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Dual long-distance QCD

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We propose an explicit form for the long-range limit of the SU(N) Yang-Mills Lagrangian expressed as a function of the dual (color-electric) vector potentials. While we cannot rigorously derive the Lagrangian, it can be made plausible that it follows from conventional Yang-Mills theory. This dual long-distance QCD Lagrangian has many of the properties of a magnetic superconductor. It has classical solutions corresponding to confined tubes of quantized electric color flux which result from a dual Meissner effect. However, the confining pressure is not produced by a scalar Higgs field, as in ordinary superconductivity, but by a magnetic condensate field which arises naturally from the nonlocal form of the dual Lagrangian. Within the classical approximation, we find the explicit distribution of color fields surrounding a flux tube. Semiclassical quantization around this solution can be expected to yield the QCD string, and the semiclassical expansion parameter is $1/N$, where N is the number of colors.

I. INTRODUCTION

Years ago 't Hooft[1] and Mandelstam[2] pointed out that many features of color confinement could be understood if continuum Yang-Mills theory possessed some of the properties of a magnetic superconductor. Since then this qualitative picture has gained considerable acceptance, but little or no progress has been made in either demonstrating that Yang-Mills theory in fact is like a magnetic superconductor, or in using the idea to compute any quantities of physical interest. Indeed, the only substantial progress toward understanding confinement has been made in the context of lattice Yang-Mills theory, where the natural variables are gauge-invariant Wilson loops rather than the vector potentials A^a_μ of the continuum theory, and where the physical picture of the Yang-Mills vacuum is difficult to see.

If the magnetic superconductor picture really is relevant to continuum Yang-Mills theory one should expect that dual vector potentials C^a_μ (i.e., electric vector potentials), instead of the A^a_μ, are the natural variables to use in the confining regime. Our principal purpose in this paper is to explicitly construct the Yang-Mills Lagrangian as a function of the dual potentials in the long-distance limit. The Lagrangian which we obtain does indeed describe something very like a magnetic superconductor: there is a spontaneous symmetry breaking leading to a nonperturbative vacuum in which color-electric fields can exist only in tubes of quantized flux.

But while many of the features of Yang-Mills theory are just like those of a magnetic superconductor, there are also essential differences. Yang-Mills theory contains no scalar fields; the nonperturbative vacuum is characterized by a nonvanishing magnetic gluon condensate instead of by the expectation value of a scalar Higgs field. The gluon condensate arises from a new set of dual tensor fields $\tilde{F}^a_{\mu\nu}$ which automatically appear because the Yang-Mills Lagrangian, as a function of C^a_μ, is nonlocal.

The dual Lagrangian depends on two parameters, one of which can be taken to be a dimensionless condensate strength b^2 and the other a mass scale, which we call M/g, specifying the long-range regime.[3] These parameters can be determined from the experimental values of the string tension and of G_2, the gluon condensate. Once they are fixed, other quantities, such as the flux-tube radius, the glueball mass, and the shape of the static quark-antiquark potential can be predicted.

These results are obtained in the classical approximation to the dual Yang-Mills theory. The semiclassical expansion about the classical flux tube has not yet been worked out in detail, but we expect that it will lead to a string theory with linearly rising Regge trajectories.[4]

While our dual Lagrangian describes Yang-Mills theory only in the long-range confining regime, it appears to define a renormalizable field theory at all ranges, which is perhaps asymptotically free at short distances. This has only been partially checked in the one-loop approximation, and the quantum aspects of the theory have not yet been systematically treated.

Some of the topics we treat here have been partially described (though with significant differences) in previous publications,[5–7] but we have nevertheless elected to make this paper essentially self-contained at the risk of a small amount of repetition.

Section II of this paper begins with a brief review of how dual potentials can be defined in a non-Abelian gauge theory.[8] It is not possible to express them explicitly in terms of the ordinary potentials; therefore the explicit form of the exact Yang-Mills Lagrangian as a function of the C_μ^a is unknown. Nevertheless it is possible to show that this Lagrangian is invariant under dual gauge transformations[8] (the magnetic gauge group) and to argue that in the long-distance confining regime it can be explicitly constructed. The construction of this long-distance limit, denoted by $\mathcal{L}(C)$, is based on the fact that in a confining theory the dual Wilson loop obeys a perimeter law[9] so that the dual potentials have only short-range correlations, which in turn implies that the dual potential propagator has a mass. The quadratic part $\mathcal{L}^{(0)}(C)$ of the dual Lagrangian is thereby determined. Since the C fields are small at large distances, the minimal extension of $\mathcal{L}^{(0)}(C)$ necessary to make it compatible with dual gauge invariance suffices to specify uniquely $\mathcal{L}(C)$ at long range.

Confinement, however, should be a consequence of Yang-Mills theory and not an extra assumption. Therefore it should be possible to derive $\mathcal{L}^{(0)}(C)$ directly from the ordinary Yang-Mills Lagrangian in the A language. Since $\mathcal{L}^{(0)}(C)$ is quadratic in C, it is also Abelian, and for an Abelian theory the translation between A and C is easy. The Lagrangian corresponding to $\mathcal{L}^{(0)}(C)$ in the A language turns out to describe an ordinary gluon with a propagator $\Delta_A^{(0)}$ behaving in momentum space as $1/q^4$ as $q^2 \to 0$. This behavior is known to arise from solving a truncated set of the Dyson equations and Ward identities of conventional A language Yang-Mills theory.[10,11] The $1/q^4$ behavior is a consequence of the resulting self-consistency which eliminates the perturbative Landau ghost pole in the gluon propagator at spacelike momentum.

$\Delta_A^{(0)}$, however, cannot be used as a starting point for any long-distance iteration scheme in the A language, since higher-order terms have low-momentum singularities even stronger than $1/q^4$. In the C language, the reverse is true. The propagator $\Delta_C^{(0)}$, which is the translation of $\Delta_A^{(0)}$, is smooth at long range, and describes the same physics as $\Delta_A^{(0)}$. From $\Delta_C^{(0)}$ we construct $\mathcal{L}^{(0)}(C)$ and then, by dual gauge invariance, $\mathcal{L}(C)$. This constitutes a convergent long-range expansion starting with $\Delta_C^{(0)}$. It is important to emphasize that $\mathcal{L}(C)$ has no simple expansion as a function of A: the exact A propagator does not behave as $1/q^4$ (we do not know how it behaves, or even if it exists). The exact C propagator, in contrast, does behave as $\Delta_C^{(0)}$, which confirms our statement that higher-order corrections in C are small at long range.

In summary, we can derive $\mathcal{L}(C)$ directly from Yang-Mills theory without assuming confinement if the dual version of the solution of the truncated Dyson equation/Ward identity system is taken as the starting point for the long-range expansion in the dual language.

The long-range dual Lagrangian we obtain in Sec. II is nonlocal. In Sec. III we express it in local form. This introduces a set of tensor fields $\hat{F}_{\mu\nu}^a$, as well as new ghost fields in addition to the usual Faddeev-Popov ghosts. Renormalizability of $\mathcal{L}(C)$ requires the introduction of two counterterms proportional to $\tilde{F}^2$ and $\tilde{F}^4$, which we denote by $-W$, and we write down the explicit form of W for SU(N) gauge theory.

Section IV is devoted to showing that the equations of motion following from $\mathcal{L}(C)$ have classical solutions describing static cylindrically symmetric tubes of color electric flux quantized in units of e/N, for SU(N), provided that the function W has a nontrivial minimum as a function of $\tilde{F}_{\mu\nu}$. (e is the Yang-Mills coupling constant.) The value $\tilde{F}_0^2$ of $\tilde{F}^2$ at the minimum gives the gluon condensate in the classical approximation and $W(\tilde{F}_0)$ is the nonperturbative vacuum energy density. The coefficient of the quadratic term in W is given by the trace anomaly, and that of the quartic term is determined by the gluon condensate G_2. In order for flux-tube solutions to exist it is necessary that $G_2 > 0$, and therefore the nonperturbative vacuum must be a magnetic, not an electric, condensate.

A number of physically interesting quantities, such as the string tension, the flux-tube radius, and the 0^{++} glueball mass are estimated for SU(N) gauge theory in Sec. V. We also, in this section, study the large-N limit, from which we recover the usual results of the large-N diagrammatic analysis of Yang-Mills theory. In particular, as $N \to \infty$, the flux tube becomes a string. Furthermore, the semiclassical expansion is the same as the large-N expansion, and the semiclassical expansion parameter is $1/N$.

The asymptotic fields, in space and color, far from an SU(N) flux tube are explicitly constructed in Sec. VI. We note in Sec. VII than the electric field $\tilde{F}_{ij}$ does not vanish exponentially asymptotically unless an additional term is added to W. Such a term is in fact present as a quantum correction to the classical approximation, though since it is not divergent it is not required by renormalizability as a counterterm.

We next arrive at Sec. VIII, where the classical flux-tube equations are solved explicitly, at all distances, for SU(2). The string tension is calculated as a function of $\tilde{F}_0^2$. The value of M/g is fixed in terms of the experimentally measured string tension and the gluon condensate. We then predict the flux-tube radius.

In the ninth and last section we briefly discuss the static potential $V(R)$ between a heavy quark and antiquark. This requires the introduction of quark sources into $\mathcal{L}(C)$. The resulting equations are too complicated to solve exactly; however we can calculate $V(R)$ in a magnetic superconductor approximation to $\mathcal{L}(C)$. This yields a Coulomb potential at short range with a rapid transition to a linear potential at long range. We also calculate $V(R)$ using a baglike approximation to the field equations. This potential agrees well with that obtained in the magnetic superconductor approximation provided the string tension is taken as a free parameter. Finally we note that making the same baglike approximation to the equations of the magnetic superconductor approximation to $\mathcal{L}(C)$ yields the equations for the static potential in the MIT bag model.[12]

In Appendix A we give more details of how $\mathcal{L}(C)$ is obtained from Yang-Mills theory.

II. CONSTRUCTION OF THE DUAL YANG-MILLS LAGRANGIAN

The Yang-Mills Lagrangian is

$$\mathcal{L}_{\rm YM}=2\,{\rm Tr}(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu})\ , \tag{2.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ie[A_\mu,A_\nu] \tag{2.2}$$

and

$$A_\mu(x)=\sum_a A_\mu^a(x)T_a\ . \tag{2.3}$$

The $A_\mu^a(x)$ are the Yang-Mills fields and T_a are the generators of the color gauge group in the fundamental representation. They satisfy the commutation relations

$$[T_a,T_b]=if_{abc}T_c\ , \tag{2.4}$$

and are normalized so that

$$2\,{\rm Tr}T_aT_b=\delta_{ab}\ . \tag{2.5}$$

We denote the Yang-Mills coupling constant by e; that is,

$$\alpha_s=e^2/4\pi\ . \tag{2.6}$$

$\mathcal{L}_{\rm YM}$ is the minimal Lagrangian invariant under the non-Abelian gauge transformation

$$A_\mu(x)\to\Omega^{-1}(x)A_\mu(x)\Omega(x)+\frac{i}{e}\Omega^{-1}(x)\partial_\mu\Omega(x)\ , \tag{2.7}$$

where $\Omega(x)$ is an element of the color gauge group.

In this section we will first review the definition of electric vector potentials C_μ given by Mandelstam.[8] We then explain why they should be useful for studying a confining theory and finally how the long-distance Lagrangian can be constructed as a function of C_μ from the dynamics of Yang-Mills theory.

To study confinement it is convenient to introduce the Wilson loop operator $W_A(l)$:

$$W_A(l)\equiv{\rm Tr}P\exp\left[ie\oint_l dx_\mu A^\mu(x)\right]\ , \tag{2.8}$$

where l is any closed curve and the symbol P represents path ordering. Let $|0\rangle$ be the vacuum state. If Yang-Mills theory confines, then, for large loops,

$$\langle 0|W_A(l)|0\rangle\sim e^{i\,{\rm area}(l)}\ , \tag{2.9}$$

where area(l) is the area enclosed by the loop l.

The operator $W_A(l)$ is gauge invariant. One may therefore define a gauge-invariant magnetic flux $\Phi_M(l)$ passing through the loop by the equation

$$W_A(l)\equiv\exp\left[\frac{2\pi i}{g}\Phi_M(l)\right]\ , \tag{2.10}$$

where

$$g\equiv2\pi/e\ . \tag{2.11}$$

We call g the magnetic Yang-Mills coupling constant. Evidently in an Abelian gauge theory Φ_M coincides with the usual definition of magnetic flux.

In Abelian theories, one can define a dual Wilson loop $W_C(l)$ in terms of the electric flux $\Phi_E(l)$ passing through the loop by the equation

$$W_C(l)\equiv\exp\left[\frac{2\pi i}{e}\Phi_E(l)\right]\ . \tag{2.12}$$

The electric flux $\Phi_E(l)$ in turn is defined in terms of the electric displacement vector $\mathbf{D}$ as

$$\Phi_E(l)=\int_S d\mathbf{S}\cdot\mathbf{D}\ , \tag{2.13}$$

where S is any surface bounded by the loop l. The electric vector potential C_μ is defined by writing the solution of the two Maxwell equations

$$\nabla\cdot\mathbf{D}=0 \tag{2.14a}$$

and

$$\nabla\times\mathbf{H}-\partial_0\mathbf{D}=0\ , \tag{2.14b}$$

as

$$\mathbf{D}=-\nabla\times\mathbf{C} \tag{2.15a}$$

and

$$\mathbf{H}=-\partial_0\mathbf{C}-\nabla C_0\ . \tag{2.15b}$$

Then combining Eqs. (2.12), (2.13), and (2.15), we obtain

$$W_C(l)=\exp\left[ig\oint_l dx_\mu C^\mu(x)\right]\ , \tag{2.16}$$

which is the dual of Eq. (2.8) for an Abelian theory.

In the non-Abelian case we cannot introduce electric vector potentials via (2.14) and (2.15). Nevertheless, 't Hooft[9] was able to define the operator $W_C(l)$ in SU(N) gauge theory by specifying its commutation relations with $W_A(l)$ as a natural generalization of the corresponding commutation relations in the Abelian theory. Subsequently Mandelstam[8] used the non-Abelian generalization of (2.16),

$$W_C(l)={\rm Tr}P\exp\left[ig\oint_l dx_\mu C^\mu(x)\right]\ , \tag{2.17}$$

to define $C_\mu(x)$. The dual Wilson loops $W_C(l)$ are invariant under gauge transformations:

$$C_\mu(x)\to\Omega_M^{-1}(x)C_\mu(x)\Omega_M(x)+\frac{i}{g}\Omega_M^{-1}(x)\partial_\mu\Omega_M(x)\ . \tag{2.18}$$

Mandelstam called the group of transformations (2.18) the magnetic-color gauge group to distinguish it from the usual electric-color gauge group, under which A_μ transforms according to (2.7). Note that the coupling constant g determining the transformations of the magnetic-color gauge group is inversely proportional to the Yang-Mills constant e. Any physical quantity, when expressed in terms of C_μ, must be invariant under the magnetic gauge group, just as it is under the electric gauge group when expressed in terms of A_μ (Ref. 13).

From a kinematical point of view Yang-Mills theory is symmetric between electric and magnetic quantities. However, there is no simple relation between C_μ and A_μ in non-Abelian gauge theory, in contrast with Abelian theory. Thus there is no dynamical electric-magnetic symmetry and although the C_μ Lagrangian is invariant under the magnetic gauge group, its explicit dependence upon the fields C_μ is in general unknown. Indeed we should expect it to be a very complicated function of C_μ since $\mathcal{L}_{\rm YM}(A)$ is a simple function of A_μ and the relationship between A_μ and C_μ is indirect and evidently complicated. Nevertheless we will argue that the long-range limit of the C_μ Lagrangian can be explicitly constructed.

We begin by recalling the following result of 't Hooft.[9] Using the definition of $W_C(l)$ via its commutation relation with $W_A(l)$, he showed that if $W_A(l)$ obeys the area law, then $W_C(l)$ necessarily satisfies a perimeter law; i.e., for large loops

$$W_C(l)\sim e^{i\,\text{perimeter}(l)} . \tag{2.19}$$

This perimeter law arises naturally if the dual potentials C_μ are weakly coupled at large separations. For, suppose that the long-distance behavior of the dual propagator

$$\Delta^{ab}_{C\mu\nu}(x,y)\equiv\langle 0\,|\,[C^a_\mu(x)C^b_\nu(y)]_+\,|\,0\rangle , \tag{2.20}$$

is that of a free massive particle, i.e., suppose that for small q^2 its Fourier transform $\Delta_C(q)$ has the behavior

$$\Delta_C(q)\sim\frac{1}{q^2-M^2} , \tag{2.21}$$

where M is some nonvanishing mass. Suppose further that the higher-order dual Green's functions have a corresponding weak-coupling long-distance behavior. Then the dual loop $W_C(l)$ will obey Eq. (2.19), just as for any weakly coupled massive field theory.

't Hooft's result is a precise way of saying that the vacuum of a confining theory has the properties of a magnetic superconductor. In an ordinary superconductor there are vortices of confined magnetic flux $\Phi_M=\int d\mathbf{S}\cdot\mathbf{B}$ producing a linear confining potential between monopoles and antimonopoles. The magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ falls exponentially with the distance of penetration into the interior of the superconducting region and the magnetic vector potential $\mathbf{A}$ thus acquires a mass. In a magnetic superconductor, correspondingly, there are tubes of quantized electric flux, $\Phi_E=\int d\mathbf{S}\cdot\mathbf{D}$, and consequently a linear potential between quarks and antiquarks. The electric displacement $\mathbf{D}=-\nabla\times\mathbf{C}$ penetrates only exponentially into the magnetic superconductor, and the dual potential $\mathbf{C}$ acquires a mass; in other words, the dual propagator has the long-distance behavior given by Eq. (2.21).

This suggests that electric vector potentials C_μ are the natural variables for describing long-distance Yang-Mills theory. Let us denote the part of the C_μ Lagrangian which is relevant for long distances by $\mathcal{L}(C)$. Since $\mathcal{L}(C)$ does not describe Yang-Mills theory at all distances it is not equivalent to $\mathcal{L}_{\rm YM}(A)$. We will see that by limiting ourselves to describing long-distance Yang-Mills theory we will obtain a simple $\mathcal{L}(C)$ which is equivalent to a complicated A_μ Lagrangian. This contrasts with the impossible task of describing Yang-Mills theory at all distances in terms of a hopelessly complicated C_μ Lagrangian equivalent to $\mathcal{L}_{\rm YM}(A)$.

We turn to the construction of $\mathcal{L}(C)$. Let us denote the quadratic terms in $\mathcal{L}(C)$ by $\mathcal{L}^{(0)}(C)$. Being quadratic in C_μ, $\mathcal{L}^{(0)}(C)$ is invariant only under magnetic Abelian gauge transformations. Furthermore because of the weak coupling at short distances the "free" C_μ propagator $\Delta^{(0)}_{C\mu\nu}$ generated by $\mathcal{L}^{(0)}(C)$ has the same long-distance behavior, Eq. (2.21), as the exact C_μ propagator except for mass and wave-function renormalization. We will show shortly that this requirement determines $\mathcal{L}^{(0)}(C)$. The Lagrangian $\mathcal{L}(C)$ can then be obtained from $\mathcal{L}^{(0)}(C)$ by adding the minimal terms necessary to make it invariant under the non-Abelian transformations, Eq. (2.18), of the magnetic gauge group. Any nonminimal terms in $\mathcal{L}(C)$ are not relevant at long distances. Thus our problem is reduced to constructing $\mathcal{L}^{(0)}(C)$ and explaining its origin in Yang-Mills theory.

We first note that an Abelian gauge theory describes the dynamics of a linear relativistic dielectric medium. Such a medium is characterized by a momentum-dependent dielectric constant $\epsilon(q^2)$ and magnetic permeability $\mu(q^2)$, which are related by the equation

$$\epsilon(q^2)\mu(q^2)=1 . \tag{2.22}$$

The equations of motion of this medium are Eqs. (2.14), along with

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}+\partial_0\mathbf{B}=0 , \tag{2.23}$$

and the constitutive equations

$$\mathbf{D}=\epsilon\mathbf{E}\ \text{ and }\ \mathbf{B}=\mu\mathbf{H} . \tag{2.24}$$

In the C_μ representation, Eqs. (2.23) and (2.24) are obtained from an action $S_{\rm max}(C)$ given by

$$S_{\rm max}(C)=-\tfrac{1}{4}\int\int G^{{\rm max}\,\mu\nu}(x)\mu(x-y)G^{\rm max}_{\mu\nu}(y)dx^4dy^4 , \tag{2.25}$$

where

$$G^{\rm max}_{\mu\nu}=\partial_\mu C_\nu-\partial_\nu C_\mu , \tag{2.26}$$

and where

$$\mu(x-y)=\int\frac{d^4q}{(2\pi)^4}e^{iq(x-y)}\mu(q^2) . \tag{2.27}$$

Equation (2.26) is just the covariant form of Eqs. (2.15) and hence the Maxwell Eqs. (2.14) are automatically satisfied. In the Landau gauge the C_μ propagator generated from Eq. (2.25) is

$$\Delta^{\rm max}_{C\mu\nu}(q)=\frac{1}{q^2\mu(q)}\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right] . \tag{2.28}$$

This discussion suggests how to write down $\mathcal{L}^{(0)}(C)$. We choose $\mu(q^2)$ so that the corresponding C_μ propagator $\Delta^{(0)}_{C\mu\nu}(q)$ has the structure given by Eq. (2.21). Setting

$$\mu(q^2) = -\frac{M^2}{q^2} + 1 \tag{2.29}$$

in Eq. (2.28) gives the desired result:

$$\Delta^{0}_{C\mu\nu}(q) = \frac{1}{q^2 - M^2}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] . \tag{2.30}$$

The corresponding Lagrangian $\mathcal{L}^{(0)}(C)$, determined from Eqs. (2.25), (2.26), (2.27), and (2.29) is

$$\mathcal{L}^{(0)}(C) = -\tfrac{1}{4}(\partial_\mu C_\nu - \partial_\nu C_\mu)\left[\frac{M^2}{\partial^2} + 1\right](\partial^\mu C^\nu - \partial^\nu C^\mu) . \tag{2.31}$$

$\mathcal{L}(C)$ is then constructed as the minimal (magnetic) gauge-invariant extension of $\mathcal{L}^{(0)}(C)$, in the same way that $\mathcal{L}_{\rm YM}(A)$ is obtained as the minimal (electric) gauge-invariant extension of the Maxwell Lagrangian. Using the usual prescription along with expression (2.31) for $\mathcal{L}^{(0)}(C)$ we obtain the following expression for $\mathcal{L}(C)$:

$$\mathcal{L}(C) = -\tfrac{1}{4}G_{\mu\nu}\left[\frac{M^2}{\mathcal{D}^2(C)} + 1\right]G_{\mu\nu} , \tag{2.32a}$$

where

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - ig[C_\mu, C_\nu] \equiv G^a_{\mu\nu}T_a , \tag{2.32b}$$

and

$$\mathcal{D}_\mu(C) = \partial_\mu - ig[C_\mu , \ ] . \tag{2.32c}$$

The matrices C_μ are related to the non-Abelian fields $C^a_\mu(x)$ in the usual way:

$$C_\mu = \sum_a C^a_\mu(x) T_a . \tag{2.33}$$

[For simplicity we will denote the dual covariant derivative $\mathcal{D}_\mu(C)$ simply by $\mathcal{D}_\mu$ in the rest of this paper.] We propose Eq. (2.32) as the Lagrangian appropriate to long-distance Yang-Mills theory.

To summarize, the form of $\mathcal{L}^{(0)}(C)$ was determined by the assumption of confinement and 't Hooft's result, Eq. (2.19). $\mathcal{L}(C)$ was then constructed as the minimal gauge-invariant extension of $\mathcal{L}^{(0)}(C)$, since nonminimal terms in the C_μ Lagrangian are suppressed at long distances.

It remains to find the dynamical origin of $\mathcal{L}^{(0)}(C)$ in long-distance Yang-Mills theory. This is in principle possible since $\mathcal{L}^{(0)}(C)$ describes an Abelian gauge theory. For an Abelian theory, in the A_μ language, the propagator $\Delta^{\rm max}_{A\mu\nu}(q)$ is given by

$$\Delta^{\rm max}_{A\mu\nu}(q) = \frac{1}{q^2\epsilon(q)}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] . \tag{2.34}$$

Equations (2.22), (2.29), and (2.34) then show that the Lagrangian $\mathcal{L}^{(0)}(C)$ yields an A_μ propagator $\Delta^{(0)}_{A\mu\nu}(q)$ having the long-distance behavior[14]

$$\Delta^{(0)}_{A\mu\nu}(q) \underset{q^2\to 0}{\sim} \left[-\frac{M^2}{q^4} + \frac{1}{q^2}\right]\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] . \tag{2.35}$$

We emphasize that $\Delta^{(0)}_A$ does not give the long-distance behavior of the exact A propagator of Yang-Mills theory, because the A fields are not weakly coupled at long range. This is in contrast with the C language, where $\Delta^{(0)}_C$ does, at long range, coincide with the exact Δ_C since the C fields couple weakly. Long-distance properties are complicated in the A language, and we do not propose to derive the exact long-distance behavior of Yang-Mills theory in terms of A. Instead we only need an approximation to long-distance Yang-Mills dynamics which is sufficient to obtain the nonperturbative zero-order propagator $\Delta^{(0)}_A$ to serve as the starting point for an iteration scheme in the dual language.

Such an approximation can in fact be found and indeed was our original motivation for suggesting Eq. (2.32). It is described in Appendix A. A truncation of the Schwinger-Dyson equations combined with the Ward identities of conventional A language Yang-Mills theory produces a nonlinear integral equation for the gluon propagator. The solution of this equation, for small q^2, coincides with $\Delta^{(0)}_A$. The origin of the M^2/q^4 is the following: because of asymptotic freedom, perturbation theory for the vacuum polarization yields a propagator having a pole for spacelike q^2 (the Landau ghost). By imposing self-consistency and compatibility with gauge invariance, through guaranteeing the Ward identities, the position of the Landau ghost is moved to $q^2 = 0$ (Ref. 15).

It is difficult to make use of this result in the A language. If one attempts to calculate the corrections to the propagator coming from the terms neglected by the truncation, using $\Delta^{(0)}_A$ as input, one finds singularities stronger than M^2/q^4 as $q^2 \to 0$. The truncation is only the first term in a nonconvergent expansion; there is no consistent way to calculate corrections. (See Appendix A.) The singular behavior of the A fields at long range destroys any systematic iteration scheme. Instead, the dual potentials C provide the avenue to evaluate the corrections systematically. From the first term $\Delta^{(0)}_A$ we construct $\Delta^{(0)}_C$ and hence $\mathcal{L}^{(0)}(C)$. The extension of $\mathcal{L}^{(0)}(C)$ to $\mathcal{L}(C)$ then incorporates the higher-order terms. The long-distance behavior of Δ_C coincides with that of $\Delta^{(0)}_C$, and corrections are calculable.[16]

III. PROPERTIES OF $\mathcal{L}(C)$

The Lagrangian $\mathcal{L}(C)$ is nonlocal. This indicates that the theory contains extra degrees of freedom corresponding to auxiliary fields which must explicitly be introduced in order to render $\mathcal{L}$ local. To find the local form of $\mathcal{L}$ we begin with the vacuum generating functional

$$Z = \int \mathcal{D}C^a_\mu \exp\left[i\int d^4x[\mathcal{L}(C) + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm ghost}]\right] . \tag{3.1}$$

Here $\mathcal{L}_{\rm gf}$ and $\mathcal{L}_{\rm ghost}$ are the usual gauge-fixing and Faddeev-Popov ghost terms necessary to specify Z in a non-Abelian gauge theory. The nonlocal factor in Z can be written as

$$\exp\left\{i\int\left[-\tfrac{1}{4}G^a_{\mu\nu}\left[\frac{M^2}{\mathcal{D}^2}\right]^{ab}G^{\mu\nu b}\right]d^4x\right\}=(\det\mathcal{D}^2)^{3(N^2-1)}\int\mathcal{D}\tilde{F}^a_{\mu\nu}\exp\left[i\int d^4x\left[\frac{M}{2}\tilde{F}^a_{\mu\nu}G^{\mu\nu a}+\tfrac{1}{4}\tilde{F}^a_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^{\mu\nu b}\right]\right], \tag{3.2}$$

where the color indices a run from 1 to N^2-1 for SU(N), and the fields $\tilde{F}^a_{\mu\nu}$ are antisymmetric in the Lorentz indices. There are then $6(N^2-1)$ fields; each of them yields a factor $(\det\mathcal{D}^2)^{-1/2}$ upon performing the functional integration in (3.2) after completing the square.

The determinants in (3.2) can be rewritten by introducing another set of ghost fields (in addition to the usual Faddeev-Popov ghosts) ψ_i^a and $\psi_i^{a\dagger}$, with $i=1,2,3$. There are thus $3(N^2-1)$ fields ψ_i^a and $3(N^2-1)$ fields $\psi_i^{a\dagger}$, and they are anticommuting Grassmann variables. Thus

$$(\det\mathcal{D}^2)^{3(N^2-1)}=\int\mathcal{D}\psi_i^a\int\mathcal{D}\psi_i^{a\dagger}\exp\left[i\int d^4x\,\psi_i^{a\dagger}(x)(\mathcal{D}^2)^{ab}\psi_i^b(x)\right]. \tag{3.3}$$

We choose a generalized Landau gauge, so that

$$\mathcal{L}_{\rm gf}=-\frac{\lambda}{2}(\partial^\mu C^a_\mu)^2 . \tag{3.4}$$

Denoting the Faddeev-Popov ghost fields by χ^a and $\chi^{a\dagger}$, we have

$$\mathcal{L}_{\rm ghost}=\chi^{a\dagger}\partial^\mu\mathcal{D}^{ab}_\mu\chi^b . \tag{3.5}$$

Finally, putting all of this together we may rewrite (3.1) as

$$Z=\int\mathcal{D}C^a_\mu\int\mathcal{D}\tilde{F}^a_{\mu\nu}\int\mathcal{D}\psi_i^{a\dagger}\int\mathcal{D}\psi_i^a\int\mathcal{D}\chi^{a\dagger}\int\mathcal{D}\chi^a\exp\left[i\int d^4x\,\mathcal{L}(C,\tilde{F},\psi^\dagger,\psi,\chi^\dagger,\chi)\right], \tag{3.6}$$

where the complete Lagrangian is

$$\mathcal{L}(C,\tilde{F},\psi^\dagger,\psi,\chi^\dagger,\chi)=\frac{M}{2}\tilde{F}^a_{\mu\nu}G^{\mu\nu a}+\tfrac{1}{4}\tilde{F}^a_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^{\mu\nu b}-\tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu a}+\psi_i^{a\dagger}(\mathcal{D}^2)^{ab}\psi_i^b+\chi^{a\dagger}\partial^\mu\mathcal{D}^{ab}_\mu\chi^b-\frac{\lambda}{2}(\partial^\mu C^a_\mu)^2 . \tag{3.7}$$

Aside from the gauge-fixing parameter λ, this Lagrangian contains two coupling constants: M and g. It is in the dual language that part of the quantum Yang-Mills Lagrangian relevant to long-range physics; that is, Eq. (3.7) is the translation of the ordinary Yang-Mills Lagrangian (2.1) from the A language to the C language, where only the terms surviving at long distances have been retained.

The fields $\tilde{F}^a_{\mu\nu}$ appearing in the local Lagrangian (3.7) represent explicitly the additional degrees of freedom that were implicit in its original nonlocal form. The physical interpretation of the variables $\tilde{F}^a_{\mu\nu}$ follows by noting that the equations of motion

$$-(\mathcal{D}^2)^{ab}\tilde{F}^b_{\mu\nu}=MG^a_{\mu\nu} \tag{3.8}$$

obtained by varying the fields $\tilde{F}^a_{\mu\nu}$ in the Lagrangian (3.7) are just the non-Abelian generalization of the constitutive equations (2.24). The non-Abelian magnetic $\mathbf{H}^a$ vector and the electric displacement vector $\mathbf{D}^a$ are the non-Abelian generalizations of Eq. (2.15), i.e.,

$$H^a_k\equiv G^a_{0k},\quad D^a_k=\tfrac{1}{2}\epsilon_{kij}G^a_{ij} , \tag{3.9}$$

where $G^a_{\mu\nu}$ is given by Eq. (2.32b). Equation (3.8) can then be written as

$$\frac{(\mathcal{D}^2)^{ab}}{M^2}\mathbf{E}^b=\mathbf{D}^a,\quad\frac{(\mathcal{D}^2)^{ab}}{M^2}\mathbf{B}^b=\mathbf{H}^a , \tag{3.10}$$

provided we identify

$$B^a_k\equiv-M\tilde{F}^a_{0k},\quad E^a_k\equiv-\frac{M}{2}\epsilon_{kij}\tilde{F}^a_{ij} . \tag{3.11}$$

Equation (3.10) has the form of the constitutive equations (2.24) with the "non-Abelian dielectric constant" $\mathcal{D}^2/M^2$, which shows that the fields defined by Eq. (3.11) can be interpreted as non-Abelian color-electric and -magnetic fields.[17] Since "the dielectric constant" $\mathcal{D}^2/M^2$ is a differential operator the constitutive equations are equations of motion. Hence it is natural that the components $\tilde{F}^a_{\mu\nu}$ of the color field tensor appear as independent variables rather than as quantities which at the outset can be eliminated in terms of the potentials C_μ.

What happens if we set the parameter M in Eq. (3.7) equal to zero? Equation (3.8) then becomes

$$(\mathcal{D}^2)^{ab}\tilde{F}^b_{\mu\nu}=0 . \tag{3.12}$$

In this situation we have no reason to identify $\tilde{F}^a_{\mu\nu}$ with the color electromagnetic field tensor since its spacetime indices are not coupled to a Lorentz tensor in $\mathcal{L}(C)$. Transformations which rotate the space-time components of $\tilde{F}^a_{\mu\nu}$ are then "internal symmetries" of $\mathcal{L}$. Equation (3.12) indicates that the theory might contain massless particles created by the operators $\tilde{F}^a_{\mu\nu}$. However one can show that the contribution to graphs containing only C^a_μ external lines arising from internal $\tilde{F}^a_{\mu\nu}$ lines are canceled by a corresponding contribution of internal ghost lines ψ_i^a and $\psi_i^{a\dagger}$. This is evident from the symmetric way the variables ψ_i^a, $\psi_i^{a\dagger}$, and $\tilde{F}^a_{\mu\nu}$ appear in $\mathcal{L}$ when $M=0$. Thus, solutions to Eq. (3.12) for $\tilde{F}^a_{\mu\nu}$ do not induce massless singularities in processes containing only C_μ external lines and hence correspond to unphysical gauge excitations. Of course the fact that the Lagrangian with $M=0$ describes only pure Yang-Mills interactions of the C_μ quanta is evident by setting $M=0$ in the original nonlocal form (2.32). The ghost fields ψ_i^a and $\psi_i^{a\dagger}$ in $\mathcal{L}(C)$ implement this requirement in the local form of the theory. The term $(M/2)\tilde{F}^a_{\mu\nu}G^{\mu\nu a}$ in $\mathcal{L}(C)$ breaks the symmetry between $\tilde{F}^a_{\mu\nu}$ and the ghost fields and, in the case of the free theory, where $g=0$, provides a mass for the C^a_μ field.

The kinetic energy terms in the Lagrangian which involve the fields $\mathbf{E}^a$ and $\mathbf{B}^a$ have opposite signs. That is,

$$\tfrac{1}{4}\tilde{F}^a_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^{\mu\nu b}=-\tfrac{1}{2}\tilde{F}^a_{0k}(\mathcal{D}^2)^{ab}\tilde{F}^b_{0k}+\tfrac{1}{4}\tilde{F}^a_{ij}(\mathcal{D}^2)^{ab}\tilde{F}^b_{ij}$$
$$\equiv -\tfrac{1}{2}\mathbf{B}^a\cdot\frac{(\mathcal{D}^2)^{ab}}{M^2}\mathbf{B}^b+\tfrac{1}{2}\mathbf{E}^a\cdot\frac{(\mathcal{D}^2)^{ab}}{M^2}\mathbf{E}^b\ . \quad (3.13)$$

The contribution of the magnetic field $\mathbf{B}^a$ to the Lagrangian has the normal sign, while the color-electric fields $\mathbf{E}^a$ are negative metric fields. The origin of this negative sign is the indefinite nature of the Lorentz metric which of course also appears in ordinary electrodynamics. We have not yet studied the general problem of possible violations of unitarity due to this negative metric.

In this paper we investigate the implications of $\mathcal{L}(C)$ in the classical approximation. A detailed study of the quantum aspects of the theory is not necessary for this purpose, but certain aspects of the quantized theory defined by $Z(C)$ will be important. In particular we must determine the form of the counterterms induced by renormalization that must added to $\mathcal{L}$.

When $g=0$, $(\mathcal{D}^2)^{ab}=\partial^2\delta^{ab}$ and $G^a_{\mu\nu}=\partial_\mu C^a_\nu-\partial_\nu C^a_\mu$ and the Lagrangian becomes a quadratic form in $\tilde{F}^a_{\mu\nu}$ and C^a_μ whose inverse is the free coupled $C,\tilde{F}$ propagator. We define

$$\Delta^{ab}_{\mu\nu}\equiv\langle(C^a_\mu C^b_\nu)_+\rangle=\delta^{ab}\Delta_{\mu\nu}\ , \quad (3.14a)$$

$$\Delta^{ab}_{\mu,\alpha\beta}\equiv\langle(C^a_\mu \tilde{F}^b_{\alpha\beta})_+\rangle=\delta^{ab}\Delta_{\mu,\alpha\beta}\ , \quad (3.14b)$$

$$\Delta^{ab}_{\alpha\beta,\gamma\delta}\equiv\langle(\tilde{F}^a_{\alpha\beta}\tilde{F}^b_{\gamma\delta})_+\rangle=\delta^{ab}\Delta_{\alpha\beta,\gamma\delta}\ . \quad (3.14c)$$

In momentum space we find that

$$\Delta_{\mu\nu}(p)=\frac{1}{p^2-M^2}\left[g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]+\frac{1}{\lambda}\frac{p_\mu p_\nu}{(p^2)^2}\ , \quad (3.15)$$

$$\Delta_{\mu,\alpha\beta}=\frac{-iM}{p^2(p^2-M^2)}(g_{\mu\alpha}p_\beta-g_{\mu\beta}p_\alpha)\ , \quad (3.16)$$

$$\Delta_{\alpha\beta,\gamma\delta}=\frac{1}{p^2}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})+\frac{M^2}{(p^2-M^2)p^2}\times\left[\frac{p_\alpha p_\gamma}{p^2}g_{\beta\delta}+\frac{p_\beta p_\delta}{p^2}g_{\alpha\gamma}-\frac{p_\beta p_\gamma}{p^2}g_{\alpha\delta}-\frac{p_\alpha p_\delta}{p^2}g_{\beta\gamma}\right]\ . \quad (3.17)$$

The Lagrangian (3.7) generates a renormalizable theory. There are divergent graphs which induce counterterms of the same structure as terms already present in (3.7) as well as two new terms quadratic and quartic in the variables $\tilde{F}^a_{\alpha\beta}$ (Ref. 18). (There are also counterterms involving the fields ψ_i^a and $\psi_i^{a\dagger}$ which we will not discuss here.) To determine the form of the counterterms we note that the $(M/2)\tilde{F}^a_{\mu\nu}G^{\mu\nu a}$ term in $\mathcal{L}$ induces a coupling $(gM/2)\tilde{F}^a_{\mu\nu}C^{\mu b}C^{\nu c}f_{abc}$ while the $\tilde{F}^a_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^{\mu\nu b}$ term induces a coupling of the form $gf_{abc}C^{\mu a}\partial^\mu\tilde{F}^b_{\alpha\beta}\tilde{F}^{\alpha\beta c}$ as well as a quartic coupling of the form $g^2 f_{abe}f_{ecd}C^{\mu a}C^d_\mu\tilde{F}^b_{\alpha\beta}\tilde{F}^{\alpha\beta c}$. The coupling proportional to M is super renormalizable and yields a divergent contribution only to $\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu a}$ graphs. The counterterm quadratic in $\tilde{F}^a_{\mu\nu}$ arises from the graphs depicted in Fig. 1. The graphs (a) are independent of M and quadratically divergent, while the graphs (b) are proportional to M^2 and are logarithmically divergent.

In order to exhibit the color dependence of these counterterms, it is convenient to use the tensor components $\tilde{F}^i_{\mu\nu j}$ of the fields which are proportional to the i,j matrix elements of $\tilde{F}_{\mu\nu}$, where

$$\tilde{F}_{\mu\nu}\equiv\sum_a\tilde{F}^a_{\mu\nu}T_a\ . \quad (3.18)$$

Hence

$$\frac{1}{\sqrt{2}}\tilde{F}^i_{\mu\nu j}\equiv(\tilde{F}_{\mu\nu})_{ij}\ . \quad (3.19)$$

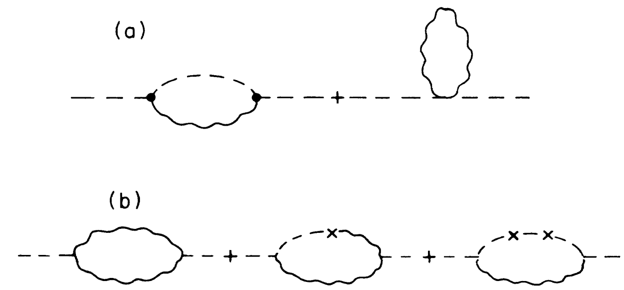

FIG. 1. Divergent $\tilde{F}^a_{\alpha\beta}$ self-energy graphs: The wiggly lines in this figure denote the C^a_μ propagator, Eq. (3.15), the dashed line represents the part of the $\tilde{F}^a_{\alpha\beta}$ propagator in Eq. (3.17) which is independent of M, the line with an $\times$ denotes the mixed propagator Eq. (3.16), while the line with $\times\times$ represents the part of the $\tilde{F}^a_{\mu\nu}$ propagator, Eq. (3.17), proportional to M^2.

The factor $1/\sqrt{2}$ is introduced into Eq. (3.19) so that

$$\sum_{i,j=1}^{N} \tilde{F}^{\,i}_{\mu\nu j}\tilde{F}^{\mu\nu j}_{i}=2\,\mathrm{tr}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$=\sum_{a=1}^{N^2-1}\tilde{F}^{\,a}_{\mu\nu}\tilde{F}^{\mu\nu a}\equiv\tilde{F}^{\,2}\ . \qquad (3.20)$$

For the case of SU(2) $T_a=\tau_a/2$, $a=1,2,3$, and the relation between the $\tilde{F}^{\,a}_{\mu\nu}$ and $\tilde{F}^{\,i}_{\mu\nu j}$ is

$$\tilde{F}^{\,2}_{\mu\nu 1}=\frac{\tilde{F}^{\,1}_{\mu\nu}+i\tilde{F}^{\,2}_{\mu\nu}}{\sqrt{2}}\ ,$$

$$\tilde{F}^{\,1}_{\mu\nu 2}=\frac{\tilde{F}^{\,1}_{\mu\nu}-i\tilde{F}^{\,2}_{\mu\nu}}{\sqrt{2}}\ , \qquad (3.21)$$

$$\tilde{F}^{\,1}_{\mu\nu 1}=-\tilde{F}^{\,2}_{\mu\nu 2}=\frac{\tilde{F}^{\,3}_{\mu\nu}}{\sqrt{2}}\ .$$

For the case of SU(3), $T_a=\lambda_a/2$, $a=1,\dots,8$, and the relation between the $\tilde{F}^{\,i}_{\mu\nu j}$, $i,j=1,2,3$ and the $\tilde{F}^{\,a}_{\mu\nu}$, $a=1,2,\dots,8$, is given in Appendix B.

By examining the color dependence of the graphs of Figs. 1 and 2 we can determine the explicit color dependence of the $\tilde{F}^{\,2}$ and $\tilde{F}^{\,4}$ counterterms for SU(N) gauge theory. Denoting the counterterms which we must add to $\mathcal{L}(C)$ by $-W$ we find

$$W=-\frac{\mu^2 N}{4}\tilde{F}^{\,i}_{\alpha\beta j}\tilde{F}^{\alpha\beta i}_{j}+\frac{N\lambda}{4!}\left[\tilde{F}^{\,i}_{\alpha\beta k}\tilde{F}^{\alpha\beta k}_{j}\tilde{F}^{\lambda\sigma j}_{l}\tilde{F}^{\,l}_{\lambda\sigma i}+\frac{1}{N}(\tilde{F}^{\,i}_{\alpha\beta j}\tilde{F}^{\alpha\beta j}_{i})^2+\frac{2}{N}(\tilde{F}^{\,i}_{\alpha\beta j}\tilde{F}^{\alpha\beta k}_{l})(\tilde{F}^{\,j}_{\lambda\sigma i}\tilde{F}^{\lambda\sigma l}_{k})\right], \qquad (3.22)$$

where μ^2 and λ are arbitrary constants. The function W thus introduces two new parameters μ^2 and λ into $\mathcal{L}$; μ^2, the coefficient of the $\tilde{F}^{\,2}$ term, has the dimension (mass)^2 and λ, the coefficient of the $\tilde{F}^{\,4}$ term, is dimensionless. The explicit factors of N which appear in Eq. (3.22) arise from the color matrices in the graphs of Figs. 1 and 2.

The complete Lagrangian is obtained by adding $-W$ to Eq. (3.7). Omitting the gauge-fixing term, the Faddeev-Popov ghosts χ^a and $\chi^{a\dagger}$, and the ghost fields ψ^a_i and $\psi^{a\dagger}_i$, the Lagrangian then takes the form

$$\mathcal{L}=\frac{M}{2}\tilde{F}^{\,a}_{\mu\nu}G^{\mu\nu a}+\tfrac{1}{4}\tilde{F}^{\,a}_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^{\mu\nu b}-\tfrac{1}{4}G^{a}_{\mu\nu}G^{\mu\nu a}-W\ , \qquad (3.23)$$

where W is given by Eq. (3.22). The equations of motion resulting from this Lagrangian are

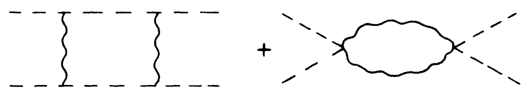

FIG. 2. Divergent $\tilde{F}^{\,4}$ graphs. Both graphs are logarithmically divergent and do not receive contributions from the $(M/2)\tilde{F}G$ term in $\mathcal{L}(C)$.

$$MG^{a}_{\mu\nu}=-(\mathcal{D}^2)_{ab}\tilde{F}^{\,b}_{\mu\nu}+\frac{\delta W}{\delta\tilde{F}^{\mu\nu a}}\ , \qquad (3.24)$$

$$(\mathcal{D}_\mu)^{ab}G^{\mu\nu b}=M(\mathcal{D}_\mu)^{ab}\tilde{F}^{\mu\nu b}-\frac{g}{2}f_{abc}(\mathcal{D}^\nu)^{bd}\tilde{F}^{\alpha\beta d}\tilde{F}^{\,c}_{\alpha\beta}\ . \qquad (3.25)$$

Note that W enters only in the constitutive equation (3.24) and that it depends upon $\tilde{F}^{\,i}_{\alpha\beta j}$ only via the quantity $\tilde{F}^{\,i}_{\alpha\beta j}\tilde{F}^{\alpha\beta k}_{l}$ or equivalently only via $\tilde{F}^{\,a}_{\alpha\beta}\tilde{F}^{\alpha\beta b}$. The constitutive equation (3.24) can then be written as

$$G^{a}_{\mu\nu}=\epsilon^{ab}(M\tilde{F}^{\,b}_{\mu\nu})\ , \qquad (3.26)$$

where ϵ^{ab} is the dielectric tensor given by

$$\epsilon^{ab}=-\frac{(\mathcal{D}^2)^{ab}}{M^2}+\frac{1}{M^2}\frac{\delta W}{\delta(\tilde{F}^{\,a}_{\alpha\beta}\tilde{F}^{\alpha\beta b})}\ . \qquad (3.27)$$

Thus the inclusion of the counterterm W gives a contribution to the dielectric constant which is a function only of $\tilde{F}^{\,a}_{\mu\nu}$. The first term in Eq. (3.27) depends only on the potentials C^a_μ.

IV. ELECTRIC FLUX-TUBE SOLUTIONS TO SU(N) GAUGE THEORY

We will now show in SU(N) gauge theory that if $\mu^2<0$ and $\lambda>0$, then the equations of motion (3.24) and (3.25) have classical solutions describing static cylindrically symmetric tubes of electric flux quantized in units of e/N (where e is the Yang-Mills coupling constant). This will demonstrate that $\mathcal{L}(C)$ possesses the essential properties of a confining theory. In particular static heavy quarks situated at the ends of the tube will experience a linear potential. However we must emphasize that we have not yet calculated the quantum fluctuations about this classical electric vortex solution and that these fluctuations could have important effects upon its properties. Thus quantities obtained from the classical solution can be identified with physical quantities only in an approximate way. The specific limitations of the classical approximation will be pointed out as we proceed in this paper.

Nevertheless the elucidation of the properties of the classical electric flux-tube solution is a necessary first step. Semiclassical quantization of $\mathcal{L}(C)$ about this classical solution should lead to a string theory of color-electric vortices, just as the semiclassical quantization around the Nielsen-Olesen solution[19] of the Abelian-Higgs model carried out by Gervais and Sakita[4] leads to a string theory of magnetic vortices.

We begin our treatment of the SU(N) flux tube by determining the behavior of the fields at large distances ρ from the center of the flux tube. (We choose the z axis to be the axis of the flux tube and use cylindrical coordinates ρ, ϕ, and z.) We will use the matrix notation, Equations (2.32), (2.33), and (3.18). At large distances from the flux tube the fields C_μ and $\tilde{F}_{\alpha\beta}$ should approach a static solution of the equations

$$G_{\mu\nu}=0,\quad \mathcal{D}_\mu\tilde{F}_{\alpha\beta}=0,\quad \frac{\delta W}{\delta\tilde{F}_{\alpha\beta}}=0\ . \qquad (4.1)$$

Solutions of Eq. (4.1) are particular solutions of Eqs. (3.24) and (3.25) and describe classical vacuum configurations. Equations (4.1) have the trivial solution $C_\mu = \tilde{F}_{\alpha\beta} = 0$ corresponding to the perturbative vacuum. In order to have flux-tube solutions it is necessary that Eq. (4.1) have a nontrivial vacuum solution,

$$C_\mu = 0, \quad \tilde{F}_{\mu\nu} = \tilde{F}_{0\mu\nu} = \text{const} \neq 0 \;, \tag{4.2}$$

where

$$\frac{\delta W}{\delta \tilde{F}_{\mu\nu}} = 0 \quad \text{for } \tilde{F}_{\mu\nu} = \tilde{F}_{0\mu\nu} \;. \tag{4.3}$$

At large distances from the flux tube C_μ and $\tilde{F}_{\alpha\beta}$ must approach a gauge transformation $\Omega_M(x)$ of the nonperturbative classical vacuum solution (4.2); i.e.,

$$C_\mu(x) \underset{\rho\to\infty}{\longrightarrow} \frac{i}{g}\Omega_M^{-1}(x)\partial_\mu \Omega_M(x) \;, \tag{4.4a}$$

$$\tilde{F}_{\alpha\beta}(x) \underset{\rho\to\infty}{\longrightarrow} \Omega_M^{-1}(x)\tilde{F}_{0\mu\nu}\Omega_M(x) \;, \tag{4.4b}$$

where $\Omega_M(x)$ is an SU(N) matrix of the magnetic gauge group. Expressions (4.4) are clearly solutions of the asymptotic equations (4.1).

Now let l be any loop which surrounds the flux tube at large distances. The loop $W_C(l)$ given by Eq. (2.17) then measures the electric flux contained in the tube [see Eq. (2.12)]. Since all the points on the loop are at a large distance from the center of the flux tube, we can use the asymptotic solution (4.4) to evaluate $W_C(l)$. Inserting Eq. (4.4a) into Eq. (2.17) we obtain

$$W_C(l) = \Omega_M^{-1}(\phi = 2\pi)\Omega_M(\phi = 0) \;, \tag{4.5}$$

where $\Omega_M(\phi=0)$ is the value of the gauge transformation at some point of the closed loop and $\Omega_M(\phi=2\pi)$ is its value at the same point after having circled once around the loop. Now from Eq. (4.4b) we obtain

$$\tilde{F}_{\mu\nu}(\phi=2\pi) = [\Omega_M^{-1}(0)\Omega_M(2\pi)]^{-1}\tilde{F}_{\mu\nu}(\phi=0)\Omega_M^{-1}(0)\Omega_M(2\pi) \;. \tag{4.6}$$

Thus in order that $\tilde{F}_{\mu\nu}$ be single valued we must have

$$\Omega_M^{-1}(2\pi)\Omega_M(0) = e^{2\pi i n/N} \;, \tag{4.7}$$

where $n = 0, 1, 2, \ldots, N-1$. The electric flux $\Phi_E(l)$ contained in the tube is determined from $W_C(l)$ by the non-Abelian generalization of Eq. (2.12). Hence from Eqs. (2.12), (4.5), and (4.6) we obtain

$$\exp\left[\frac{2\pi i}{e}\Phi_E(l)\right] = e^{2\pi i n/N} \;, \tag{4.8}$$

where $n = 0, 1, \ldots, N-1$. That is, the electric flux $\Phi_E(l)$ is quantized in units of the SU(N) quark charge e/N. The flux-tube solution is therefore a Z_N vortex.

The above argument is essentially a repetition of an argument given by 't Hooft and others who noted that an SU(N) gauge theory coupled to scalar Higgs fields in the adjoint representation has classical Z_N vortex solutions. There are two differences: (1) The Z_N vortices obtained from $\mathcal{L}(C)$ are color electric, not color magnetic; (2) no scalar fields are arbitrarily introduced in $\mathcal{L}(C)$. The role of the scalar Higgs fields is played by the fields $\tilde{F}_{\mu\nu}$ (**E** and **B**) which appear automatically when the Lagrangian $\mathcal{L}(C)$ is rendered local.

We next study Eq. (4.3) defining $\tilde{F}_{0\mu\nu}$. It is convenient to use the tensor T_{jl}^{ik} given by

$$T_{jl}^{ik} = \sum_{\mu\nu} \tilde{F}^{\,i}_{\mu\nu j}\tilde{F}^{\,\mu\nu k}_{l} \;. \tag{4.9}$$

Then since

$$\sum_{i=1}^{N} \tilde{F}^{\,i}_{\mu\nu i} = 0 \;,$$

we have

$$T_{il}^{ik} = T_{jk}^{ik} = 0 \;, \tag{4.10}$$

as well as

$$T_{jl}^{ik} = T_{lj}^{ki} \;. \tag{4.11}$$

Furthermore if we define M_l^i by the equation

$$M_l^i = T_{kl}^{ik} \equiv \tilde{F}^{i}_{\mu\nu k}\tilde{F}^{\,\mu\nu k}_{l} \;, \tag{4.12}$$

then

$$M_i^i = \tilde{F}^{\,i}_{\mu\nu k}\tilde{F}^{\,\mu\nu k}_{i} \equiv \tilde{F}^{\,2} \;. \tag{4.13}$$

We can write W in the form

$$W = -\frac{N\mu^2\tilde{F}^{\,2}}{4} + \frac{N\lambda}{4!}\left[M_k^i M_i^k + \frac{(\tilde{F}^{\,2})^2}{N} + \frac{2}{N}T_{jl}^{ik}T_{ik}^{jl}\right] . \tag{4.14}$$

Differentiating Eq. (4.14) with respect to $\tilde{F}_{\mu\nu}$ and using Eq. (4.3) yields the following equation for $\tilde{F}^{\,i}_{0\mu\nu j}$:

$$\mu^2\tilde{F}^{\,i}_{0\mu\nu j} = \frac{\lambda}{6}\left[\tilde{F}^{\,i}_{0\mu\nu k}M^k_{0j} + M^i_{0k}\tilde{F}^{\,k}_{0\mu\nu j} + \frac{2\tilde{F}^{\,2}_{0}}{N}\tilde{F}^{\,i}_{0\mu\nu j} + \frac{4}{N}T^{ik}_{0jl}\tilde{F}^{\,l}_{0\mu\nu k}\right] . \tag{4.15}$$

The quantities M^k_{0j} and T^{ik}_{0jl} are the values of M_j^k and T_{jl}^{ik} for $\tilde{F}^{\,m}_{\mu\nu n} = \tilde{F}^{\,m}_{0\mu\nu n}$. Multiplying Eq. (4.15) by $\tilde{F}^{\,\mu\nu m}_{0n}$ gives

$$\mu^2 T^{im}_{0jn} = \frac{\lambda}{6}\left[T^{mi}_{0nk}M^k_{0j} + M^i_{0k}T^{km}_{0jn} + \frac{2\tilde{F}^{\,2}_{0}}{N}T^{mi}_{0nj} + \frac{4}{N}T^{ik}_{0jl}T^{lm}_{0kn}\right] . \tag{4.16}$$

Equation (4.16) determines the values of the tensor T^{im}_{0jn} at the extrema of W. Inserting Eq. (4.16) into Eq. (4.14) yields the value W_0 of W at the extrema:

$$W_0 = -\frac{N\mu^2\tilde{F}^{\,2}_{0}}{8} \;. \tag{4.17}$$

The Hamiltonian density evaluated at the vacuum solution is just W_0. There are many solutions of Eq. (4.16) and we seek the one that yields a minimum value for W_0. The value of W_0 at this minimum is the vacuum energy density. Thus we must minimize Eq. (4.17) subject to the constraint (4.16). At the classical level the value of T^{ik}_{0jl} at this minimum must also be compatible with the definition (4.9) of T^{ik}_{jl} as a sum of products of $\tilde{F}^{\,i}_{\mu\nu j}$; when quantum effects are included, however, Eq. (4.9) becomes an operator equation and the expectation value of this operator is no longer expressible in terms of expectation values of the operators, $\langle \tilde{F}_{\mu\nu}\rangle$. If we do not impose Eq. (4.9) we are therefore going beyond the classical approximation and are including some of the effects of quantum corrections.

Except in the case of SU(2), the classical formula (4.9) is incompatible with the color-singlet nature of the vacuum. For $N \geq 3$, the singlet vacuum energy lies somewhat lower than that obtained in the classical approximation, though for $N=3$ the difference is only a few percent.

In Sec. VI we will obtain the explicit form for $\tilde{F}_{0\mu\nu}$ and calculate $\tilde{F}^{\,2}_0$ making use of Eq. (4.9). Here, however, we will drop Eq. (4.9). It is then easy to see that the minimum of W subject to the constraint (4.16) is realized when T^{ik}_{0jl} is a color singlet. Using Eqs. (4.10) and (4.11), we conclude that

$$T^{ik}_{0jl}=A\left[\delta^i_l\delta^k_j-\frac{1}{N}\delta^i_j\delta^k_l\right], \tag{4.18}$$

where A is a constant.[20]

Equations (4.18), (4.12), and (4.13) then give

$$M^i_j=\frac{\tilde{F}^{\,2}_0}{N}\delta^i_j \tag{4.19}$$

and

$$T^{ik}_{0jl}=\frac{\tilde{F}^{\,2}_0}{N^2-1}\left[\delta^i_l\delta^k_j-\frac{1}{N}\delta^i_j\delta^k_l\right], \tag{4.20}$$

where $\tilde{F}^{\,2}_0$ is determined from Eq. (4.16). Equation (4.20) yields

$$T^{mi}_{0nj}T^{jk}_{0il}=\frac{\tilde{F}^{\,2}_0}{N^2-1}T^{mk}_{0nl}\;. \tag{4.21}$$

Inserting Eqs. (4.19) and (4.20) into Eq. (4.16) and using Eq. (4.21) we obtain

$$\mu^2=\frac{\lambda}{6}\left[\frac{\tilde{F}^{\,2}_0}{N}+\frac{\tilde{F}^{\,2}_0}{N}+\frac{2\tilde{F}^{\,2}_0}{N}+\frac{4}{N}\frac{\tilde{F}^{\,2}_0}{N^2-1}\right] \tag{4.22}$$

or

$$\tilde{F}^{\,2}_0=\frac{3\mu^2}{2\lambda}\left[\frac{N^2-1}{N}\right], \tag{4.23}$$

and then Eq. (4.17) gives

$$W_0=-\frac{3}{16}\frac{(\mu^2)^2}{\lambda}(N^2-1)\;. \tag{4.24}$$

The interpretation of Eq. (4.24) is evident. The gluon field has N^2-1 independent degrees of freedom. Each degree of freedom contributes equally to the energy of a gauge invariant vacuum and $-\frac{3}{16}(\mu^2)^2/\lambda$ is the vacuum energy density per degree of freedom. Likewise, from Eq. (4.23) we see that $\frac{3}{2}\mu^2/\lambda N$ is the contribution of each color degree of freedom to $\tilde{F}^{\,2}_0$.

Note that the trivial solution, $\tilde{F}^{\,i}_{0\mu\nu j}=0$, of Eq. (4.15) corresponds to the perturbative vacuum and the corresponding perturbative vacuum energy density $(W_0)_{\rm pert}=0$. The physically relevant vacuum energy density $\epsilon_{\rm vac}$ is the difference between the energy density of the nonperturbative vacuum and that of the perturbative vacuum:

$$\epsilon_{\rm vac}=W_0-(W_0)_{\rm pert}=-\frac{3}{16}\frac{(\mu^2)^2}{\lambda}(N^2-1)\;. \tag{4.25}$$

Now in order to have a flux-tube solution the nonperturbative vacuum with $\tilde{F}^{\,2}_0\neq 0$ must be the physically realized solution. This means that we must have $\epsilon_{\rm vac}<0$ which implies

$$\lambda>0\;. \tag{4.26}$$

Next we can use Eq. (4.23) to calculate the gluon condensate G_2. G_2 is defined as

$$G_2=\frac{\alpha_s}{\pi}\sum_{a=1}^{N^2-1}[\langle(\mathbf{B}^{a2}-\mathbf{E}^{a2})\rangle-\langle(\mathbf{B}^{a2}-\mathbf{E}^{a2})\rangle_{\rm pert}]\;, \tag{4.27}$$

where $\alpha_s=e^2/4\pi$. The first term in Eq. (4.27) is the expectation value of the operator $\mathbf{B}^{a2}-\mathbf{E}^{a2}$ in the physical Yang-Mills vacuum and the second term is its expectation value in the perturbative vacuum. Using Eq. (3.11) we obtain in the classical approximation[21]

$$\langle(\mathbf{B}^{a2}-\mathbf{E}^{a2})\rangle\approx-\frac{M^2}{2}\tilde{F}^{\,a}_{0\mu\nu}\tilde{F}^{\,\mu\nu a}_0\;. \tag{4.28}$$

The classical approximation (4.28) replaces the expectation value of a product of operators by the product of their expectation values and neglects quantum-mechanical correlations as discussed above. Then using Eqs. (2.11), (4.27), (4.23), and (4.28), we find

$$G_2\approx-\frac{M^2}{g^2}\tilde{F}^{\,2}_0=-\frac{M^2}{g^2}\frac{3\mu^2}{2\lambda}\frac{N^2-1}{N}\;. \tag{4.29}$$

It is known from experiment that G_2 is positive. Hence

$$\mu^2<0\;. \tag{4.30}$$

The QCD vacuum is thus magnetic ($\tilde{F}^{\,2}_0<0$) as expected for a dual superconductor. The pressure from the large-distance vacuum magnetic field confines the lines of electric flux, just as electrically charged Cooper pairs confine magnetic flux lines in a superconductor.

We can demonstrate that the condition $\mu^2<0$ is necessary for producing confined tubes of electric flux. To see qualitatively how this comes about let $\tilde{F}_{\mu\nu}$ be proportional $\tilde{F}_{0\mu\nu}$, i.e., choose

$$\tilde{F}^{\,a}_{\mu\nu}=\tilde{F}^{\,a}_{0\mu\nu}f\;. \tag{4.31}$$

Inserting Eq. (4.31) into (3.22) then gives

$$W=W_0+\frac{N\tilde{F}_0^2\mu^2}{8}(f^2-1)^2 \; . \tag{4.32}$$

Since $\tilde{F}_0^2\mu^2>0$ we see that W has an absolute minimum at $f=1$ for either sign of μ^2. However the ansatz (4.31) yields a kinematic term in $\mathcal{L}$ of the form

$$\frac{\tilde{F}^a_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^{\mu\nu b}}{4}=\tilde{F}_0^2 f(\mathcal{D}^2)^{aa}f \; . \tag{4.33}$$

If $\tilde{F}_0^2<0$, the kinematic term (4.33) has the normal sign. If $\tilde{F}_0^2>0$, the sign of the kinematic term is reversed and the kinetic energy associated with the excitation is negative. There will then be no flux-tube solutions to Eqs. (3.24) and (3.25). On the other hand, in the case of a magnetic QCD vacuum ($\tilde{F}_0^2<0$), both the kinetic and the potential-energy terms are positive and flux-tube solutions exist. [Of course the argument given here is crude since in addition to using the simple ansatz (4.31) the effect of the $(M/2)\tilde{F}^a_{\mu\nu}\tilde{G}^{\mu\nu a}$ term in $\mathcal{L}$ was not considered. However, the result remains true as will be seen in Sec. VIII.] Note that Eq. (4.33) is just a reflection of the fact that the kinetic energy associated with electric excitations is negative [see Eq. (3.13)].

Next note that Eqs. (4.25) and (4.29) give the following relation between ϵ_{vac} and G_2:

$$\epsilon_{\text{vac}}=-\frac{g^2N}{8}\frac{\mu^2}{M^2}G_2 \; . \tag{4.34}$$

Equation (4.34) can be compared with relation between G_2 and the trace of the energy-momentum tensor T^μ_μ obtained from the trace anomaly,[22]

$$T^\mu_\mu=\frac{\beta(e)}{2e}F^{\mu\nu}F_{\mu\nu} \; , \tag{4.35}$$

where for pure SU(N) Yang-Mills theory

$$\beta(e)=\mu\frac{de}{d\mu}=-\frac{e^3}{16\pi^2}(\tfrac{11}{3}N)+O(e^5) \; . \tag{4.36}$$

Dropping the higher-order terms and identifying

$$\epsilon_{\text{vac}}=\tfrac{1}{4}T^\mu_\mu \tag{4.37}$$

we obtain the relation[23]

$$\epsilon_{\text{vac}}=-\frac{11N}{96}G_2 \; . \tag{4.38}$$

Equation (4.38) constrains the parameters entering into the relation (4.34) between ϵ_{vac} and G_2 obtained in the classical approximation. Comparing Eq. (4.34) and (4.38) we find

$$-\mu^2=\frac{11}{12}\left[\frac{M}{g}\right]^2 . \tag{4.39}$$

Thus the trace anomaly determines the coefficient of $\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu a}$ in the Lagrangian (3.23). We emphasize that Eq. (4.39) is a relation between coupling constants defined at a given mass scale. Quantum effects will induce corrections to Eq. (4.39) at other scales. Using Eqs. (4.39) and (4.29), we obtain the following expression for λ in terms of M/g and G_2:

$$\lambda=\frac{(11/8N)(M/g)^4}{G_2/(N^2-1)} \; . \tag{4.40}$$

A measurement of G_2 at a given scale determines λ in terms of (M/g).

V. ESTIMATES OF THE SU(N) STRING TENSION AND THE $1/N$ LIMIT

We can write W as

$$W=-\frac{\mu^2N}{4}W_2(\tilde{F}_{\mu\nu})+\frac{N\lambda}{4!}W_4(\tilde{F}_{\mu\nu}) \; , \tag{5.1}$$

where

$$W_2(\tilde{F}_{\mu\nu})=\tilde{F}^i_{\mu\nu k}\tilde{F}^{\mu\nu k}_i \tag{5.2a}$$

and

$$W_4(\tilde{F}_{\mu\nu})=\tilde{F}^i_{\alpha\beta k}\tilde{F}^{\alpha\beta k}_j\tilde{F}^{\lambda\sigma l}_i\tilde{F}^j_{\lambda\sigma l}+\frac{1}{N}(\tilde{F}^i_{\alpha\beta j}\tilde{F}^{\alpha\beta j}_i)^2$$
$$+\frac{2}{N}(\tilde{F}^i_{\alpha\beta j}\tilde{F}^{\alpha\beta k}_l)(\tilde{F}^j_{\lambda\sigma i}\tilde{F}^{\lambda\sigma l}_k) \; . \tag{5.2b}$$

Then

$$\mathcal{L}=\frac{M}{2}\tilde{F}^{\mu\nu a}G^a_{\mu\nu}+\tfrac{1}{4}\tilde{F}^a_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^b_{\mu\nu}-\tfrac{1}{4}G^{\mu\nu a}G^a_{\mu\nu}$$
$$+\frac{\mu^2N}{4}W_2(\tilde{F}_{\mu\nu})-\frac{N\lambda}{4!}W_4(\tilde{F}_{\mu\nu}) \; . \tag{5.3}$$

It is convenient to rescale the variables in Eq. (5.3) for the purposes of studying the resulting classical equations of motion. We let

$$\tilde{F}^a_{\mu\nu}\to\frac{M}{g}\tilde{F}^a_{\mu\nu}, \quad C^a_\mu\to\frac{M}{g}C^a_\mu, \quad x_\mu\to\frac{x_\mu}{M} \; . \tag{5.4}$$

Then

$$G^a_{\mu\nu}\to\frac{M^2}{g}G^a_{\mu\nu}, \quad \mathcal{D}^{ab}_\mu\to M\mathcal{D}^{ab}_\mu \; , \tag{5.5a}$$

where

$$G^a_{\mu\nu}\equiv\partial_\mu C^a_\nu-\partial_\nu C^a_\mu+f_{abc}C^b_\mu C^c_\nu \tag{5.5b}$$

and

$$\mathcal{D}^{ab}_\mu\equiv\delta^{ab}\partial_\mu+f^{adb}C^d_\mu \; . \tag{5.5c}$$

Henceforth in this paper, unless indicated otherwise, x_μ, C^a_μ, $\tilde{F}^a_{\mu\nu}$, $G^a_{\mu\nu}$, and $\mathcal{D}^{ab}_\mu$ will represent the rescaled variables defined in Eqs. (5.4) and (5.5). We can then rewrite $\mathcal{L}$:

$$\mathcal{L}=\frac{M^4}{g^2}[\tfrac{1}{2}\tilde{F}^{\mu\nu a}G^a_{\mu\nu}+\tfrac{1}{4}\tilde{F}^a_{\mu\nu}(\mathcal{D}^2)^{ab}\tilde{F}^{\mu\nu b}$$
$$-\tfrac{1}{4}G^{\mu\nu a}G^a_{\mu\nu}-W] \; , \tag{5.6}$$

where the dimensionless W is now expressed in terms of dimensionless parameters

$$\lambda_1=-\frac{N\mu^2}{4M^2},\quad \lambda_2=\frac{N\lambda}{4!g^2}, \tag{5.7}$$

as

$$W=\lambda_1 W_2(\tilde{F}_{\mu\nu})+\lambda_2 W_4(\tilde{F}_{\mu\nu}). \tag{5.8}$$

Equations (4.23) and (4.24) can be written in terms of dimensionless variables as

$$\tilde{F}_0^2=-2b^2(N^2-1), \tag{5.9}$$

$$W_0=-\lambda_1 b^2(N^2-1), \tag{5.10}$$

where

$$b^2=\frac{1}{8N}(\lambda_1/\lambda_2). \tag{5.11}$$

Equations (4.25) and (4.29) become

$$\epsilon_{\text{vac}}=\frac{M^4}{g^2}\frac{\lambda_1\tilde{F}_0^2}{2}=-\frac{M^4}{g^2}\lambda_1 b^2(N^2-1) \tag{5.12}$$

and

$$G_2=-\left[\frac{M}{g}\right]^4\tilde{F}_0^2=2\left[\frac{M}{g}\right]^4 b^2(N^2-1), \tag{5.13}$$

while the approximate relation Eq. (4.39) is

$$\frac{g^2\lambda_1}{N}\approx\frac{11}{48}. \tag{5.14}$$

We next expand W about $\tilde{F}^{\,i}_{0\mu\nu j}$ writing

$$\tilde{F}^{\,i}_{\mu\nu j}=\tilde{F}^{\,i}_{0\mu\nu j}+\tilde{F}^{\,i}_{1\mu\nu j}. \tag{5.15}$$

Retaining terms in W which are quadratic in $\tilde{F}^{\mu\nu i}_{1j}$ we find

$$W=W_0+4\lambda_2\left[S^{ik}_{kn}S^{nl}_{li}+\frac{2}{N}S^{im}_{jn}S^{jn}_{im}+2N\frac{(\tilde{F}^{\,i}_{0\mu\nu j}\tilde{F}^{\mu\nu j}_{1i})^2}{N^2-1}\right], \tag{5.16}$$

where the tenor S^{im}_{jn} is given by

$$S^{im}_{jn}=\frac{\tilde{F}^{\,i}_{0\mu\nu j}\tilde{F}^{\mu\nu m}_{1n}+\tilde{F}^{\,i}_{1\mu\nu j}\tilde{F}^{\mu\nu m}_{0n}}{2}-\frac{\delta^i_n\delta^m_j-(1/N)\delta^i_j\delta^m_n}{N^2-1}(\tilde{F}^{\mu\nu k}_{0l}\tilde{F}^{\,l}_{1\mu\nu k}). \tag{5.17}$$

It satisfies the equation

$$S^{in}_{ni}=0.$$

Since $\lambda_2>0$, Eq. (5.16) makes explicit the relative minimum of W at $\tilde{F}^{\,i}_{0\mu\nu j}$. Furthermore we see that in the quadratic approximation W depends only upon the component of $\tilde{F}^{\,i}_{1\mu\nu j}$ along $\tilde{F}^{\,m}_{0\mu\nu n}$, symmetrized with respect to tensor indices. Note also that if we choose $\tilde{F}^{\,i}_{1\mu\nu j}=A\tilde{F}^{\,i}_{0\mu\nu j}$, then $S^{ik}_{jl}=0$.

Now (assuming that a flux-tube solution exists) let us estimate the string tension σ, which is the energy per unit length of a flux tube measured relative to the energy of the vacuum. Denoting the Hamiltonian density calculated from the Lagrangian density, Eq. (3.7), by $\mathcal{H}$ we have

$$\sigma=\frac{1}{M^2}\int d^2\mathbf{x}[\mathcal{H}(x)-\epsilon_{\text{vac}}], \tag{5.18}$$

where x is the rescaled coordinate, Eq. (5.4), and $\mathcal{H}$, as the Lagrangian $\mathcal{L}$, Eq. (5.6), can be written as

$$\mathcal{H}=\frac{M^4}{g^2}\mathcal{H}_d, \tag{5.19}$$

where $\mathcal{H}_d$ is a function only of the rescaled variables. Equation (5.8) then assumes the form

$$\sigma=\frac{M^2}{g^2}\sigma_d, \tag{5.20}$$

where

$$\sigma_d=\int d^2\mathbf{x}\,|\,\mathcal{H}_d(x)-\tfrac{1}{2}\lambda_1\tilde{F}_0^2\,|. \tag{5.21}$$

The dimensionless string tension, σ_d, is a function of the dimensionless parameters λ_1 and λ_2 or equivalently of $\tilde{F}_0^2$ and $W_0\equiv\frac{1}{2}\lambda_1\tilde{F}_0^2$. The explicit form for $\mathcal{H}_d$ is given in Eq. (3.24) of Ref. 5 (Ref. 24). In Eq. (5.21), $\mathcal{H}_d(x)\equiv\mathcal{H}_d(\tilde{F}_{\mu\nu}(x),\ C_\mu(x))$, where $\tilde{F}_{\mu\nu}(x)$ and $C_\mu(x)$ are the static z-independent flux-tube solutions of Eqs. (3.24) and (3.25). The integral in Eq. (5.21) is in the xy plane.

We can estimate σ as follows. The magnitude of the integral (5.18) is of order $-\epsilon_{\text{vac}}=-(M^4/g^2)W_0$, and from Eq. (5.16) the range of $\mathbf{x}$ over which it varies is of order

$$\left[\frac{8\lambda_2 N\tilde{F}_0^2}{N^2-1}\right]^{-1/2}=(2\lambda_1)^{-1/2}.$$

Using $\pi/2\lambda_1$ as the effective volume of integration in (5.18) we obtain the estimate

$$\sigma\sim-\frac{1}{M^2}\left[\frac{M^4}{g^2}W_0\right]\frac{\pi}{2\lambda_1}=\frac{\pi M^2}{2g^2}b^2(N^2-1). \tag{5.22}$$

The same rough estimate (5.22) can be obtained somewhat more systematically by calculating the contribution of the potential energy density $\mathcal{H}_{\text{potential}}=(M^4/g^2)W$ to Eq. (5.18). (The kinetic energy contribution should be of the same order.) Replacing $\mathcal{H}$ by $(M^4/g^2)W$ in Eq. (5.18) then gives

$$\sigma\approx\frac{M^2}{g^2}\int d^2\mathbf{x}(W-W_0). \tag{5.23}$$

We estimate the difference $W-W_0$ by making the quadratic expansion (5.18), assuming that $\tilde{F}_{1\mu\nu}$ is proportional to $\tilde{F}_{0\mu\nu}$, i.e.,

$$\tilde{F}^{\,i}_{1\mu\nu j}=f(\mathbf{x})\tilde{F}^{\,i}_{0\mu\nu j}. \tag{5.24}$$

Inserting Eq. (5.24) into (5.16) and using Eqs. (5.17) and (5.11) we obtain

$$W-W_0=\frac{\lambda_1}{b^2}\frac{(\tilde{F}_{1j}^{\mu\nu i}\tilde{F}_{0\mu\nu i}^{j})^2}{N^2-1}\ . \tag{5.25}$$

The form of the function $f(\mathbf{x})$in Eq. (5.24) for $\rho>>1$ can be obtained by linearizing Eq. (3.24). This is compatible with the quadratic expansion, (5.16) for W. Equation (3.24) then becomes $\partial^2\tilde{F}_{1\mu\nu j}^{i}=\delta W/\delta\tilde{F}_{1i}^{\mu\nu j}$ which combined with Eq. (5.25) gives

$$\partial^2\tilde{F}_{0\mu\nu i}^{j}\tilde{F}_{1j}^{\mu\nu i}=-8\lambda_1\tilde{F}_{0\mu\nu i}^{j}\tilde{F}_{1j}^{\mu\nu i}\ . \tag{5.26}$$

Inserting Eq. (5.24) into (5.26) and taking f independent of ϕ, we obtain the following equation for $f(\rho)$ valid for $\rho>>1$:

$$(-\nabla^2+8\lambda_1)f=0\ , \tag{5.27}$$

which has the solution

$$f\approx\frac{Ae^{-\sqrt{8\lambda_1}\rho}}{\rho^{1/2}}\quad\text{for } \rho>>(8\lambda_1)^{-1/2}\ . \tag{5.28}$$

Now the part of the field $\tilde{F}_{\mu\nu}$ which is proportional to $\tilde{F}_{0\mu\nu}$ becomes small inside the flux tube, i.e., $\tilde{F}_{1\mu\nu}\to\tilde{F}_{0\mu\nu}$ or $f\to 1$. We can approximate this situation by integrating Eq. (5.23) from infinity to $\rho=(8\lambda_1)^{-1/2}$ using the asymptotic expression (5.28). We determine the constant A by requiring $f=1$ at this point. The contribution to the string tension from smaller distances (i.e., inside the flux tube) will be neglected. Combining Eqs. (5.23), (5.25), and (5.28) then gives the expression (5.22) for σ, which, when combined with Eq. (5.13) for G_2, gives

$$\sigma\sim\frac{\pi}{4}\left[\frac{M}{g}\right]^2(-\tilde{F}_0^2)=\frac{\pi}{4}\left[\frac{g^2}{M^2}\right]G_2\ . \tag{5.29}$$

We emphasize that Eq. (5.29) relating the SU(N) string tension to the gluon condensate is a rough estimate which accounts for the long-distance contribution to the string tension in an approximation where only the leading long-distance exponential is considered.[25] The asymptotic solutions of the exact equations (3.24) and (3.25) will be linear combinations of exponentials. However, neglect of the nonleading contributions should not change the estimate (5.29) qualitatively since for fixed b^2 it is independent of the coefficient λ_1, which determines the rate of exponential decay.

Next let us make a similar estimate of the mass of a spherically symmetric static solution of Eqs. (3.24) and (3.25) which can be interpreted as a glueball. [In a previous publication we have explicitly obtained such a solution for the case of SU(2).] The magnetic pressure of the QCD vacuum keeps the gluon distribution in the glueball confined to a finite region of space. Estimating the mass of this glueball by the same long-distance approximation that yielded Eq. (5.29), we obtain

$$M_{\text{glueball}}\sim 2\pi\sigma(8\lambda_1M^2)^{-1/2}=2\pi\sigma\left[\frac{6}{11N}\right]^{1/2}\frac{g}{M}\ . \tag{5.30}$$

This result is even cruder than Eq. (5.29) since it depends explicitly upon λ_1 and is therefore very sensitive to the assumption of a single exponential. The origin of this result is evident. The factor $(8\lambda_1M^2)^{-1/2}$ is the natural length scale of the problem [see Eqs. (5.27) and (5.4)] and the factor 2π arises from the transition from cylindrical to spherical symmetry. [σ is defined by the two-dimensional cylindrically symmetric integration, Eq. (5.23), while M_{glueball} is determined by a corresponding three-dimensional spherically symmetric integration.]

Finally we estimate the mean-square radius R_{FT}^2 of the flux tube. Using Eqs. (5.20) and (5.21) we can write

$$R_{\text{FT}}^2=\frac{1}{M^2}\frac{\int \mathbf{x}^2d^2\mathbf{x}[\mathcal{H}_d(x)-\frac{1}{2}\lambda_1\tilde{F}_0^2]}{\int d^2\mathbf{x}[\mathcal{H}_d(x)-\frac{1}{2}\lambda_1\tilde{F}_0^2]}\ . \tag{5.31}$$

The approximation (5.23), (5.24), and (5.25) then gives

$$R_{\text{FT}}^2=\frac{1}{M^2}\frac{\int d^2\mathbf{x}f^2(\mathbf{x})\mathbf{x}^2}{\int d^2\mathbf{x}f^2(\mathbf{x})}\ , \tag{5.32}$$

where f is given by Eq. (5.28) for $\rho<<1$. Taking $\rho^{1/2}f=e^{-\beta_0\rho}$, with $\beta_0=\sqrt{8\lambda_1}$ gives

$$R_{\text{FT}}^2\sim\frac{1}{2\beta_0M^2}=\frac{1}{16\lambda_1M^2}=\frac{3}{11N}\left[\frac{g}{M}\right]^2\ . \tag{5.33}$$

This estimate is also very crude. However it gives the correct dependence of R_{FT} upon N. In Sec. VIII we will calculate R_{FT}^2 exactly for SU(2).

The $(M/2)\tilde{F}_{\mu\nu}^{a}G_{\mu\nu}^{a}$ term in $\mathcal{L}(C)$ played no role in obtaining the long-distance estimate (5.29). Yet it is essential in $\mathcal{L}(C)$ since it reflects the presence of the $M^2/\mathcal{D}^2(C)$ term in the original nonlocal form of the dual Lagrangian. To understand its effect on the string tension let us see what happens if it is not present.

We first note that if $M=0$, then σ is proportional to b^2. To see this, we set $M=0$ in $\mathcal{L}(C)$, but retain μ as an independent parameter [instead of being related to M by Eq. (4.39)] so that flux-tube solutions can exist. Next make the rescaling transformation (5.4) where M is now a mass scale. The Lagrangian then takes the form (5.6), except that the $\frac{1}{2}\tilde{F}^{\mu\nu a}G_{\mu\nu}^{a}$ term is absent. The string tension will still have the form (5.20) where σ_d is a function of λ_1 and λ_2 and hence depends upon the arbitrary mass scale M^2 only via its dependence on $\lambda_1=-N\mu^2/4M^2$. Since σ cannot depend upon the arbitrary scale M^2, σ_d must equal λ_1 multiplied by a function only of λ_2. Using Eq. (5.11) we can then write

$$\sigma_d=b^2f(\lambda_2)\ , \tag{5.34a}$$

or equivalently

$$\sigma=\frac{-3\mu^2}{4N\lambda}f\left[\frac{N\lambda}{4!g^2}\right]\ . \tag{5.34b}$$

Thus with $M=0$ in $\mathcal{L}(C)$ we see that the string tension is proportional to b^2, i.e., to μ^2/λ, provided that λ_2 is fixed.[26]

However, when $M\neq 0$, the Lagrangian $\mathcal{L}(C)$ does not yield a string tension satisfying this scaling law. Instead (5.34) is replaced by a more complicated relation from which one can only conclude that σ is proportional to b^2 for large b^2. We will see in Sec. VIII that numerical solutions of the SU(2) flux-tube equations verify the b^2 dependence of the string tension for $b^2 \gg 1$, but that it decreases less rapidly with b^2 for small b^2. For $b^2 \ll 1$, σ is more nearly linear in b^2. Thus for $b^2 \leq 1$, significant contributions to the string tension appear to arise from short distances, inside the flux tube itself. The presence of the $(M/2)\tilde{F}^{\mu\nu a}G^a_{\mu\nu}$ term in $\mathcal{L}(C)$ thus produces an important change in the dynamics of flux-tube formation.

We note that the long-distance expressions (5.29), (5.30), and (5.33) for σ, M_{glueball}, and R_{FT}, as well as Eq. (5.13) for G_2, indicate that long-distance contributions to physical quantities depend on M and g only in the combination (M/g) (Ref. 27). The parameter μ^2 is, we recall, fixed in terms of (M/g) by requiring that the classical approximation to the trace anomaly is satisfied. The remaining parameter in $\mathcal{L}(C)$, namely, λ, is related to b^2 according to Eq. (4.40), which we may rewrite as $\lambda = 11/8Nb^2$.

Yang-Mills theory is specified in terms of a single dimensionful parameter Λ_{QCD}. The appearance of two parameters, M/g and b^2, in $\mathcal{L}(C)$ reflects the degree to which our "derivation" of $\mathcal{L}(C)$ from Yang-Mills theory is imprecise. We must take b^2 as a free parameter, though in principle it is determined from Yang-Mills theory. In practice we can determine it by calculating σ_d as a function of b^2 and fitting the experimental string tension. Since b^2 is the dimensionless gluon condensate per color degree of freedom, we can anticipate that b^2 will be of order unity. If $b^2 \ll 1$, short distances contribute enough to σ so that use of $\mathcal{L}(C)$ may be unreliable.

It is interesting to use the above results to see what happens when we take the large-N limit of QCD in which $N \to \infty$ and $e \to 0$, such that Ne^2 remains fixed. There are arguments that QCD becomes a string theory in this limit.[28,29] The usual large-N diagrammatic analysis[28,29] shows that in this limit

$$G_2 \sim N, \quad \epsilon_{\text{vac}} \sim N^2, \quad \sigma \sim N, \quad M_{\text{glueball}} \sim \sqrt{N}, \quad R_{\text{FT}} \sim 1/\sqrt{N} \; . \tag{5.35}$$

Let us now see how the parameters of $\mathcal{L}(C)$ depend upon N in this limit. Since $g = 2\pi/e$ we must take

$$g \sim \sqrt{N} \; . \tag{5.36}$$

Equation (5.14) then implies

$$\lambda_1 \sim \text{independent of } N \; . \tag{5.37}$$

From Eq. (5.33) we obtain $R_{\text{FT}} \sim 1/\sqrt{N}$ provided

$$M \sim \sqrt{N} \; . \tag{5.38}$$

Using Eqs. (5.36), (5.37), and (5.38), we find from Eqs. (5.29) and (5.30) that $\sigma/G_2 \sim$ independent of N and $M_{\text{glueball}}/\sigma \sim 1/\sqrt{N}$ in accordance with Eq. (5.35). Finally Eq. (5.13) gives $G_2 \sim N$ provided that

$$b^2 \sim \frac{1}{N} \; . \tag{5.39}$$

Equation (5.39) follows from Eq. (5.11) if

$$\lambda_2 \sim \text{independent of } N \; . \tag{5.40}$$

Equation (5.7) then shows that the original unscaled parameters λ and μ^2 are independent of N. To summarize, we obtain expressions for G_2, ϵ_{vac}, σ, M_{glueball}, and R_{FT} which have the desired large-N behavior, Eq. (5.35), provided M behaves as $\sqrt{N}$.

From Eqs. (5.6) and (5.8) we note that $\mathcal{L}(C)$ depends explicitly upon M and g only via a multiplicative factor M^4/g^2. Thus the semiclassical expansion $\hbar \to 0$ of $\exp[(i/\hbar)\int \mathcal{L}]$ can be achieved by keeping $\hbar$, λ_1, and λ_2 fixed and letting g^2/M^4 become small. On the other hand, from Eqs. (5.36), (5.37), (5.38), and (5.40), we see that the large-N expansion corresponds to the limit

$$\frac{g^2}{M^2} \sim \frac{1}{N}, \quad \lambda_1 \text{ fixed}, \quad \lambda_2 \text{ fixed} \; . \tag{5.41}$$

Thus we conclude that the semiclassical expansion around our flux-tube solution coincides with the large-N expansion and that the semiclassical expansion parameter is $1/N$.

Furthermore, since the width of the flux tube is $1/\sqrt{N}$, we must let the radius of the flux tube go to zero at the same time that we make the semiclassical expansion. The leading term in the semiclassical expansion about this zero width vortex should yield the Nambu-Goto string[30] according to arguments given by Gervais and Sakita[4] in the context of the Nielsen Olesen vortex solution of the Abelian Higgs model. This string theory has the familiar spectrum of linearly rising Regge trajectories of stable meson resonances. The next order of the semiclassical expansion ($1/N$ expansion) must at the same time account for the finite radius of the classical flux-tube solution. To order $1/N$ the meson spectrum will be altered and the meson will acquire a width. These general conclusions are in accordance with the expectations of the usual large-N analysis.

Our work therefore provides further justification that the large-N limit of QCD is a string theory. It goes beyond previous analyses by explicitly constructing a classical flux-tube solution as the starting point and by showing that the expansion in $1/N$ is just the semiclassical expansion around this solution. In principle the first $1/N$ corrections should correct some of the unphysical features of the Nambu-Goto string, and a physically satisfactory string theory could result. Of course this program remains to be carried out and we have no concrete knowledge of the nature of the string theory that might finally emerge. However we might hope that a string theory arising from a well-behaved field theory (Yang-Mills theory) will possess only physically acceptable properties. The unphysical features of retaining only the leading term in the $1/N$ expansion can then be attributed to the inadequacy of the approximation rather than of the theory itself.

VI. SYMMETRY STRUCTURE OF THE ASYMPTOTIC FLUX-TUBE FIELDS

In Sec. IV we determined the value of $\tilde{F}_0^2$, but did not specify the detailed Lorentz or gauge structure of $\tilde{F}_{0\mu\nu}$. Given any particular choice of $\tilde{F}_{0\mu\nu}$ we can of course make a gauge rotation or Lorentz transformation without changing the value of $\tilde{F}_0^2$. However in order to explicitly construct a flux-tube solution we must make a particular choice of $\tilde{F}_{0\mu\nu}$. Furthermore we must specify the gauge rotation $\Omega_M(x)$ from which the asymptotic flux-tube fields are constructed via Eq. (4.4). Since $\tilde{F}_0^2<0$, we will choose $\tilde{F}_{0\mu\nu}$ to be purely magnetic. We define rescaled color-electric and -magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in terms of the rescaled fields $\tilde{F}_{\mu\nu}$ by Eqs. (3.11) with M replaced by unity. Henceforth $\mathbf{E}$ and $\mathbf{B}$ will represent the rescaled fields.

We then choose our vacuum fields $\mathbf{B}_0$ and $\mathbf{E}_0$ to be

$$\mathbf{E}_0=0,\quad \mathbf{B}_0=\mathbf{B}_{01}J_x+\mathbf{B}_{02}J_y+\mathbf{B}_{03}J_z\ , \tag{6.1}$$

where $\mathbf{B}_{0i}, i=1,2,3$ are three orthogonal vectors having a common magnitude. We take $\mathbf{B}_{03}$ along the axis of the flux tube. The matrices J_x, J_y, and J_z are three SU(N) generators which form an N-dimensional irreducible representation of an SU(2) subalgebra. They satisfy the equation

$$J_x^2+J_y^2+J_z^2=\frac{N^2-1}{4}\ . \tag{6.2}$$

It is easy to show that one can always construct such matrices as linear combinations of the generators T_a of SU(N). For example, for SU(2) we choose

$$J_x=\frac{\tau_1}{2},\quad J_y=\frac{\tau_2}{2},\quad J_z=\frac{\tau_3}{2}\ , \tag{6.3}$$

while for SU(3) we choose

$$J_x=\lambda_5,\quad J_y=\lambda_7,\quad J_z=\lambda_2\ , \tag{6.4}$$

where λ_i are the SU(3) generators in the notation of Gell-Mann. In the case of SU(3), had we chosen $J_x=\lambda_1/2$, $J_y=\lambda_2/2$, $J_z=\lambda_3/2$, we would not have obtained Z_3 flux-tube solutions. Instead we would have found a solution which could have been deformed into the vacuum via intermediate SU(3) configurations.

The choice (6.1) for $\mathbf{B}_0$ yields a vacuum which is rotationally invariant because a rotation in ordinary space of the three spatial vectors $\mathbf{B}_{0i}$ can always be compensated by an SU(N) rotation of the matrices J_x, J_y, and J_z such that the matrix $\mathbf{B}_0$ is left invariant. This vacuum is, however, not invariant under pure Lorentz transformations. Since $\tilde{F}_{0\mu\nu}$ is used to construct the asymptotic field surrounding a static flux tube, there is a natural preferred Lorentz system, namely, the system in which the classical solution is static. To understand the restoration of Lorentz invariance requires going beyond the static classical approximation.

The magnitudes of the vectors $\mathbf{B}_{0i}$ are fixed by Eq. (5.9) which, when expressed in terms of the matrix $\mathbf{B}_0$, takes the form

$$-2\,\mathrm{tr}\mathbf{B}_0\cdot\mathbf{B}_0=\frac{\tilde{F}_0^2}{2}=-b^2(N^2-1)\ . \tag{6.5}$$

On the other hand, Eqs. (6.1) and (6.2) give

$$\begin{aligned}-2\,\mathrm{tr}\mathbf{B}_0\cdot\mathbf{B}_0&=-2\mathbf{B}_{01}^2\mathrm{tr}(J_x^2+J_y^2+J_z^2)\\ &=-\frac{\mathbf{B}_{01}^2}{2}(N^2-1)N\ ,\end{aligned} \tag{6.6}$$

where $\mathbf{B}_{01}^2$ is the common magnitude squared of vectors $\mathbf{B}_{0i}$. Combining Eqs. (6.5) and (6.6) we obtain

$$\mathbf{B}_{01}^2=\frac{2b^2}{N}\ . \tag{6.7}$$

Choosing coordinate axes along $\mathbf{B}_{0i}$ we can write (6.1) as

$$\mathbf{E}_0=0,\quad \mathbf{B}_0=b\left[\frac{2}{N}\right]^{1/2}(J_x\hat{\mathbf{e}}_x+J_y\hat{\mathbf{e}}_y+J_z\hat{\mathbf{e}}_z)\ . \tag{6.8}$$

Next we construct the gauge transformation $\Omega_M(x)$ which, via Eqs. (4.4) and (6.8), determine the field configurations at large distances from the center of the flux tube. We take $\Omega(0)=1$, so that $\Omega(2\pi)$ belongs to Z_N. For SU(2) we choose

$$\Omega_M(\varphi)=e^{-i\tau_3\phi/2}\ . \tag{6.9}$$

We have

$$\begin{aligned}\Omega_M^{-1}\nabla\Omega_M&=-\frac{i\hat{\mathbf{e}}_\phi}{\rho}\frac{\tau_3}{2}\ ,\\ \Omega_M^{-1}\tau_1\Omega&=\tau_1\cos\varphi-\tau_2\sin\varphi\ ,\\ \Omega_M^{-1}\tau_2\Omega_M&=\tau_2\cos\varphi+\tau_1\sin\varphi\ .\end{aligned} \tag{6.10}$$

Then from Eqs. (6.3), (6.8), and (6.10) we obtain the following large-distance fields surrounding an SU(2) flux tube:

$$\begin{aligned}C_0&=\mathbf{E}=0\ ,\\ \mathbf{C}&=\frac{1}{ig}\Omega_M^{-1}\nabla\Omega_M=-\frac{1}{g}\frac{\hat{\mathbf{e}}_\phi}{\rho}\frac{\tau_3}{2}\ ,\\ \mathbf{B}&=\Omega_M^{-1}(\phi)\mathbf{B}_0\Omega_M(\phi)\\ &=b\left[\hat{\mathbf{e}}_\rho\frac{\tau_1}{2}+\hat{\mathbf{e}}_\phi\frac{\tau_2}{2}+\hat{\mathbf{e}}_z\frac{\tau_3}{2}\right]\ .\end{aligned} \tag{6.11}$$

The 1 and 2 color components of the magnetic field point in the radial and tangential directions, respectively, while the 3 component of the $\mathbf{B}$ remains aligned along the z axis. Their magnitudes remain equal. Furthermore we have

$$\Omega_M^{-1}(\varphi=2\pi)=e^{2\pi i/2}\ . \tag{6.12}$$

Comparing Eq. (6.12) with Eqs. (4.7) and (4.8) we see that the configuration (6.11) describes an $n=1$, Z_2 flux tube. We can also show this by evaluating $W_C(l)$ directly. Using Eq. (6.11) we have

$$\oint dx_\mu C^\mu = \frac{2\pi}{g}\frac{\tau_3}{2} = e\frac{\tau_3}{2} \ . \tag{6.13}$$

The tube then contains a unit $e/2$ of color-electric flux. Since the large-distance potential Eq. (6.11) is Abelian we have

$$W_C(l) = \exp\left[ig\oint dx_\mu C^\mu\right] = \exp\left[2\pi i\frac{\tau_3}{2}\right] = e^{2\pi i/2} \ , \tag{6.14}$$

which coincides with Eq. (6.12).

For the case of SU(3) we choose

$$\Omega_M^{(n)}(\varphi) = e^{-inY\varphi}, \quad n=1,2 \ , \tag{6.15}$$

where Y is the hypercharge matrix:

$$Y = \frac{\lambda_8}{\sqrt{3}} = \frac{1}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix} . \tag{6.16}$$

Then

$$\begin{aligned}
&\Omega_M^{(n)-1}\nabla\Omega_M^{(n)} = -in\frac{\hat{\mathbf{e}}_\phi}{\rho}Y, \quad n=1,2, \ ,\\
&\Omega_M^{(n)-1}\lambda_2\Omega_M^{(n)} = \lambda_2 \ ,\\
&\Omega_M^{(n)-1}\lambda_5\Omega_M^{(n)} = \lambda_5\cos n\varphi + \lambda_4\sin n\varphi \ ,\\
&\Omega_M^{(n)-1}\lambda_7\Omega_M^{(n)} = \lambda_7\cos n\varphi + \lambda_8\sin n\varphi \ .
\end{aligned} \tag{6.17}$$

Thus at large distances from a flux tube containing n units of Z_3 flux we have

$$\begin{aligned}
&\mathbf{C} = -\frac{n}{g}\frac{\hat{\mathbf{e}}_\phi}{\rho}Y, \quad n=1,2 \ ,\\
&\mathbf{B} \equiv b\left(\tfrac{2}{3}\right)^{1/2}[\hat{\mathbf{e}}_x(\lambda_5\cos n\varphi + \lambda_4\sin n\varphi)\\
&\qquad + \hat{\mathbf{e}}_y(\lambda_7\cos n\varphi + \lambda_6\sin n\varphi) + \hat{\mathbf{e}}_z\lambda_2] \ .
\end{aligned} \tag{6.18}$$

The color-electric flux contained in the tube is

$$\oint dx_\mu C^\mu = \frac{e}{3}n(3Y) \ . \tag{6.19}$$

Hence

$$\begin{aligned}
W_C(l) &= \exp\left[ig\frac{e}{3}n3Y\right] = \exp\left[2\pi i\frac{n}{3}(3Y)\right]\\
&= e^{2\pi i n/3} \ .
\end{aligned} \tag{6.20}$$

The result (6.20) also follows directly from (4.7) and (6.15), i.e.,

$$W_C(l) = \Omega^{-1}(\varphi=2\pi) = e^{2i\pi nY} = e^{2\pi i n/3} \ . \tag{6.21}$$

A flux-tube solution having the asymptotic behavior Eq. (6.18) thus contains n units of Z_3 color-electric flux.

In the case of SU(N) we can choose

$$\Omega_M^{(n)}(\varphi) = e^{-in\varphi Y_N} \ , \tag{6.22}$$

where Y_N is a traceless diagonal matrix satisfying the equation

$$e^{2\pi i Y_N} = e^{2\pi i/N} \ . \tag{6.23}$$

The resulting long-distance electric vector potential,

$$\mathbf{C} = -\frac{n}{g}\frac{\hat{\mathbf{e}}_\phi}{\rho}Y_N \ , \tag{6.24}$$

corresponds to a Z_N flux tube containing n units of Z_N flux.

The explicit form of the matrix T_{jl}^{ik}, corresponding to the vacuum configuration, Eq. (6.8), is[31]

$$T_{jl}^{ik} = \frac{2\tilde{F}_0^2}{N}\left[\frac{1}{N^2-1}\right]\mathbf{J}_j^i\cdot\mathbf{J}_l^k \ . \tag{6.25}$$

Then

$$M_l^i \equiv T_{kl}^{ik} = 2\left[\frac{\tilde{F}_0^2}{N}\right]\left[\frac{1}{N^2-1}\right]\mathbf{J}_k^i\cdot\mathbf{J}_l^k = \frac{\tilde{F}_0^2}{N}\delta_l^i \ , \tag{6.26}$$

in agreement with Eq. (4.19). In obtaining Eq. (6.26) we have used the fact that in our notation $\mathbf{J}_j^i/\sqrt{2}$ are the conventionally normalized angular momentum matrices. [See Eq. (3.19).] Now for $N=2$ it is easy to verify that T_{jl}^{ik} has the singlet form Eq. (4.20). As pointed out earlier this is only true for SU(2). For example, for SU(3) we find, using Eqs. (6.4), (6.25), and Appendix B, that

$$T_{jl}^{ik} = \frac{\tilde{F}_0^2}{6}(\delta_l^i\delta_j^k - \delta_k^i\delta_l^j) \ , \tag{6.27}$$

which clearly does not have the color-singlet structure. In fact Eq. (6.27) implies for all SU(N) that

$$T_{jn}^{im}T_{ml}^{nk} = \frac{\tilde{F}_0^2}{3}T_{jl}^{ik} \ , \tag{6.28}$$

which agrees with Eq. (4.21) only for $N=2$. These results make explicit, as pointed out in Sec. IV, that for $N\neq 2$ the classical expression, Eq. (4.9), cannot have the color-singlet form.[32]

To obtain the value of $\tilde{F}_0^2$ and W_0 for the SU(N) vacuum having the structure of Eq. (6.8) we insert Eq. (6.28) into Eq. (4.16). This yields the following equation for $\tilde{F}_0^2$:

$$\mu^2 = \frac{\lambda}{6}\left[\frac{\tilde{F}_0^2}{N} + \frac{\tilde{F}_0^2}{N} + \frac{2\tilde{F}_0^2}{N} + \frac{4}{N}\frac{\tilde{F}_0^2}{3}\right] , \tag{6.29}$$

instead of Eq. (4.22), obtained from the color-singlet expression, Eq. (4.21). Equation (6.29) then gives

$$\tilde{F}_0^2 = \frac{9}{8}\frac{\mu^2}{\lambda}N \ . \tag{6.30}$$

This value coincides with Eq. (4.23) for $N=2$ and is smaller (in magnitude) for $N\geq 3$. The value of W at an extremum is given in terms of the value of $\tilde{F}_0^2$ by Eq. (4.17). For $\tilde{F}_0^2$ given by Eq. (4.23), W has the value W_0, Eq. (4.24), which is an absolute minimum, while for $\tilde{F}_0^2$

given by (6.30) W has the value

$$W_0=-\frac{9}{64}\frac{(\mu^2)^2}{\lambda}N^2\ , \tag{6.31}$$

which is only a relative minimum of W unless $N=2$. For SU(3) we obtain

$$\tilde{F}_0^2=\frac{27}{8}\frac{\mu^2}{\lambda}\quad\text{instead of}\quad\frac{4\mu^2}{\lambda} \tag{6.32}$$

and

$$W_0=-\frac{81}{64}\frac{(\mu^2)^2}{\lambda}\quad\text{instead of}\quad-\frac{3}{2}\frac{(\mu^2)^2}{\lambda}\ . \tag{6.33}$$

The relation (4.29) between G_2 and the parameters of the theory will be correspondingly modified. These changes are small compared to the experimental uncertainties in G_2.

VII. SU(2)

For the case of SU(2) we use Eq. (3.21) to express W in terms of the fields $\tilde{F}^a_{\mu\nu}$ and obtain the expression

$$W=-\frac{\mu^2}{2}\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu a}+\frac{\lambda}{12}[(\tilde{F}^a_{\alpha\beta}\tilde{F}^{\alpha\beta a})^2+(\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu b})(\tilde{F}^a_{\alpha\beta}\tilde{F}^{\alpha\beta b})]\ . \tag{7.1}$$

Let $\tilde{F}_{\mu\nu}\to(M/g)\tilde{F}_{\mu\nu}$ and $W\to(M/g^2)W$. Then

$$W=\lambda_1\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu a}+\lambda_2[(\tilde{F}^a_{\alpha\beta}\tilde{F}^{\alpha\beta a})^2+(\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu b})(\tilde{F}^a_{\alpha\beta}\tilde{F}^{\alpha\beta b})]\ , \tag{7.2}$$

where, from Eq. (5.7),

$$\lambda_1=-\frac{\mu^2}{2M^2},\quad \lambda_2=\frac{\lambda}{12g^2}\ . \tag{7.3}$$

The vacuum value of $\tilde{F}^2$, given by Eqs. (5.9) and (5.11) is

$$\tilde{F}_0^2=-6b^2,\quad\text{where } b^2=\frac{\lambda_1}{16\lambda_2}\ . \tag{7.4}$$

The quadratic expansion of W about its minimum takes the form

$$W=-3\lambda_1b^2+\tfrac{16}{3}\lambda_2(\tilde{F}^a_{0\mu\nu}\tilde{F}_1^{\mu\nu a})^2+4\lambda_2\left[\frac{\tilde{F}^a_{0\mu\nu}\tilde{F}_1^{\mu\nu b}+\tilde{F}^a_{1\mu\nu}\tilde{F}_0^{\mu\nu b}}{2}-\tfrac{1}{3}\delta^a_b\tilde{F}^d_{1\mu\nu}\tilde{F}_0^{\mu\nu d}\right]\left[\frac{\tilde{F}^a_{0\alpha\beta}\tilde{F}_1^{\alpha\beta b}+\tilde{F}^a_{1\alpha\beta}\tilde{F}_0^{\alpha\beta b}}{2}-\tfrac{1}{3}\delta^a_b\tilde{F}^d_{1\alpha\beta}\tilde{F}_0^{\alpha\beta d}\right]\ . \tag{7.5}$$

The expression (6.8) for the vacuum color-magnetic field can be written as

$$B^a_{0i}=b\delta^a_i\ , \tag{7.6}$$

where the upper index refers to color and the lower index to space. If we write the expansion of B^a_i about its vacuum value as

$$B^a_i=B^a_{0i}+B^a_{1i}\ , \tag{7.7}$$

then Eq. (7.5) can be written as

$$W=-3\lambda_1b^2+\frac{\beta_0}{2}S^{ab}S^{ab}+\frac{\beta_2}{2}T^{ab}T^{ab}\ , \tag{7.8}$$

where

$$S^{ab}=\frac{\delta^a_b}{3}\left[\sum_d B^d_{1d}\right]\ , \tag{7.9}$$

$$T^{ab}=\frac{B^a_{1b}+B^b_{1a}}{2}-\frac{\delta^a_b}{3}\left[\sum_d B^d_{1d}\right]\ , \tag{7.10}$$

and

$$\beta_0=8\lambda_1,\quad \beta_2=32\lambda_2b^2=2\lambda_1\ . \tag{7.11}$$

The "scalar" field S^{ab} has a single independent component $\sum_d B^d_{1d}$ which is invariant under a simultaneous rotation in color space and ordinary space. The traceless "tensor" field T^{ab} has five independent components which transform as an irreducible second-rank tensor under a simultaneous rotation in color space and ordinary space.

That part of the kinematic term in the Lagrangian, $\tilde{F}^a_{\mu\nu}[(\mathcal{D}^2)^{ab}/4]\tilde{F}^{\mu\nu b}$, which is quadratic in B^a_i can be expressed in terms of S^{ab} and T^{ab} as

$$-\frac{B^a_i\partial^2B^a_i}{2}=-\tfrac{1}{2}S^{ab}\partial^2S^{ab}-\tfrac{1}{2}T^{ab}\partial^2T^{ab}-\tfrac{1}{2}A^{ab}\partial^2A^{ab}\ , \tag{7.12}$$

where

$$A^{ab}\equiv\tfrac{1}{2}(B^a_{1b}-B^b_{1a})\ . \tag{7.13}$$

The "antisymmetric tensor" field A^{ab} accounts for the remaining three independent components of B^a_i. Comparing Eqs. (7.12) and (7.8) we see that $(\beta_0M^2)^{1/2}$ is the "mass" of the scalar field and that $(\beta_2M^2)^{1/2}$ is the "mass" of the tensor field. The massless field A^{ab} is the "Goldstone boson" which arises because the vacuum is not invariant under spatial rotations. However because of the invariance of the vacuum configuration under a combined color and spatial rotation, the field A^{ab} can be eliminated from the Lagrangian by a gauge transformation, and we can set $A^{ab}=0$. The color-magnetic field B^a_{1i} will then be a linear combination of S^{ab} and T^{ab}, which will be exponentially damped at distances ρ from the center of the flux tube which are greater than $(\beta_0)^{-1/2}$ and $(\beta_2)^{-1/2}$, respectively.

The color-electric field $\mathbf{E}$ vanishes at large distances from the flux tube [see Eq. (6.11)]. However it cannot vanish everywhere since the ij component of the constitutive equation (3.24) relates E_k to the color-electric displacement vector D_k. The field $D_k=\frac{1}{2}\epsilon_{kij}G_{ij}$ vanishes at large distances since the tensor G_{ij} constructed from the asymptotic potential Eq. (6.11) via Eq. (5.5b) is zero. This is just the dual of the Meissner effect. The electric displacement vector cannot penetrate into the interior of the QCD vacuum just as a magnetic field cannot penetrate into the interior of a superconductor. On the other hand, $\mathbf{D}$ cannot vanish in the interior of a flux tube, since the tube contains a unit $e/2$ of flux of $\mathbf{D}$ [Eq. (6.13)]. From Eq. (3.24) there must then also be nonvanishing component of $\mathbf{E}$ inside the flux tube.

Since the color-electric field $\mathbf{E}$ does not vanish in the interior of the flux tube there must be a damping mechanism to cause it to vanish at large distances. However from Eqs. (7.8), (7.9), and (7.10), we see that the quadratic terms in $\tilde{F}_{1\mu\nu}$, which are those which are important at long distances, involve only the color-magnetic field $\mathbf{B}_1$. There is no term in Eq. (7.2) to produce the necessary large-distance damping of $\mathbf{E}$. However, the cubic coupling $gMf_{abc}\tilde{F}^a_{\mu\nu}C^{\mu b}C^{\nu c}$ induced by the term $(M/2)\tilde{F}^{\mu\nu a}G^a_{\mu\nu}$ in $\mathcal{L}$ gives rise to $\tilde{F}^4$ graphs which are not included in Fig. 2. An example of such a graph is shown in Fig. 3. Since it is ultraviolet finite it is not necessary to introduce a counterterm in $\mathcal{L}$ having the corresponding spacetime and color structure. Nevertheless we are free to add such a term so that some of the quantum effects of these graphs are accounted for in the classical approximation. We find that the quadratic approximation to W contains an additional term which is quadratic in $\mathbf{E}$ and of the same structure as Eq. (7.8). It thus produces a damping term in the equations of motion for $\mathbf{E}$ so that the electric field dies off exponentially at large distances. There results an everywhere finite confined flux-tube solution of Eqs. (3.24) and (3.25)

It is natural to include such a contribution to W since it arises from couplings induced by the $(M/2)G^a_{\mu\nu}\tilde{F}^{\mu\nu a}$ term in $\mathcal{L}$. It is just such a term which is responsible for giving $\mathbf{E}$ physical degrees of freedom and for producing a nonvanishing $\mathbf{E}$ inside the flux tube. However, in the next section, where we solve the equations of motion (3.24) and (3.25) with this additional term in W, we find that the field $\mathbf{E}$ makes a small contribution to the string tension and the glueball mass. The inclusion of this additional term in W therefore leaves the arguments presented in previous sections essentially unchanged. In particular the estimate (5.29) for the string tension comes from the region outside the flux tube and does not depend upon the details of what happens inside, except for the assumption that a solution exists. It is important that this estimate does not differ substantially from the results of the numerical calculations presented in the next section. If it did, then the value of the string tension would depend sensitively upon the details of the field distribution inside the flux tube and the long-distance Lagrangian $\mathcal{L}(C)$ could not be used reliably.

In the next section we will examine in more detail the sensitivity of σ to the details of the short-distance structure. We will do this by including in W all possible quartic SU(2)-invariant, Lorentz-invariant, terms in $\tilde{F}_{\mu\nu}$ so that W has the form

$$W=\lambda_1\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu a}+\lambda_2(\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu a})^2+\lambda_3(\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu b})(\tilde{F}^a_{\lambda\sigma}\tilde{F}^{\lambda\sigma b})+\lambda_4(\tilde{F}^a_{\mu\nu}\tilde{F}^{\nu\lambda b})(\tilde{F}^{\mu\beta a}\tilde{F}^b_{\beta\lambda})+\lambda_5(\tilde{F}^a_{\mu\nu}\tilde{F}^{\nu\lambda a})(\tilde{F}^{\mu\beta b}\tilde{F}^b_{\beta\lambda})\ . \tag{7.14}$$

With $\lambda_3=\lambda_2$ and $\lambda_4=\lambda_5=0$, the general form (7.14) reduces to Eq. (7.2). Deviations from these values are in principle calculable from quantum corrections. In particular, the terms involving λ_4 and λ_5 in W are induced by graphs involving the $gMf_{abc}\tilde{F}^a_{\mu\nu}C^{\mu b}C^{\nu c}$ coupling such as that of Fig. 3. The vacuum determined by the minimum of W still has $\mathbf{E}_0=0$ and $B^a_{0i}=b\delta^a_i$, where b is now given by

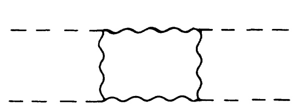

FIG. 3. Ultraviolet finite $\tilde{F}^4$ graph arising from the $(gM/2)\tilde{F}^a_{\mu\nu}\tilde{G}^{\mu\nu a}$ term in $\mathcal{L}$.

$$b^2=\frac{\lambda_1}{12\lambda_2+4\lambda_3+2\lambda_4+4\lambda_5}\ , \tag{7.15}$$

which is the generalization of Eq. (7.4).

Furthermore, expanding W [Eq. (7.14)] about this minimum and retaining quadratic terms in $\mathbf{B}_1$ and $\mathbf{E}$ we obtain the generalization of Eq. (7.8)

$$W=-3\lambda_1b^2+\frac{\beta_0}{2}S^{ab}S^{ab}+\frac{\beta_2}{2}T^{ab}T^{ab}-\frac{\alpha_0}{2}S^{ab}_ES^{ab}_E -\frac{\alpha_2}{2}T^{ab}_ET^{ab}_E\ , \tag{7.16}$$

where S^{ab}, T^{ab}, and b^2 are given by Eqs. (7.9), (7.10), and (7.15), respectively, and where

$$S_E^{ab}=\frac{\delta_b^a}{3}\left[\sum_d E_d^d\right],\quad T_E^{ab}=\frac{E_b^a+E_a^b}{2}-\frac{\delta_b^a}{3}\left[\sum_d E_d^d\right], \tag{7.17}$$

with

$$\beta_0=8\lambda_1,\quad \beta_2=8b^2(4\lambda_3+2\lambda_4+\lambda_5), \tag{7.18}$$

and

$$\alpha_0=-8b^2(3\lambda_4+\lambda_5),\quad \alpha_2=-8b^2\lambda_5 . \tag{7.19}$$

With $\lambda_2=\lambda_3$ and $\lambda_4=\lambda_5=0$, the above equations reduce to Eqs. (7.8)–(7.11) with b^2 given by Eq. (7.4).

Next note that the quadratic term in E_i^a arising from the kinematic term in the Lagrangian can be written as

$$\tfrac{1}{2}E_i^a\partial^2E_i^a=\tfrac{1}{2}S_E^{ab}\partial^2S_E^{ab}+\tfrac{1}{2}T_E^{ab}\partial^2T_E^{ab}+\tfrac{1}{2}A_E^{ab}\partial^2A_E^{ab}, \tag{7.20}$$

where

$$A_E^{ab}=\tfrac{1}{2}(E_b^a-E_a^b) . \tag{7.21}$$

Comparing Eqs. (7.20) and (7.16) we see that if $\alpha_0>0$, and $\alpha_2>0$, the fields S_E^{ab} and T_E^{ab} will be exponentially damped at large distances. Furthermore one can set $A_E^{ab}=0$ everywhere when seeking static solutions of Eqs. (3.24) and (3.25) (Ref. 33).

Finally we note that the estimates for physical quantities given in Sec. V remain the same, provided b^2 is defined by Eq. (7.15). The parameter λ_1 is, as before, determined in terms of g^2 by Eq. (5.14). Fixing b^2 then constrains the remaining λ_i. We will see in the next section that, unless b^2 is small, σ is not very sensitive to the values of $\lambda_2-\lambda_3$, λ_4, and λ_5 which are in principle calculable. Furthermore the string tension is then in the range given by the estimate (5.29), which verifies its insensitivity to the details of the short-distance structure of the flux tube.

VIII. EQUATIONS OF MOTION AND SOLUTIONS FOR SU(2) FLUX TUBES

Equations (6.11) give the behavior of the fields **E**,**B** and the potentials **C** as the cylindrical radius ρ approaches ∞ in an $n=1$ SU(2) flux tube. We now seek the simplest ansatz for the fields at finite distances for which the field equations (3.24) and (3.25) close. We assume for all ρ

$$\mathbf{B}=B_1(\rho)\hat{\mathbf{e}}_\rho T_1+B_2(\rho)\hat{\mathbf{e}}_\phi T_2+B_3(\rho)\hat{\mathbf{e}}_z T_3 . \tag{8.1a}$$

This is the simplest possibility, since the asymptotic **B** field Eq. (6.11) already has all these components. Then it is not hard to see that the field equations will close if we postulate, in addition, that

$$\mathbf{C}=C_3(\rho)\hat{\mathbf{e}}_\phi T_3+C_2(\rho)\hat{\mathbf{e}}_z T_2 \tag{8.1b}$$

$$C_0=C_0(\rho)T_1 \tag{8.1c}$$

and

$$\mathbf{E}=E(\rho)\hat{\mathbf{e}}_\rho T_1+E_2(\rho)\hat{\mathbf{e}}_\phi T_2+E_3(\rho)\hat{\mathbf{e}}_z T_3 . \tag{8.1d}$$

When the ansatz (8.1) is inserted into the field equations (3.24) and (3.25) (rescaled), they become the following set of differential equations for the rescaled variables.

"Gauss's law":

$$\nabla^2C_0-(C_2^2+C_3^2+B_2^2+B_3^2-E_2^2-E_3^2)C_0 -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_1-C_3B_2+C_2B_3=0 . \tag{8.2a}$$

"Ampere's law":

$$\tilde{\nabla}^2C_3-(C_2^2-C_0^2+B_1^2+B_2^2-E_1^2-E_2^2)C_3 -\frac{\partial E_3}{\partial\rho}-\frac{2B_1B_2}{\rho}+\frac{2E_1E_2}{\rho}+C_0B_2+C_2E_1=0 \tag{8.2b}$$

and

$$\nabla^2C_2-(C_3^2-C_0^2+B_1^2+B_3^2-E_1^2-E_3^2)C_2 +\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_2-C_0B_3+C_3E_1=0 . \tag{8.2c}$$

The electric constitutive equations:

$$\tilde{\nabla}^2E_1-(C_2^2+C_3^2)E_1-\frac{2C_3E_2}{\rho}-C_2C_3+\frac{\partial W}{\partial E_1}=0 , \tag{8.2d}$$

$$\tilde{\nabla}^2E_2-(C_3^2-C_0^2)E_2-\frac{2C_3E_1}{\rho}+\frac{\partial C_2}{\partial\rho}+\frac{\partial W}{\partial E_2}=0 , \tag{8.2e}$$

$$\nabla^2E_3-(C_2^2-C_0^2)E_3-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho C_3+\frac{\partial W}{\partial E_3}=0 . \tag{8.2f}$$

The magnetic constitutive equations:

$$\tilde{\nabla}^2B_1-(C_2^2+C_3^2)B_1-\frac{2C_3B_2}{\rho}-\frac{\partial C_0}{\partial\rho}-\frac{\partial W}{\partial B_1}=0 , \tag{8.2g}$$

$$\tilde{\nabla}^2B_2-(C_3^2-C_0^2)B_2-\frac{2C_3B_1}{\rho}+C_0C_3-\frac{\partial W}{\partial B_2}=0 , \tag{8.2h}$$

$$\nabla^2B_3-(C_2^2-C_0^2)B_3-C_0C_2-\frac{\partial W}{\partial B_3}=0 . \tag{8.2i}$$

In the above equations we use the notation

$$\nabla^2=\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho},\quad \tilde{\nabla}^2=\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho .$$

The last six of the above equations are the constitutive equations (3.24) while the first three are the equations of motion (3.25). This system of nine coupled nonlinear second-order differential equations describes in detail the field configurations associated with the $n=1$ SU(2) flux tube.

The ansatz (8.1) also gives us the explicit form of the function W, and therefore of the various derivatives of W appearing in the field equations. We find, using the general expression (7.14) for W, that

$$W=-2\lambda_1(A_1+A_2+A_3)+(4\lambda_2+4\lambda_3+2\lambda_4+2\lambda_5)(A_1^2+A_2^2+A_3^2)+(8\lambda_2+2\lambda_5)(A_1A_2+A_2A_3+A_3A_1)$$
$$+8\lambda_4(B_1E_1B_2E_2+B_2E_2B_3E_3+B_3E_3B_1E_1)+(4\lambda_4+4\lambda_5)(B_1^2E_1^2+B_2^2E_2^2+B_3^2E_3^2)\ , \tag{8.3}$$

where we use the notation

$$A_i=B_i^2-E_i^2\ . \tag{8.4}$$

As $\rho\to\infty$, the asymptotic behavior of the solutions to the differential equations (8.2) is just that given by Eq. (6.11). That is, in terms of the functions defined in Eq. (8.1) we have, as $\rho\to\infty$,

$$C_3\to-\frac{1}{\rho},\quad C_0\to 0,\quad C_2\to 0\ , \tag{8.5a}$$

$$B_1\to b,\quad B_2\to b,\quad B_3\to b\ , \tag{8.5b}$$

$$E_1\to 0,\quad E_2\to 0,\quad E_3\to 0\ . \tag{8.5c}$$

The approach to these asymptotic values is exponential. To study the approach to the asymptotic regime, the field equations can be linearized around the asymptotic values with W replaced by its quadratic expansion Eq. (7.16). As pointed out in Sec. VII, the constants α_i and β_i in Eq. (7.16) must all be positive to assure exponential approach to the vacuum.

Solutions to the nine coupled nonlinear differential equations (8.2) are obtained numerically for various values of the parameters λ_1 through λ_5. Boundary conditions are imposed at a large cutoff radius R, simulating infinity, at which we require $B_1=B_2=B_3=b$, $C_3=-1/R$, and $C_0=C_2=E_1=E_2=E_3=0$. The solutions are of course tested for sensitivity to the cutoff radius.

At the origin, the field equations show that C_3, E_1, E_2, B_1, and B_2 vanish while C_0, C_2, E_3, and B_3 approach constants. Accordingly, in the numerical solutions, the functions C_3, E_1, E_2, B_1, and B_2 are set equal to zero at the origin, while the first derivatives of C_0, C_2, E_3, and B_3 are made to vanish.

To calculate the string tension we use Eqs. (5.20) and (5.21). Substituting our ansatz (8.2) into the explicit expression[24] for $\mathcal{H}_d$ given in Ref. 5 and using the field equations (8.2), we obtain

$$\mathcal{H}_d=\Bigg[\tfrac{1}{2}\Bigg[C_0^2(C_3^2+C_2^2)-C_3^2C_2^2+C_0^2(B_2^2+B_3^2-E_2^2-E_3^2)-C_2^2(B_1^2+B_3^2-E_1^2-E_3^2)-C_3^2(B_1^2+B_2^2-E_1^2-E_2^2)$$
$$+\frac{2C_3}{\rho}(E_1E_2-B_1B_2)\Bigg]+W-\tfrac{1}{2}B_i\frac{\partial W}{\partial B_i}-\tfrac{1}{2}E_i\frac{\partial W}{\partial E_i}\Bigg]\ . \tag{8.6}$$

Next we must choose values for the parameters appearing in W. The parameter λ_1 is determined in terms of the coupling constant g by Eq. (5.14), while the value of g can be estimated from the $1/R$ contribution to phenomenologically determined potentials between heavy quarks as follows. The $1/R$ part of the potential is the dominant contribution at shorter distances, $R\simeq(1\ \text{GeV})^{-1}$, which is already inside the flux tube. We can estimate the potential at such distances from the exchange of a C_μ gluon between quarks in the perturbative vacuum, where the gluon propagator is given by Eqs. (3.15), (3.16), and (3.17). The quark C_μ gluon coupling can be found by generalizing arguments given in Ref. 7. However since no non-Abelian effects enter into the calculation it can be equally well carried out using the original A_μ variables. Exchanging a gluon with a propagator Δ_A^0, Eq. (2.35), between quarks gives rise to a vector interaction mediated by the sum of a Coulomb potential and a linear potential. Inside the flux tube where such a calculation is applicable the linear term generated by the M^2/q^4 propagator is unimportant and the potential is essentially Coulombic with the usual coefficient in SU(3):

$$V(R)=-\frac{4}{3}\frac{\alpha_s}{R}=-\frac{\pi}{g^2R}\ . \tag{8.7}$$

We emphasize that this is an estimate approximately valid inside the flux tube. The single Δ_A^0 gluon exchange bears no relation to the potential outside the flux tube where the nonvanishing magnetic field confines the electric flux and their results a Lorentz-scalar linear potential between quarks whose strength is given by the string tension σ. The long-distance potential is not directly associated with the M^2/q^4 small-q^2 behavior of Δ_A^0. Comparing Eq. (8.7) to a phenomenological potential,[34] we find

$$g^2\approx 6.6\quad\text{for } R\simeq(1\ \text{GeV})^{-1}\ . \tag{8.8}$$

Using Eqs. (5.14) and (8.8) we then obtain

$$\lambda_1\approx 0.1\ . \tag{8.9}$$

We choose $b^2=-\tilde{F}_0^2/6$ as a second independent parameter. In the long-distance approximation, Eq. (5.29), $\sigma_d=3\pi b^2/2$, i.e., it is independent of all λ_i provided b^2 is fixed. We choose $\lambda_2=\lambda_3$ and $\lambda_4=0$ so that W, Eq.

(7.14), has a structure similar to its original form Eq. (7.1). (We have also considered other sets of λ_i not satisfying these conditions and arrive at the same conclusions.) Equation (7.15) determines λ_3 in terms of b^2, λ_1, and λ_5; and takes the form

$$(4\lambda_3+\lambda_5)=\frac{\lambda_1}{4b^2} . \tag{8.10}$$

Equations (7.18) and (7.19) become

$$\beta_0=4\beta_2=8\lambda_1, \quad \alpha_0=\alpha_2=-8b^2\lambda_5 , \tag{8.11}$$

so that if λ_5 is negative then the electric field **E** damps exponentially at large distances.

The values of $\lambda_2-\lambda_3$, λ_4, and λ_5 can in principle be calculated from Feynman graphs such as the one shown in Fig. 3 (Ref. 35). In this paper, however, we shall take these to be free parameters. We have computed σ_d as a function of b^2 for $\lambda_5=-0.025$, -0.125, and -0.625. The results are shown in Fig. 4. The curves have essentially the same shape, those for larger values of λ_5 lying lower. For comparison, the long-distance estimate $\sigma_d=3\pi b^2/2$ is shown in the same figure. The exact σ_d decreases less rapidly than b^2 for small b^2, reflecting contributions to σ_d coming from small distances inside the flux tube. The sensitivity of σ_d to λ_5 is also larger for small b^2. Both of these facts suggest that our predictions may be somewhat less reliable for $b^2\leq 1$.

Experimentally, the string tension σ, determined by the slope of Regge trajectories, is

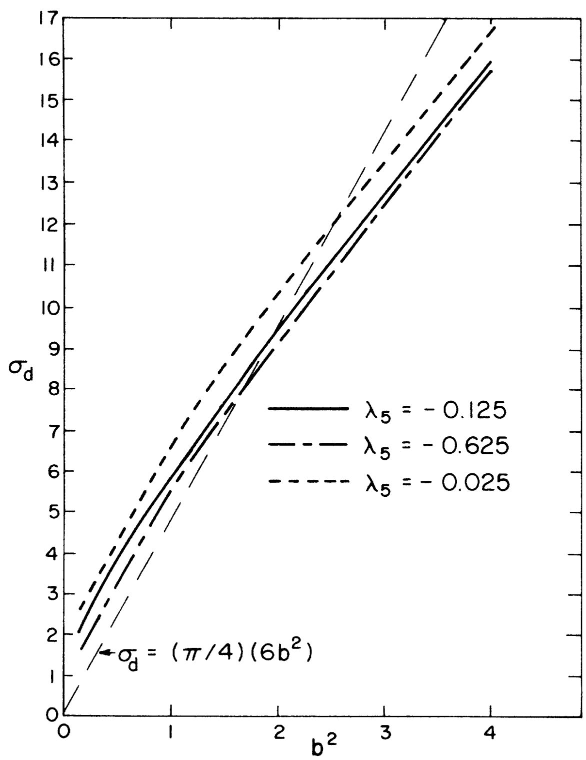

FIG. 4. σ_d vs b^2 for different values of λ_5. The straight line is the long-distance estimate, $\sigma_d=\pi/4(6b^2)$.

$$\sigma=(420\text{ MeV})^2 \tag{8.12}$$

while G_2, found from an analysis of QCD sum rules for heavy-quark–antiquark systems, is of order[36]

$$G_2\sim(330\text{ eV})^4 . \tag{8.13}$$

(The uncertainty in G_2 is about a factor of 2.) Thus the measured value of σ^2/G_2 is around 2.6.

Theoretically, $\sigma^2/G_2=\sigma_d^2/6b^2$, so at $b^2=1$ the "experimental" value of σ_d is

$$\sigma_d(b^2=1)_{\text{expt}}\sim 4 . \tag{8.14}$$

Our calculated value, for $\lambda_5=-0.125$, is

$$\sigma_d(b^2=1)_{\text{theory}}\approx 5.9 . \tag{8.15}$$

Decreasing λ_5 by a factor of 10 increases σ_d by 20%, increasing it by a factor of 10 decreases σ_d by 10%. For $b^2>1$, the ratio $\sigma_d(b^2)_{\text{theory}}/\sigma_d(b^2)_{\text{expt}}$ increases. For $b^2<1$, as indicated above, we have less confidence in the theoretical estimates because short-range effects begin to become important. Altogether, then, b^2 of order unity appears optimum. We should emphasize the uncertainties in all of these estimates. The experimental value (8.14) of $\sigma_d(b^2=1)$ is from SU(3) data on σ and G_2, while the theoretical value (8.15) is from an SU(2) calculation. There are also modifications due to quantum fluctuations in Eq. (5.13) for G_2, and, as we have mentioned, G_2 is not very well known experimentally.

Taking the above into account we select the following values for the parameters in W:

$$b^2=1, \quad \lambda_1=0.1, \quad \lambda_5=-0.125 ,$$

which yields $\sigma_d\approx 5.9$. From the corresponding numerical solutions to Eq. (8.2) we calculate C_μ and $\tilde{F}_{\mu\nu}$ from Eq. (8.1), **D** and **H** from Eq. (3.9), and the energy density $\mathcal{H}_d$ from Eq. (8.6). In Fig. 5(a) we plot $-C_3(\rho)$, the ϕ component of the vector potential, as a function of the scaled radius ρ. The potential $-C_3$ vanishes at the origin, increases reaching a maximum at $\rho=2$, and then decreases approaching its asymptotic value $1/\rho$ exponentially. In Fig. 5(b) we plot the ρ and ϕ components of the color-magnetic field: $B_1(\rho)$ and $B_2(\rho)$. These functions are large only outside the flux tube and decrease rapidly in its interior. In Fig. 5(c) we plot the z components of the electric field and the electric displacement vector, both of which lie in the 3 direction in color space. We see that these fields are large only when the one and two color components of the magnetic field B_1 and B_2 are small. The color-electric flux is then confined by the magnetic pressure produced by color-magnetic fields lying in directions orthogonal to E_z and D_z both in ordinary space and color space. Finally in Fig. 5(d) we plot the energy density $\mathcal{H}_d$, from which the confined nature of the flux tube is evident.

Using Eqs. (5.20), (8.15) and the experimental value (8.12) for σ we obtain

$$(M/g)^2\sim(175\text{ MeV})^2 . \tag{8.16}$$

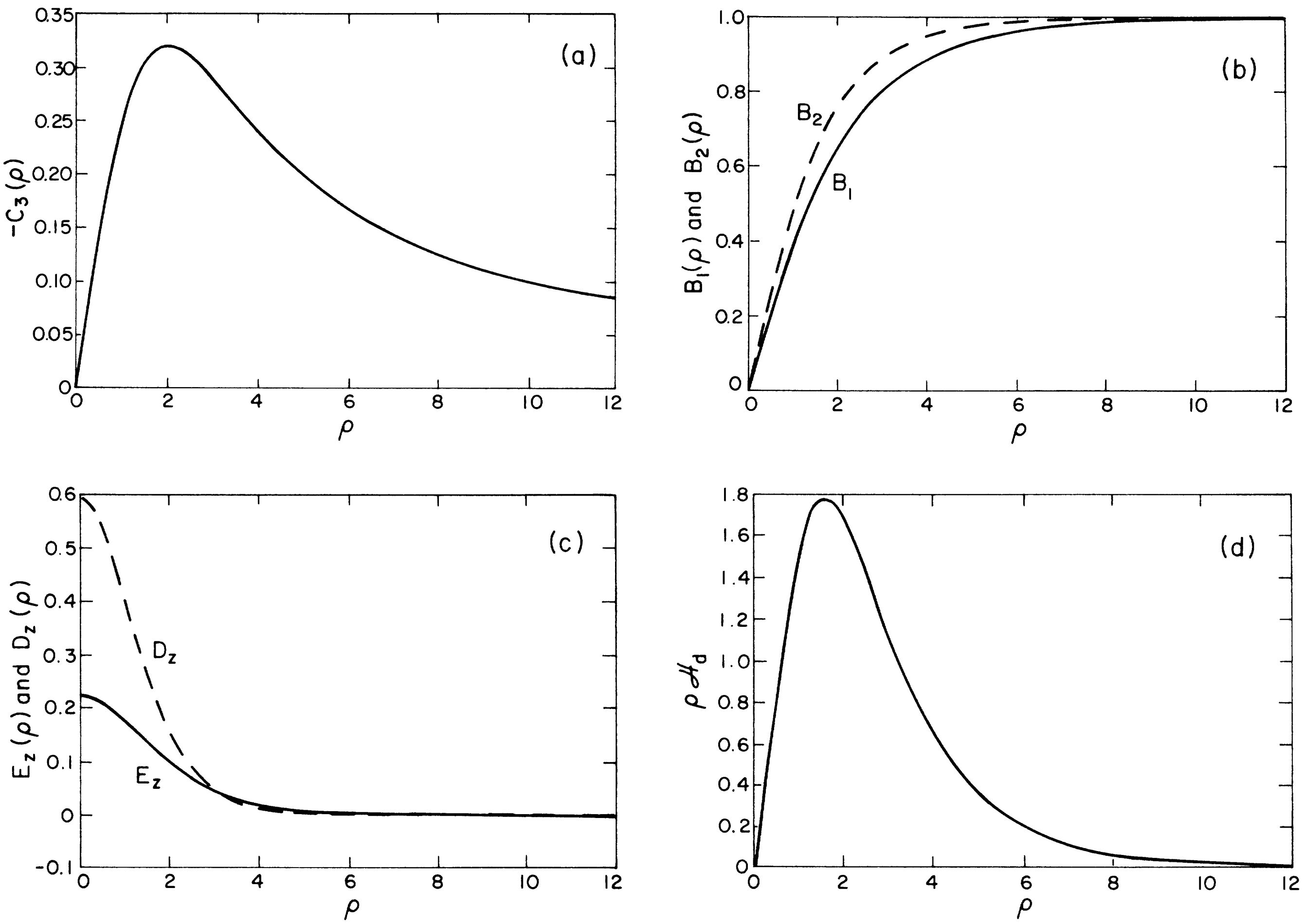

FIG. 5. Numerical solutions to Eqs. (8.2) with $b^2=1$, $\lambda_1=0.1$, $\lambda_5=-0.125$. (a) shows $-C_3$ as a function of the (scaled) cylindrical radius. (b) shows B_1 and B_2. (c) shows E_z and D_z and (d) shows $\rho\mathcal{H}_d$.

The classical approximation for G_2, Eq. (5.13) with $b^2=1$ then gives

$$G_2 \approx (270\ \text{MeV})^4\ . \tag{8.17}$$

This value is insensitive to the choice of b^2, and is about a factor of 2 below the experimental value. Next, using Eq. (5.31) for the radius of the flux tube,[37] we obtain

$$R_{\text{FT}} \approx 1.5\ \text{fm}\ . \tag{8.18}$$

In Ref. 6 we obtained spherically symmetric static solutions to Eqs. (3.24) and (3.25) which we interpreted as a glueball. Using the above parameters we obtain

$$M_{\text{glueball}} \approx 4.5\ \text{GeV}\ . \tag{8.19}$$

The detailed Lorentz structure and color structure of this solution is not important for our present discussion. We only point out that this solution, unlike our flux-tube solution, is not topologically distinct from the vacuum. Thus although classically stable it is not topologically stable and it is not clear that such an object manifests itself as an unstable physical particle. On the other hand it is complicated to solve our equations for gluon distributions having the geometry of closed flux tubes, and we have made no attempt to do so. We only remark here that the long-distance estimate (5.30) makes no reference to the internal geometry and hence in principle is applicable to closed flux-tube glueballs as well. Indeed, we can use the crude estimate (5.33), to write Eq. (5.30), in the form $M_{\text{glueball}} \sim (2\pi\sigma)\sqrt{2}(R_{\text{FT}})$, which aside from the factor $\sqrt{2}$ is the naive geometrical estimate for the mass of a closed flux-tube glueball.

IX. THE STATIC POTENTIAL: THE LANDAU-GINZBURG APPROXIMATION AND THE BAG APPROXIMATION

As a final application let us briefly discuss the static potential $V(R)$ between a heavy quark and antiquark fixed at a separation R. We know at large R that $V(R)$ is a linear potential and that at short distances it is a Coulomb potential. To calculate $V(R)$ at all distances requires the introduction of sources into Eqs. (3.24) and (3.25). How this is done is described in Ref. 7. However, the resulting equations are sufficiently complicated that they must be simplified. Two approximation schemes to Eqs. (3.24) and (3.25), which permit the calculation of $V(R)$, are outlined in what follows.

A. Magnetic superconductor approximation $V^{\mathrm{MS}}(R)$

As noted in Sec. V, with $M=0$, and for fixed μ, $\mathcal{L}(C)$ reduces to a Landau-Ginzburg-type Lagrangian. When $M=0$, Eqs. (3.24) and (3.25) close if we set

$$E_1=E_2=E_3=C_0=C_2=0\ ,\quad B_1=B_2\equiv B\ ,\quad B_3=b=\text{const}\ , \tag{9.1}$$

in our SU(2) ansatz (8.1). Equations (8.2) for the SU(2) flux tube then reduce to just two equations, for the functions B and $C_3\equiv C$:

$$\hat{\nabla}^2 C-\frac{2B^2}{\rho}-2B^2C=0 \tag{9.2a}$$

and

$$\nabla^2 B-\left[C+\frac{1}{\rho}\right]^2 B-\frac{1}{2}\frac{\partial W}{\partial B}=0\ , \tag{9.2b}$$

where the function W is given by

$$W=-\frac{3x}{4}(B^2-b^2)^2\ , \tag{9.3}$$

with

$$x\equiv-\frac{\lambda_1}{b^2}\ . \tag{9.4}$$

With the substitution $C\rightarrow A$, $B\rightarrow\phi$, Eqs. (9.2) become the Landau-Ginzburg equations for an ordinary superconductor characterized by a Landau-Ginzburg parameter $\xi=\sqrt{-3x/2}$. They thus describe a magnetic superconductor.

Now although, as noted in Sec. V, there are important differences between the physics of dual QCD as described by $\mathcal{L}(C)$ and the physics of a magnetic superconductor, the constraints (9.1), valid in the magnetic superconductor approximation, are approximately satisfied by the solutions of Eq. (8.2). The fact that the solution of the exact Eqs. (8.2) are fairly well described by the Landau-Ginzburg approximation[38] may be understood by referring to Fig. 4. We note that the physical value of b^2 is close to the region in which σ is linear in b^2. This linearity is an indication of the validity of the Landau-Ginzburg approximation. In fact we can obtain a good estimate of the string tension in the Landau-Ginzburg approximation from the straight line in Fig. 4. We thus expect that one can obtain a good estimate of $V(R)$ by introducing sources into Eqs. (9.2) and then calculating the resulting static approximation potential in the magnetic superconductor approximation.

Indeed Ball and Caticha[39] have calculated the potential between a heavy monopole and antimonopole in an ordinary superconductor. Their results may be taken over directly, with the relabeling and rescaling indicated above, to determine $V^{\mathrm{MS}}(R)$, the magnetic superconductor approximation to $V(R)$. They find a pure Coulomb potential at short range with a rapid transition to a linear potential at long range. In Fig. 6 we plot the potential $V^{\mathrm{MS}}(R)$ between a heavy quark and antiquark in the magnetic superconductor approximation to dual long-distance QCD. At short distances $V^{\mathrm{MS}}(R)$ behaves according to Eqs. (8.7) and at large-distance behavior is fixed by the value Eq. (8.12) of the string tension.

In the same paper Ball and Caticha also calculated the monopole antimonopole potential using a linearized approximation to Eqs. (9.2). In this approximation their equations [which we denote as $V_{\mathrm{bag}}^{\mathrm{MS}}(R)$] reduce to those of the MIT bag-model static potential.[12] Although $V_{\mathrm{bag}}^{\mathrm{MS}}(R)$ does not yield the correct string tension, it can be made to agree quite accurately with $V^{\mathrm{MS}}(R)$ if the string tension is allowed to be a free parameter. Thus the MIT bag-model static quark-antiquark potential can be viewed as an approximate solution to the magnetic superconductor approximation to dual long-distance QCD as described by $\mathcal{L}(C)$.

B. The bag approximation to the exact Eqs. (8.2): $V_{\mathrm{bag}}(R)$

Of course, since Ball and Caticha calculated the Landau-Ginzburg potential exactly, there was no need for them to make a bag approximation. However, we want to find the potential $V(R)$ obtained by introducing sources into Eqs. (8.2) generated by $\mathcal{L}(C)$. In this case, in contrast with the magnetic superconductor case (9.2), the problem is too difficult to solve exactly and an approximation is required. We have calculated[7] the potential $V(R)$ using a linearized Abelian approximation to Eqs. (8.2) analogous to the approximation to Eqs. (9.2)

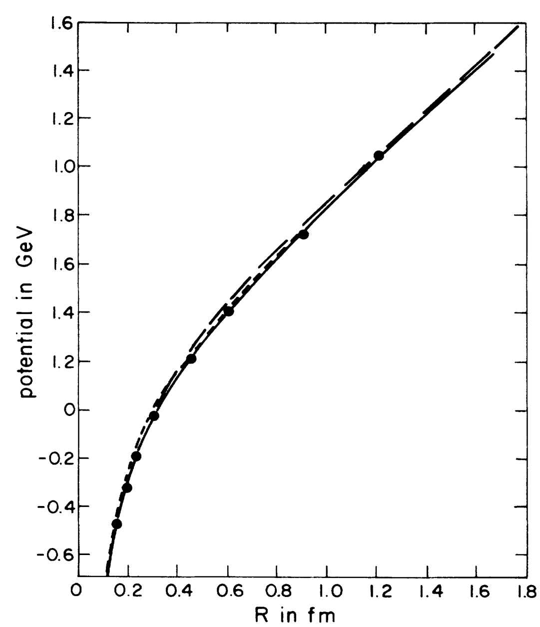

FIG. 6. The heavy-quark potential from various models. The solid curve is the linear plus Coulomb potential of Ref. 34. The dashed line is the MIT static bag-model potential $V_{\mathrm{bag}}^{\mathrm{MS}}$. The long- and short-dashed lines are the bag-model potential V_{bag} of Ref. 7. The dots are the magnetic superconductor potential V^{MS} from Ref. 39.

which yielded the MIT bag-model potential. Again the resulting potential [denoted as $V_{\text{bag}}(R)$] does not give the correct string tension. However, the comparison between $V^{\text{MS}}(R)$ and $V_{\text{bag}}^{\text{MS}}(R)$ suggests that $V_{\text{bag}}(R)$ can be an adequate approximation to $V(R)$ provided the string tension is taken as a free parameter. When this is done the potential $V_{\text{bag}}(R)$ is almost indistinguishable from $V^{\text{MS}}(R)$.

The basic difference between the MIT bag-model potential and the potential obtained from the bag approximation to Eqs. (8.2) is that they give a different dependence of the string tension on the flux Φ contained in the flux tube.[40] The string tension in the bag approximation to the exact Eqs. (8.2) is given in terms of Φ and ϵ_{vac} by the equation

$$\sigma=\Phi\sqrt{-2\epsilon_{\text{vac}}}\left[1+\frac{\sqrt{2}M^2\Phi}{32\pi\sqrt{-\epsilon_{\text{vac}}}}\right], \tag{9.5}$$

where ϵ_{vac} must now be taken as a parameter fixed by the string tension. This dependence on Φ can be compared with that given by lattice calculations[41] which suggest that in SU(2) the ratio between the string tension in the adjoint representation (i.e., between gluons) and the fundamental representation (i.e., between quarks) is close to 8/3, for small enough lattices that the screening of gluon sources by gluon pairs (which eventually destroys the linear potential in the adjoint case), has not yet come into play. The ratio 8/3 is, of course, for SU(2), just the ratio of Casimir eigenvalues for the two representations.

It is conventional in bag models to take the effective Abelian charge on the source to be proportional to the square root of the Casimir eigenvalue.[42] Thus the color-electric flux in a bag flux tube is also proportional to this value. The string tension in the MIT bag model is proportional to the flux; therefore the ratio of adjoint to fundamental string tension in the MIT bag[43] will be $\sqrt{8/3}$, for SU(2), rather than the close to 8/3 value found in the lattice calculation.[41] In contrast with the MIT bag, where σ is linear in Φ, the bag approximation to Eqs. (8.2) has a term in Φ^2 as well. For SU(2) it will therefore give a result between $\sqrt{8/3}$ and 8/3 for the adjoint to fundamental ratio. We obtain a ratio

$$\frac{\sigma_{\text{adjoint}}}{\sigma_{\text{fundamental}}}\approx 1.83$$

to be compared with approximately $\sqrt{8/3}\simeq 1.63$ for the MIT bag model and $8/3\sim 2.67$ for the lattice. However since the bag approximation to Eqs. (8.2) does not give the correct absolute σ, it can only be used qualitatively to understand the ratio.

X. CONCLUSION

We have begun with a propagator Δ_A^0 obtained from a truncated system of equations based on the Dyson equations and Ward identities of Yang-Mills theory and constructed the free Lagrangian $\mathcal{L}^{(0)}(C)$ which, in the C_μ representation, describes the same physics as Δ_A^0. Gauge invariance then fixes the long-range limit $\mathcal{L}(C)$ of the interacting dual Yang-Mills Lagrangian. This Lagrangian turns out to describe a system with properties resembling those of a magnetic superconductor.

The physical origin of these properties is the following; it was pointed out a number of years ago by Nielsen[44] that one can understand asymptotic freedom as a consequence of the paramagnetic nature of the perturbative Yang-Mills vacuum. When this perturbative feature is extended beyond its domain of applicability, there results an infinite magnetic permeability $\mu(q^2)$ at a spacelike momentum (the Landau ghost). Imposing self-consistency and compatibility with gauge invariance moves this singularity in the permeability out to infinite distance; i.e., $\mu(q^2)\to M^2/q^2$ as $q^2\to 0$. This is the physics described by $\mathcal{L}^{(0)}(C)$, the free Lagrangian constructed from $\mu(q^2)$. Although it gives a mass to the dual gluon and a linear potential between quarks (indicative of confinement), it contains unphysical long-distance singularities and allows electric flux to spread out. In the C_μ language these feature manifest themselves in the massless field $\tilde{F}_{\mu\nu}$ which, because of the nonlocal nature of $\mu(q^2)$, are implicitly contained in $\mathcal{L}^{(0)}(C)$.

But when interactions are included by extending $\mathcal{L}^{(0)}(C)$ to $\mathcal{L}(C)$ so as to maintain invariance under the transformations of the magnetic gauge group, the fields $\tilde{F}_{\mu\nu}$ assume a role analogous to that of Higgs fields. The resulting interactions induced between the fields $\tilde{F}_{\mu\nu}$ produce a condensation of the $\tilde{F}_{\mu\nu}$ quanta yielding a magnetic physical vacuum with $\tilde{F}^2<0$, which is energetically favorable to the perturbative vacuum in which $\tilde{F}_{\mu\nu}=0$. This modification of the vacuum structure is the most important consequence of the interactions present in $\mathcal{L}$ and eliminates the two problems present in $\mathcal{L}^{(0)}$: namely, (1) long-distance singularities are eliminated via a dual Higgs mechanism where masses are generated from a "spontaneous symmetry breaking" induced by a nonvanishing vacuum expectation value of the fields $\tilde{F}_{\mu\nu}$; (2) the resulting vacuum magnetic pressure prevents color-electric flux lines between distant quark sources from spreading out into space.

Thus the magnetic superconducting properties of long-distance QCD appear in two stages: the singular long-range magnetic permeability $\mu(q^2)$ results from the self-consistent solution of the truncated Dyson equations. The confinement of electric flux lines results from the vacuum condensation of the $\tilde{F}_{\mu\nu}$ quanta already present in $\mathcal{L}^{(0)}$.

Almost all the applications considered in this paper have been carried out in the classical approximation from which, after using the string tension and gluon condensate to fix the coupling strength, we predict about 1.5 fm for the flux-tube radius. Further applications require going beyond the classical approximation. Semiclassical quantization around the classical flux-tube solution should lead to a string theory with linearly rising Regge trajectories,[4] but this quantization has not yet been carried out for $\mathcal{L}(C)$. Furthermore, perturbative calculations in g around the nonperturbative vacuum are necessary for determining the range of applicability of $\mathcal{L}(C)$; that is, to determine whether $\mathcal{L}(C)$ can describe hadron physics in a large part of the long-range regime not described by short-distance asymptotically free perturbation theory. Dual gluons can be coupled to quarks; however this coupling complicates calculations substantially. Finally one can use $\mathcal{L}(C)$ to study long-distance QCD at finite temperature. In particular one can calculate the deconfining transition temperature.

To conclude, we wish to point out a number of important problems that remain. We have seen how combined gauge and rotational invariance is maintained in the presence of a nonvanishing classical vacuum field $\tilde{F}^{a}_{0\mu\nu}$; however it still remains to understand how Lorentz boost invariance is restored by quantum corrections. It is also necessary to study the quantum theory defined by $\mathcal{L}(C)$ to make sure that no violations of unitarity in physical amplitudes are induced by the negative metric states resulting from the indefinite Lorentz metric. We have noted the cancellation between the negative metric $\tilde{F}_{ij}$ fields and the ghost fields ψ_i^a in lowest order, but we have no general argument that this continues in all orders.

All of these points need to be cleared up in order to know whether $\mathcal{L}(C)$ describes a consistent quantum field theory. While it is true that $\mathcal{L}(C)$ is supposed to coincide with Yang-Mills theory only at long range, it is defined at all ranges, and it would certainly be desirable for it to be a consistent field theory everywhere.

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APPENDIX A

The iteration scheme is based on the system of Dyson equations and Ward identities characterizing Yang-Mills theory. The form of these relations is gauge dependent; it is convenient to select axial gauge. In this gauge, the Dyson equations E_n relate the n-legged ordinary gluon proper vertex functions Γ_{An} to each other as

$$e^{n-2}\Gamma_{An}=E_n(\gamma_2,e\gamma_3,e^2\gamma_4;\Gamma_{A2},e\Gamma_{A3},\ldots,e^{n-1}\Gamma_{An+1},e^2\Gamma_{An+2}) . \tag{A1}$$

We suppress color and Lorentz indices for ease of writing. We have also explicitly displayed the powers of the Yang-Mills coupling constant e which appear. γ_2, γ_3, and γ_4 refer to the bare two, three, and four legged vertex functions. We use the index A on the Γ_{An} to emphasize that these are the usual A language vertex functions of Yang-Mills theory.

The functions E_n can all be uniquely broken up with two separate parts $E_n^{(1)}$ and $E_n^{(2)}$ having the following properties: (i) $E_n^{(1)}$ involves vertex functions only up to Γ_{An+1}; (ii) $E_n^{(1)}$ and $E_n^{(2)}$ separately satisfy the Ward identities; and (iii) $E_n^{(2)}$ is one order of e^2 higher than $E_n^{(1)}$.

Figure 7 illustrates this decomposition for Γ_{A2}. [In general, $E_n^{(1)}$ and $E_n^{(2)}$ are defined by Eq. (A17) of Nucl. Phys. **B186**, 531 (1981).]

The Ward identities relate the divergence of Γ_{An} to Γ_{An-1}; they can be solved to express Γ_{An} as a function of Γ_{An-1} plus an undetermined transverse part. Thus

$$\Gamma_{An}=I_n(\Gamma_{An-1})+\Gamma^T_{An} , \tag{A2}$$

where I_n is the kinematic singularity-free longitudinal part of Γ_{An}, and Γ^T_{An} gives zero when we take the divergence of Eq. (A2). We know that any set of Γ_{An} exactly satisfying all of the Dyson equations also automatically satisfies the Ward identities.

The basic idea of the iteration scheme is now the following. We first drop $E_n^{(2)}$ in the Dyson equations (A1), and define vertices $\Gamma^{(1)}_{An}$ satisfying the truncated system

$$e^{n-2}\Gamma^{(1)}_{An}=E_n^{(1)}(\gamma_2,e\gamma_3,e^2\gamma_4,\Gamma_{A2},\ldots,e^{n-1}\Gamma^{(1)}_{An+1}) . \tag{A3}$$

We know[11] that if the vertices on the right-hand side of (A3) satisfy their Ward identities, than the vertex on the left-hand side does too; thus this set of vertices is guaranteed to describe a gauge-invariant approximation.

In particular, for $n=2$, Eq. (A3) becomes

$$\Gamma^{(1)}_{A2}=E_2^{(1)}(\gamma_2,e\gamma_3,e^2\gamma_4;\Gamma^{(1)}_{A2},e\Gamma^{(1)}_{A3}) , \tag{A4}$$

and $E_2^{(1)}$ is represented by the diagrams on the first line of Fig. 7. $\Gamma^{(1)}_{A3}$ is in turn expressed in terms of $\Gamma^{(1)}_{A4}$ by Eq. (A3) for $n=3$. Since it therefore satisfies its Ward

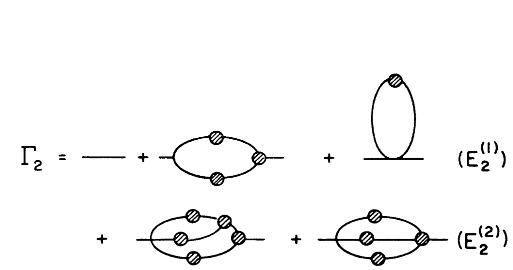

FIG. 7. Graphical representation of Dyson equation E_2 for Γ_{A2}.

identity, it can be written

$$\Gamma^{(1)}_{A3}=I_3(\Gamma^{(1)}_{A2})+\Gamma^{(1)T}_{A3} \ . \tag{A5}$$

Now suppose that the transverse part $\Gamma^{(1)T}_{A3}$ is unimportant in Eq. (A4) in the infrared limit. Then we can replace $\Gamma^{(1)}_{A3}$ in (A4) by the first term in (A5), and (A4) becomes a closed equation for $\Gamma^{(1)}_{A2}$.

To be specific we can write (introducing explicit Lorentz indices)

$$\Gamma^{(1)}_{A3\mu\nu\lambda}(q,k,k')=\delta_{\nu\lambda}(\epsilon(k)k_\mu-\epsilon(k')k'_\mu)-\frac{\epsilon(k)-\epsilon(k')}{k^2-k'^2}(k\cdot k'\delta_{\nu\lambda}-k_\lambda k'_\nu)(k-k')_\mu+\text{cyclic permutations} \ , \tag{A6}$$

where $\epsilon(q)$ is the dielectric constant defined in terms of $\Gamma^{(1)}_{A2}$ by the equation

$$\Gamma^{(1)}_{A2\mu\nu}(q)=\epsilon(q)(q^2\delta_{\mu\nu}-q_\mu q_\nu) \ . \tag{A7}$$

With these forms, (A4) yields an integral equation for the dielectric constant:

$$\epsilon(q)=1+e^2\int\frac{dk\,K(k,q)}{\epsilon(k)}+e^2\epsilon(q)\int\frac{dk\,L(k,q)}{\epsilon(k)\epsilon(k')} \ , \tag{A8}$$

where K and L are kinematic factors.

Setting $\epsilon=1$ on the right-hand side of Eq. (A8) yields the usual perturbative solution with its inconsistent low-momentum behavior. One can show[11] that the only possible self-consistent low-momentum behavior of $\epsilon(q)$ compatible with Eq. (A8) is

$$\epsilon(q)\sim -q^2/M^2 \quad \text{as } q^2\to 0 \ , \tag{A9}$$

where M^2 is an undetermined scale parameter. We have solved Eq. (A8) numerically and have obtained a self-consistent solution for all q^2 having the low-q^2 behavior given by Eq. (A9) (see Fig. 8). Denoting by $\Delta^{(0)}_A(q)$ the gluon propagator obtained from solving (A8), we obtain Eq. (2.35) (transcribed to the axial gauge).

To verify the self-consistency of dropping $\Gamma^{(1)T}_{A3}$, we must study Eq. (A3) for $n=3$, in which we replace $\Gamma^{(1)}_{A4}$ on the right-hand side by $I_4(\Gamma^{(1)}_{A3})$. Equations (A3) for $n=2$ and $n=3$ then form a set of coupled integral equations for $\Gamma^{(1)}_{A2}$ and $\Gamma^{(1)}_{A3}$. This system of equations is too complicated to study in detail but it has a structure analogous to that of our previous Eq. (A8) for $\epsilon(q)$.

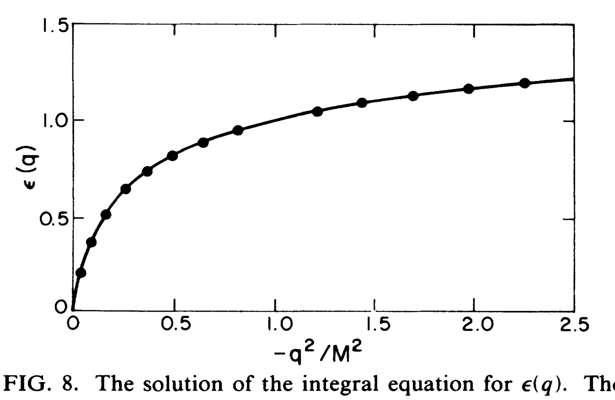

FIG. 8. The solution of the integral equation for $\epsilon(q)$. The solid curve is the input $\epsilon(q)$. The circles are the output $\epsilon(q)$ calculated from the right-hand side of the integral equation (A8).

Consistency will occur if these equations have solutions with the low-momentum behavior given by (A7) and (A9) for $\Gamma^{(1)}_{A2}$ and by (A6) and (A9) for $\Gamma^{(1)}_{A3}$, namely,

$$\Gamma^{(1)}_{A3}\sim q^3/M^2 \ . \tag{A10}$$

In the same way the low-momentum limit of all the $\Gamma^{(1)}_{An}$ should be just that determined by repeated application of the Ward identities to $\Gamma^{(1)}_{A2}$. Since the Ward identities are linear, the $\Gamma^{(1)}_{An}$ will all be proportional to $1/M^2$. Furthermore, each $\Gamma^{(1)}_{An}$ is one power of momentum lower than $\Gamma^{(1)}_{An-1}$; thus

$$\Gamma^{(1)}_{A4}\sim q^2/M^2, \quad \Gamma^{(1)}_{A5}\sim q/M^2, \quad \Gamma^{(1)}_{A6}\sim 1/M^2 \ , \tag{A11}$$

and $\Gamma^{(1)}_{An}$ vanishes for $n\geq 7$.

The exact infrared behavior of the A language vertex functions can now be obtained by iterating the full Dyson equations, including the $E^{(2)}_n$ parts, using the M^2/q^4 propagator $\Delta^{(0)}_A$ and the vertices $\Gamma^{(1)}_{An}$ as input. Thus, it is easy to see by power counting, will generate a series expansion of the form

$$\Gamma_{An}=\sum_{i=0}^{\infty}(e^2M^2)^i I^{(i)}_n(q) \ , \tag{A12}$$

where the $I^{(i)}_n$ are $4i$-dimensional integrals depending only on the external momenta and not on the mass scale M. Dimensionally, therefore, they behave as $(q^2)^{-i}(q)^{4-n}$, so that the expansion parameter in (A12) is (e^2M^2/q^2), q being a characteristic external momentum. There results an expansion dominated by low momentum with a dimensionful coupling constant e^2M^2 whose value reflects the effect of high momentum on low momentum. The expansion diverges in the infrared, and the infrared behavior of the exact Γ_{An} differs completely from that of the $\Gamma^{(1)}_{An}$. This reflects the singular long-distance behavior of the variables A_μ.

The way to construct a convergent infrared expansion is to avoid the A language altogether. We must instead translate the expansion into the C language. To do this, we first note that the purely Abelian theory described by

$$\Gamma^{(1)}_{A2}\sim q^4/M^2 \tag{A13}$$

corresponds in the C language to the Abelian theory having

$$\Gamma^{(1)}_{C2}\sim M^2 \ . \tag{A14}$$

Furthermore, since we know that the C Lagrangian is gauge invariant, the Γ_{Cn} must all satisfy the same Ward identities as do the Γ_{An}. Consequently a set of C_μ vertices, $\Gamma_{Cn}^{(1)}$, can be explicitly constructed from (A14) by repreated application of the Ward identities, just as the $\Gamma_{An}^{(1)}$ were constructed from $\Gamma_{A2}^{(1)}$. These vertices $\Gamma_{Cn}^{(1)}$ are in fact precisely the bare vertices generated by the $M^2/\mathcal{D}^2$ term in $\mathcal{L}(C)$. The complete $\mathcal{L}(C)$, Eq. (2.32), is obtained by including the first low-momentum correction to $\Gamma_{C2}^{(1)}$.

Because in each order the Ward identities are respected, each term in the expansion (A12) is gauge invariant. The corresponding C language expansion of which the $\Gamma_{Cn}^{(1)}$ constitute the first term, is also gauge invariant. Furthermore the higher-order terms in this C_μ expansion produce vertices Γ_{Cn} which satisfy unitarity. The quantum corrections to $\Gamma_{Cn}^{(1)}$ generated by $\mathcal{L}(C)$ produce a series having both these properties. Nonminimal additions to $\mathcal{L}(C)$ can, at low momentum, be accounted for by renormalizing the parameters appearing in $\mathcal{L}(C)$. The quantum Lagrangian $\mathcal{L}(C)$ then describes in the C_μ language the long-distance physics of the series (A12) and is the appropriate Lagrangian for describing long-distance Yang-Mills theory. The expansion parameter is g^2q^2/M^2 instead of e^2M^2/q^2.

APPENDIX B

In this appendix we write down the explicit relation between the components $\tilde{F}^{a}_{\mu\nu}$ of $\tilde{F}_{\mu\nu}$ and its tensor components $\tilde{F}^{i}_{\mu\nu j}$ for SU(3). We have

$$\begin{aligned}
\tilde{F}^{1}_{\mu\nu 2}&=\frac{1}{\sqrt{2}}(\tilde{F}^{1}_{\mu\nu}-i\tilde{F}^{2}_{\mu\nu})=\tilde{F}^{2*}_{\mu\nu 1}\ ,\\
\tilde{F}^{1}_{\mu\nu 3}&=\frac{1}{\sqrt{2}}(\tilde{F}^{4}_{\mu\nu}-i\tilde{F}^{5}_{\mu\nu})=\tilde{F}^{3*}_{\mu\nu 1}\ ,\\
\tilde{F}^{2}_{\mu\nu 3}&=\frac{1}{\sqrt{2}}(\tilde{F}^{6}_{\mu\nu}-i\tilde{F}^{7}_{\mu\nu})=\tilde{F}^{3*}_{\mu\nu 2}\ ,\\
\tilde{F}^{3}_{\mu\nu 3}&=-(\tfrac{2}{3})^{1/2}\tilde{F}^{8}_{\mu\nu}\ ,\\
\tilde{F}^{1}_{\mu\nu 1}&=\frac{1}{\sqrt{2}}\left[\tilde{F}^{3}_{\mu\nu}+\frac{\tilde{F}^{8}_{\mu\nu}}{\sqrt{3}}\right]\ ,\\
\tilde{F}^{2}_{\mu\nu 2}&=\frac{1}{\sqrt{2}}\left[-\tilde{F}^{3}_{\mu\nu}+\frac{\tilde{F}^{8}_{\mu\nu}}{\sqrt{3}}\right]\ .
\end{aligned}\tag{B1}$$

[1]G. 't Hooft, in *High Energy Physics*, proceedings of the European Physical Society International Conference on High Energy Physics, 1975, edited by A. Zichichi (Editrice Compositori, Bologna, 1976), p. 1225.

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[3]The parameter M/g is in principle related to the dimensionful parameter $\Lambda_{\rm QCD}$ characterizing short-distance QCD, but to explicitly connect the two requires knowing the dual Lagrangian not only in the confining regime but in the short-distance regime as well. Expressing M/g in terms of $\Lambda_{\rm QCD}$ is therefore beyond our present capabilities.

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[13]In fact, Eq. (2.17) is not sufficient to define the C_μ including all of their color degrees of freedom. Mandelstam had to construct additional loop operators, as well as $W_C(l)$. Like $W_C(l)$, his loop operators are all invariant under the transformations of the magnetic-color gauge group.

[14]This equivalence of the two propagators Δ_A^0 and Δ_C^0 was first noted by V. P. Nair and C. Rosenzweig, Phys. Lett. **135B**, 450 (1984).

[15]This result was also obtained by Mandelstam (Ref. 10) using a different truncation, also compatible with gauge invariance, of the Schwinger-Dyson equations. The M^2/q^4 behavior has also been considered by C. Nash and R. L. Stuller, Proc. R. Ir. Acad. **78**, 22 (1978); A. I. Alekseev, B. A. Arbuzov, and V. A. Baykov, Teor. Mat. Fiz. **52**, 187 (1982); F. T. Brandt, J. Frenkel, Phys. Rev. D **36**, 1247 (1987).

[16]We emphasize that the use of the solution of the truncated Dyson equations to determine $\mathcal{L}^{(0)}(C)$ is not rigorous. However it provides a physical foundation for $\mathcal{L}^{(0)}(C)$ based upon Yang-Mills dynamics. Furthermore the basic assumption that the singular long-distance corrections to $\Delta_A^{(0)}$ arise from the use of the wrong variables is compatible with the result that the corrections to $\Delta_C^{(0)}$ are small. Indeed had the long-distance corrections to $\Delta_C^{(0)}$ been important, this identification would not have been self-consistent.

[17]Strictly speaking, these fields **E** and **B** are polarization fields; the actual color-electric and -magnetic fields are **E**+**D** and **B**+**H**, respectively.

[18]Cubic terms in $\tilde{F}$ turn out to have no divergences because of the symmetries of the couplings; higher powers of $\tilde{F}$ are finite by power counting.

[19]H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1974).

[20]It is clear that the color-singlet expression Eq. (4.18) is compatible with Eq. (4.9) only for $N=2$. The right-hand side of Eq. (4.18) is essentially the unit operator in a space of dimension N^2-1 and the completeness relation represents it as a sum of N^2-1 simple products. Equation (4.9) is such a representation but it has at most six terms. We will see that for SU(2) Eq. (4.9) gives exactly the expression (4.18) for T^{ik}_{0jl}.

[21]Recall that the electric- and magnetic-color fields are really $E+D$ and $B+H$, respectively. But D and H vanish in the vacuum, so that (4.28) is valid.

[22]N. K. Nielsen, Nucl. Phys. **B120**, 212 (1977); J. C. Collins, A. Duncan, and S. D. Joglekar, Phys. Rev. D **16**, 438 (1977).

[23]Although Eq. (4.35) as an operator equation is only approximate, we can define the coupling constant so that it is exactly valid at a particular mass scale. We choose this scale to be that at which we have experimental knowledge of G_2, i.e., about 1 GeV. Equation (4.38) then remains approximately

valid at other scales to the extent that the running coupling constant is determined by the one-loop β function. See S. L. Adler, Rev. Mod. Phys. **54**, 729 (1982).

[24]This reference has some different notations. The factor $M^2M_f^2$ appearing in Eq. (3.24) is M^4/g^2 in the notation of this paper.

[25]One can prove that σ is proportional to $\tilde{F}_0^2$ for large $\tilde{F}_0^2$ without recourse to Eq. (5.29).

[26]Note that the Lagrangian (5.3) with $M=0$ has the structure of a generalized Landau-Ginzburg Lagrangian for which $\lambda_2=N\lambda/4!g^2$ plays the role of the Landau-Ginzburg parameter.

[27]This is plausible from the point of view of the A_μ description where the propagator Δ_A^0 contributes to physical quantities only in the combination $e^2\Delta_A^0=e^2M^2/q^4\sim(M/g)^2/q^4$.

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[29]E. Witten, Nucl. Phys. **B160**, 57 (1979).

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[31]In the remainder of this section where we compare with the results of Sec. IV, $\tilde{F}_0^2$ and W_0 will denote the unscaled variables used there.

[32]Note that Eq. (6.28) is obtained from Eq. (4.21), by replacing N^2-1 by 3. This reflects the fact that with the color-singlet expression Eq. (4.20), the energy is equally distributed over all N^2-1 degrees of freedom while in the expression (6.28) for T_{jl}^{ik} the energy is distributed over only the three color degrees of freedom represented by the generators J_x, J_y, J_z.

[33]When one carries out semiclassical quantization about this classical solution it will be important to verify that Lorentz invariance is maintained.

[34]E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D **21**, 203 (1980).

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[38]We emphasize that we are only comparing the $n=1$ Z_2 flux tube of SU(2) gauge theory with the $n=1$ flux tube of a U(1) gauge theory.

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