truncated Fock-space basis. Comparison with the integral-equation approach and our conclusions are presented in Sec. VI. The light-cone ladder approximation to the Bethe-Salpeter equation is reviewed in the Appendix.

II. REVIEW OF LIGHT-FRONT QUANTIZATION

We start from the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m^2\phi^2-\frac{\lambda}{3!}\phi^3 .$$

The equation of motion and the commutation relation are derived from Schwinger's action principle.[6]

In $1+1$ dimensions, the equation of motion is

$$\partial^+\partial^-\phi+m^2\phi+\frac{\lambda}{2!}\phi^2=0 .$$

Here

$$\partial^+=2\frac{d}{dx^-}$$

and

$$\partial^-=2\frac{d}{dx^+} ,$$

where

$$x^+=x^0+x^1$$

and

$$x^-=x^0-x^1 .$$

The metric tensor is given by $g^{++}=g^{--}=0$, $g^{+-}=g^{-+}=2$. The commutation relation is given by

$$[\phi(x^+,x^-),\partial^+\phi(x^+,y^-)]\,|_{x^+}=i\delta(y^--x^-) .$$

The stress tensor is given by

$$T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial^\nu\phi-g^{\mu\nu}\mathcal{L} .$$

Thus

$$T^{++}=\partial^+\phi\partial^+\phi$$

and

$$T^{+-}=m^2\phi^2+\frac{2\lambda}{3!}\phi^3 .$$

From the stress tensor $T^{\mu\nu}$ we construct the energy-momentum operator P^μ:

$$P^\mu=\tfrac{1}{2}\int dx^- T^{+\mu} .$$

In terms of free field annihilation and creation operators a and $a^\dagger$, the free field solution can be written as

$$\phi_0(x^+,x^-)=\frac{1}{2\pi}\int\frac{dk^+}{2k^+}[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}] .$$

We choose the interacting field ϕ to coincide at $x^+=0$ with the free field solution. Then the Hamiltonian operator P^- in the continuum version is given by

$$P^-=P_0^-+V^- ,$$

where

$$P_0^-=\int\frac{dk^+}{2\pi 2k^+}a^\dagger(k^+)a(k^+)\frac{m^2}{k^+} ,$$

$$V^-=\frac{\lambda}{2}\int[dk]\frac{1}{k_1^++k_2^+}[a^\dagger(k_1^++k_2^+)a(k_1^+)a(k_2^+)+a^\dagger(k_1^+)a^\dagger(k_2^+)a(k_1^++k_2^+)] ,$$

where

$$[dk]=\frac{dk_1^+}{2\pi 2k_1^+}\frac{dk_2^+}{2\pi 2k_2^+} .$$

The momentum operator is given by

$$P^+=\int\frac{dk^+}{2\pi 2k^+}k^+a^\dagger(k^+)a(k^+) .$$

We shall return to the continuum version in the Appendix where we discuss the relationship of our method to the Bethe-Salpeter equation in the ladder approximation.

III. DISCRETIZATION

In this section, following the conventions of Ref. 2, we construct the light-front momentum and Hamiltonian operator in the discretized version. Discretization is introduced by the replacement

$$k^+\to k_n^+=\frac{2\pi}{L}n,\quad n=1,2,3,\ldots,\Lambda .$$

It is convenient to introduce the dimensionless variable

$$\xi=\frac{\pi x^-}{L} .$$

Then

$$\tfrac{1}{2}k^+x^-=n\xi .$$

The interacting field ϕ at $x^+=0$ is given by

$$\phi(\xi)=\frac{1}{\sqrt{4\pi}}\sum_1^\Lambda\frac{1}{\sqrt{n}}(a_ne^{-in\xi}+a_n^\dagger e^{in\xi}) .$$

a_n and $a_m^\dagger$ obey the commutation relation

$$[a_n,a_m^\dagger]=\delta_{n,m} .$$

One also introduces operators K and H such that

$$P^+=\frac{2\pi}{L}K$$

and

$$P^-=\frac{L}{2\pi}H .$$

Thus K is the dimensionless momentum operator and H is the Hamiltonian operator with dimensions of mass squared. In the discretized version, the momentum operator K and the Hamiltonian operator H are given by

$$K=\sum_n na_n^\dagger a_n$$

and

$$H=H_0+H_1 ,$$

where

$$H_0=\sum_n\frac{1}{n}a_n^\dagger a_nm^2 ,$$

$$H_1=\frac{\lambda}{\sqrt{4\pi}}\frac{1}{2}\sum_{klm}\left[\frac{a_k^\dagger a_la_m+a_m^\dagger a_l^\dagger a_k}{\sqrt{klm}}\right]\delta_{l+m,k} .$$

IV. SOLUTION IN THE FULL FOCK-SPACE BASIS

In this section we discuss the solution in the full Fock-space basis and the manifestation of the pathology of the interaction under study.

We adopt the notation that the states are identified by their $\lambda=0$ structure. We present the mass gap as a function of K for different values of coupling in Fig. 1. Here we have chosen the mass parameter in the Lagrangian $m=1.0$. As has been discussed before,[2,3,7] $K\rightarrow\infty$ gives the continuum limit. For fixed coupling one detects the pathology of the ϕ^3 model by observing that the square of the mass gap becomes negative with increasing K. Small values of K are in fact sufficient to detect this pathology by taking a large value of the coupling. For example, at $\lambda=7.09$, the square of the mass gap becomes negative already at $K=2$. As is obvious we have to go to higher values of K to detect the pathology for smaller values of the coupling.

It should also be noted that in this calculation the mass gap is defined with respect to the perturbative vacuum. To solve a field theory one should first determine the dynamical vacuum. In the light-front scheme, however, if we neglect $k^+=0$ states the vacuum decouples from all the other states. Thus the instability of the ground state can manifest itself only indirectly. This also indicates the dangers of introducing a fixed mass gap with respect to the perturbative vacuum.

For the problem we consider, the truncation of the basis states yields bound states whose behavior converges well for increasing K.

V. THE TWO- AND THREE-BODY PROBLEM IN THE TRUNCATED SECTOR

For the study of the relativistic bound-state problem, one approach is to solve the Bethe-Salpeter equation. However, in practice, one solves this equation by invoking one or more approximations. Usually one adopts the ladder approximation which is the lowest-order approximation. Second, one converts the four-dimensional equation to a three-dimensional one by means of either covariant or noncovariant reductions. Thus whatever confidence we have in the resulting solutions is limited to the weak-coupling limit. Further, one may question the reliability of the whole scheme when the exact solution differs drastically from the approximate one even in the weak-binding case. In this work we set aside all such well-founded worries. One of our aims is to show the utility of the Hamiltonian matrix diagonalization as opposed to the integral equation approach in solving the approximated problem. We are particularly interested in solving this model in order to develop some intuition for the relative contribution of nonvalence states to the momentum distribution function as we move from weak to moderately strong couplings.

The utility of the light-front scheme (more precisely, the infinite-momentum approach) to the study of the two-body scattering equation was first discussed by Weinberg[8] in the case of ϕ^3 interaction in $3+1$ dimensions. For the model of two massive scalar particles exchanging a massless scalar particle, Feldman, Fulton, and Townsend[9] calculated the eigenvalues of the integral equation for the bound state (in the infinite-momentum frame) in the weak-coupling limit. The relation between the light-cone bound-state equation and the Bethe-Salpeter equation has been discussed by Brodsky, Ji, and Sawicki.[10] For recent work on the light-cone ladder equation see Ji and Furnstahl[11] and Ji.[12] For related work of interest, see Celenza, Ji, and Shakin.[13] In this work we restrict our discussion to $1+1$ dimensions but we expect the general features of our results to persist into higher dimensions.

First we present our results for the two-particle bound state. We consider the mass operator squared M^2 in the discretized version and diagonalize this operator in the restricted Fock space of two and three particles for different values of the coupling. The mass of the lowest state as a function of even values of K is calculated and extrapolated to large K in order to obtain an estimate of the invariant mass of this state. For $\lambda=1.0$ and 3.0, the invariant mass is estimated to be 1.94 and 1.29, respectively. For the values of the coupling constants we consider here, the results we present appeared to be well converged at $K=30$.

Next let us consider the wave function and the momentum distribution function for these states. The state vector $|\Psi\rangle$ is given by

$$|\Psi\rangle=\sum_j C_j\,|\phi_j\rangle\ ,$$

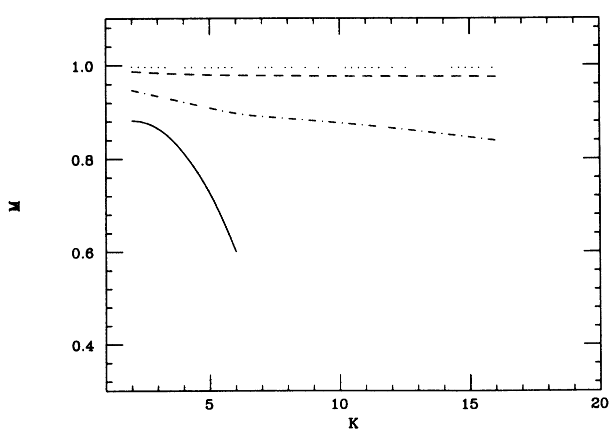

FIG. 1. The mass gap as a function of K for different values of the coupling constant λ. m is chosen to be 1.0. Smooth lines are drawn through results obtained at even values of K. The curves correspond to the following values of λ: dotted (0.5), dashed (1.0), dotted dashed (2.0), and solid (3.0).

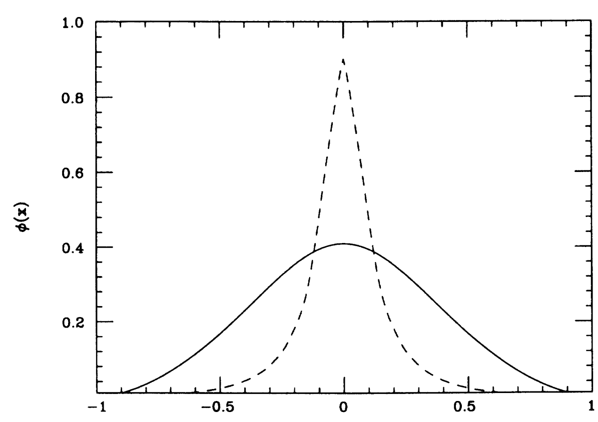

FIG. 2. The two-particle amplitude of the two-body wave function at $K=30$ as a function of $x=x_1-x_2$. Dashed and solid lines correspond to $\lambda=1.0$ and 3.0, respectively.

where the $|\phi_j\rangle$ are the Fock-space basis functions and the coefficients C_j's result from matrix diagonalization. Let us introduce the variable x_i which denotes the light-cone momentum fraction carried by the ith constituent: $x_i = k_i^+/K$. For simplicity we present the two-particle (valence) component of the many-body wave function for the two-particle state. This amplitude is defined by

$$\phi(x_1,x_2) = \langle 0 | a(x_1) a(x_2) | \Psi \rangle ,$$

i.e.,

$$= \sum_j C_j \langle 0 | a(x_1) a(x_2) | \phi_j \rangle .$$

This amplitude is shown in Fig. 2 for coupling strengths $\lambda = 1.0$ and 3.0 at $K = 30$ as a function of the relative momentum fraction $x = x_1 - x_2$. As expected this amplitude is broader for a strongly coupled state as compared to a weakly coupled state. As $\lambda \to 0$ we expect this amplitude to approach a δ function at $x = 0$.

Next we look at the momentum distribution function for the bound states as the coupling goes from weak to strong. The momentum distribution function $F(x)$ is the probability density to find a parton in the bound state with momentum fraction between x and $x + dx$. Thus

$$F(x)\Delta x = \sum_j C_j^2 \langle \phi_j | a_n^\dagger a_n | \phi_j \rangle \frac{\Delta n}{N_j} ,$$

where N_j is the number of particles in the Fock-space component $|\phi_j\rangle$ and $x = n/K$. Thus

$$F(x)\Delta x = K \sum_j C_j^2 \langle \phi_j | a_n^\dagger a_n | \phi_j \rangle \frac{\Delta x}{N_j} .$$

This choice of the momentum distribution function leads to unit normalization which is written in the continuum limit as

$$\int_0^1 F(x) dx = 1 .$$

The momentum distribution function is plotted in Figs. 3(a) and 3(b) for coupling strengths $\lambda = 1.0$ and 3.0, respectively. The function $F(x)$ is peaked around $x = 0.5$, irrespective of the value of the coupling. The three-particle Fock space or "nonvalence" contribution to $F(x)$ is significant only in the region $x < 0.4$. Also the contribution due to the nonvalence component increases as coupling increases. For example, for $\lambda = 1.0$ the nonvalence states contribute about 2.4% to the normalization integral, whereas for $\lambda = 3.0$ this contribution is about 17.2%.

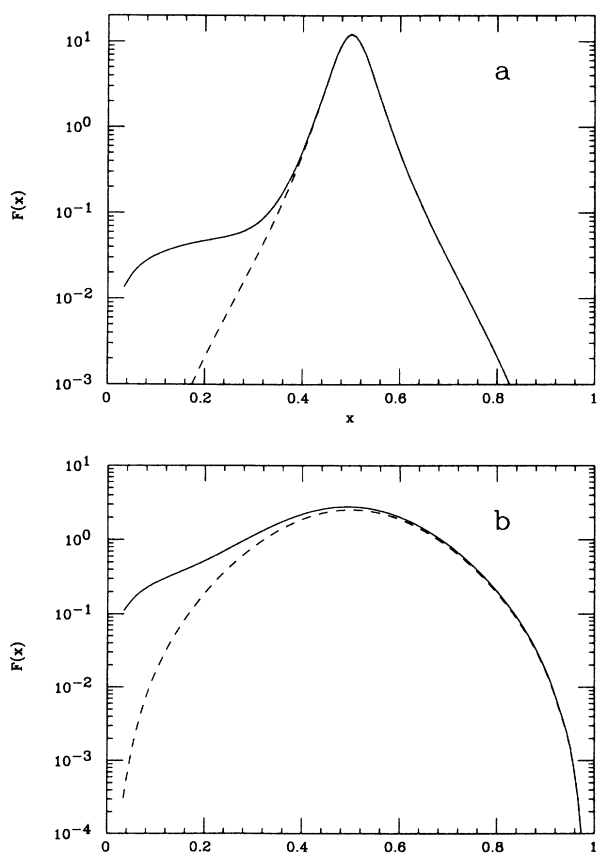

FIG. 3. The momentum distribution function for the two-body state at $K = 30$ as a function of the light-cone momentum fraction x. Dashed line denotes the valence and solid line denotes the full distribution. (a) $\lambda = 1.0$, (b) $\lambda = 3.0$.

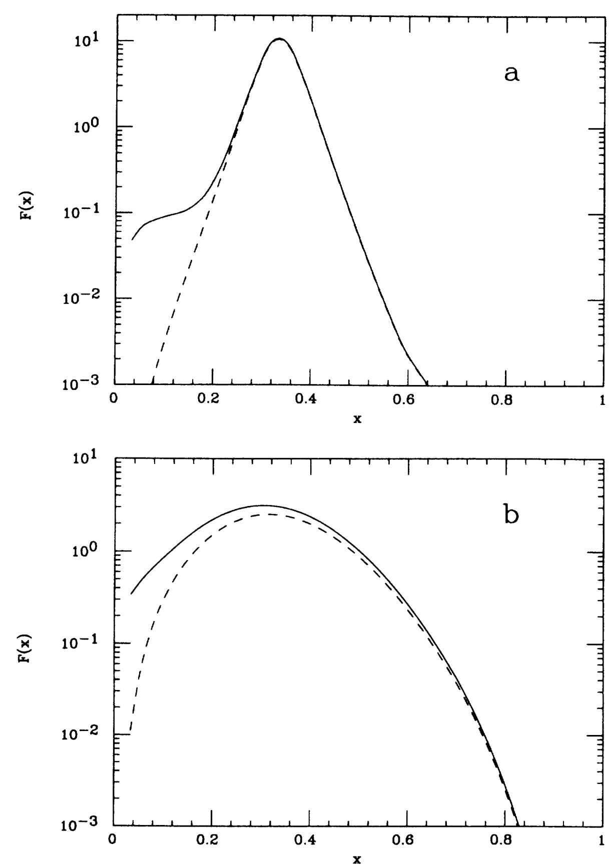

FIG. 4. Same as in Fig. 3 but for the three-body state.

Next let us consider the three-particle bound state. Here we diagonalize the mass operator in the restricted space of three and four particles for K values which are integer multiples of 3. For $\lambda=1.0$ and 3.0, the invariant mass of the lowest state in the continuum limit is estimated to be 2.90 and 1.47, respectively. The associated momentum distribution functions evaluated at $K=30$ are shown in Figs. 4(a) and 4(b), respectively. Here as may be expected they peak about $x=\frac{1}{3}$. The nonvalence contribution again dominates the low-x behavior. The nonvalence contribution to the normalization integral is 5.3% and 25.6% for $\lambda=1.0$ and 3.0, respectively.

VI. DISCUSSION AND CONCLUSIONS

Using an interaction which is attractive in the two-body sector we are able to study the two- and three-body relativistic bound states in the discretized version of the light-front Hamiltonian approach. One should note that since we are dealing with a relativistic field theory, the qualifiers "two body" and "three body" only refer to the composition of the state at zero coupling. Once we make restrictions on the allowed Fock-space states for a given value of the mass parameter m and coupling λ we are able to extract the invariant mass of the state without further approximations. We are driven to the restriction on Fock-space states only because we are dealing with an unphysical interaction. In general we need not put any restrictions on the Fock-space states. The valence and the nonvalence states are visible at each stage of the calculation, which enables us to separate out the contribution of these states to the momentum distribution function. We are unable to study the very-strong-coupling limit only because of the present choice of the unphysical interaction.

We may compare our approach with the conventional integral-equation approach (see the Appendix). For the two-body problem in the latter method one eliminates the three-body wave function in terms of the kernel and the two-body wave function to arrive at an integral equation. Thus the information on the nonvalence component of the wave function is no longer explicit. For the three-body problem, one has to solve a multidimensional integral equation. In the present method, however, one evaluates the matrix elements of the Hamiltonian and solves by matrix diagonalization. The limit of our approach is governed by the size of the Hamiltonian matrix required to obtain a good estimate of the continuum limit.

In this work we have dealt with a toy model which has its own inherent limitations. We note that this method has been applied to a gauge theory[3] before but the role of higher Fock-space states has not been discussed. For pedagogical and practical reasons one eventually hopes to calculate the structure function of multiparticle states in models involving fermions.

ACKNOWLEDGMENTS

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APPENDIX

The derivation of the integral equation for the bound state starting from the light-front Hamiltonian has been discussed before by Brodsky and collaborators.[10,14] For completeness of discussion we outline this derivation here.

The state vector $|\psi\rangle$ for the two-particle bound state obeys the eigenvalue equation

$$P^{+}P^{-}|\psi\rangle=M^{2}|\psi\rangle .$$

Here P^{+} is the light-front momentum, P^{-} is the light-front Hamiltonian, and M is the invariant mass of the bound state. We choose $P^{+}=1$. Here $P^{-}=P_0^{-}+V^{-}$ Then we have

$$(M^{2}-P_0^{-})|\psi\rangle=V^{-}|\psi\rangle .$$

We expand the state vector $|\psi\rangle$ in terms of Fock-space states:

$$|\psi\rangle=\int\frac{dk_1'^{+}}{2\pi 2k_1'^{+}}\frac{dk_2'^{+}}{2\pi 2k_2'^{+}}|k_1'^{+},k_2'^{+}\rangle\psi(k_1'^{+},k_2'^{+})\delta(1-k_1'^{+}-k_2'^{+})$$

$$+\int\frac{dk_1'^{+}}{2\pi 2k_1'^{+}}\frac{dk_2'^{+}}{2\pi 2k_2'^{+}}\frac{dk_3'^{+}}{2\pi 2k_3'^{+}}|k_1'^{+},k_2'^{+},k_3'^{+}\rangle\psi(k_1'^{+},k_2'^{+},k_3'^{+})\delta(1-k_1'^{+}-k_2'^{+}-k_3'^{+})+\cdots .$$

Here $\psi(k_1'^{+},k_2'^{+})$ is the valence wave function of the two-body bound state and so on. We neglect the higher Fock-space components in the following and utilize the expressions for P_0^{-} and V^{-} given in Sec. II.

Multiplying the wave equation from the left by $\langle k_1^{+},k_2^{+}|$ we arrive at

$$\left[M^{2}-\frac{m^{2}}{k_1^{+}}-\frac{m^{2}}{k_2^{+}}\right]\psi(k_1^{+},k_2^{+})=\int\frac{dk_1'^{+}}{2\pi 2k_1'^{+}}\frac{dk_2'^{+}}{2\pi 2k_2'^{+}}\frac{dk_3'^{+}}{2\pi 2k_3'^{+}}$$

$$\times\langle k_1^{+},k_2^{+}|V^{-}|k_1'^{+},k_2'^{+},k_3'^{+}\rangle\psi(k_1'^{+},k_2'^{+},k_3'^{+})\delta(1-k_1'^{+}-k_2'^{+}-k_3'^{+}) .$$

Multiplying the wave equation from the left by $\langle k_1^+,k_2^+,k_3^+ \mid$ we arrive at

$$\left[M^2-\frac{m^2}{k_1^+}-\frac{m^2}{k_2^+}-\frac{m^2}{k_3^+}\right]\psi(k_1^+,k_2^+,k_3^+)=\int\frac{dk_4^+}{2\pi 2k_4^+}\frac{dk_5^+}{2\pi 2k_5^+}\times\langle k_1^+,k_2^+,k_3^+ \mid V^- \mid k_4^+,k_5^+\rangle\psi(k_4^+,k_5^+)\delta(1-k_4^+-k_5^+)\,.$$

Eliminating the "nonvalence" wave function using the above equation we arrive at the light-cone ladder equation

$$\left[M^2-\frac{m^2}{x_1}-\frac{m^2}{x_2}\right]\psi(x_1,x_2)=\frac{\lambda^2}{4\pi}\int\frac{dx_3}{x_3}\frac{dx_4}{x_4}\delta(1-x_3-x_4)\psi(x_3,x_4)K(M^2,m^2,x_1,x_2,x_3,x_4)\,,$$

where

$$K=\frac{\theta(x_1-x_3)}{x_1-x_3}\frac{1}{M^2-\dfrac{m^2}{x_3}-\dfrac{m^2}{x_2}-\dfrac{m^2}{x_1-x_3}}+\frac{\theta(x_2-x_4)}{x_2-x_4}\frac{1}{M^2-\dfrac{m^2}{x_4}-\dfrac{m^2}{x_1}-\dfrac{m^2}{x_2-x_4}}\,.$$

Here $x_i=k_i^+/P^+$.

The three-dimensional version of this equation has been discussed in Refs. 9–14.

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