PHYSICAL REVIEW D VOLUME 37, NUMBER 4 15 FEBRUARY 1988

Nontrivial homotopy and tunneling by topological instantons

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(Received 16 July 1987; revised manuscript received 30 September 1987)

Tunneling by topological instantons is described as a consequence of nontrivial homotopy among field histories and not of barrier penetration. A derivation of the Yang-Mills θ vacua, with finite-action (weak) boundary conditions, is given from this perspective which clarifies certain weaknesses of the barrier-penetration approach. The treatment of nontrivial homotopy in field-theory path integrals is discussed with special attention to the roles of finite action, compactification, continuity of paths, and the justification of the use of Euclidean instantons in a Minkowski-time path integral.

The origin of the Yang-Mills θ vacua[1] is conventionally described as tunneling between vacuum sectors of different winding numbers by topological instantons. The analogy[2] is drawn to tunneling by barrier penetration in a periodic potential and the phenomena of Bloch waves. This barrier-penetration argument is not completely satisfactory. An alternative derivation of the θ vacua in terms of nontrivial homotopy in the space of field histories will be given. This will be used as a backdrop to presenting the general treatment of nontrivial homotopy in field-theory path integrals.

The main difficulty in the barrier-penetration description of the θ vacua is that in order to show that the initial and final configurations are in different topological sectors, one must compactify the field configurations. This is done[2–4] by imposing a condition on the configurations which is stronger than the assumption of finite action. The justification of this compactification has always been recognized as weak but it had seemed necessary.

Another difficulty is that in general finite-action paths cannot change the homotopy class of a configuration because the path must be continuous.[4] This might lead one to imagine that a tunneling mechanism is at work but the difficulty remains because Euclidean paths cannot change the homotopy class of configurations either. The impression is easily formed that the tunneling which gives rise to the θ vacua is in contradiction to this claim, and it is important (for peace of mind, if nothing else) to understand why in fact it is not.

A satisfactory resolution of these problems may be obtained if one allows that there are other mechanisms of tunneling than barrier penetration. In a path-integral description, tunneling occurs when there are paths in configuration space to be included in the path integral which are not continuously related to a Minkowski-time solution to the equations of motion. In a barrier-penetration problem, the extent of Minkowski-time solutions is limited by the barrier and there are paths which reach beyond this range. In a problem like the θ vacua, it will be shown that there are paths whose boundary values are not consistent with Minkowski-time solutions of the equations of motion. It is not necessary to understand this fact in terms of a "barrier" as it is a natural consequence of nontrivial homotopy.

In both cases, when evaluating the Minkowski path integral by stationary phase, one finds that these paths which cannot be reached by deviations from Minkowski time solutions are reached from Euclidean- (or complex-) time solutions of the equations of motion. To account for their contribution to the path integral, one must deform the sum over paths to pass through the Euclidean-time stationary point(s). This leads to the familiar exponential damping which is characteristic of tunneling phenomena.

The role of nontrivial homotopy in a quantum-mechanical path integral was first considered by Schulman[5] and was later studied by Laidlaw and DeWitt.[6] These authors observe that if one restricts one's attention to continuous paths, then the set of paths between an initial point a and a final point b decomposes into homotopy classes, that is, into classes of paths which may be continuously deformed one into another. This induces a decomposition of the full amplitude for transition between a and b into a sum over partial amplitudes, each partial amplitude being a sum over paths in a particular homotopy class:

$$K(a,b;t)=\sum_{\alpha\in\{\text{homotopy classes}\}}\chi(\alpha)K^{\alpha}(a,b;t)\ . \qquad (1)$$

Since the paths in different homotopy classes are not continuously related, there is apparently no *a priori* weighting of the partial amplitudes and these "homotopy factors" must be determined. The problem of finding all sets of homotopy factors consistent with unitarity in quantum mechanics has been solved by Laidlaw and DeWitt.[6] They find that the homotopy factors must form a one-dimensional unitary representation of the fundamental group of the configuration space.

The situation is analogous in field theory. Indeed, Dowker[7] has shown that the Laidlaw-DeWitt argument applies directly to field theory when one works in the configuration space of the field theory in which each point represents the value of the field on a spacelike hypersurface (a field configuration). There are a few subtleties with this however. In quantum mechanics, the path integral is a sum over continuous paths so it may be taken to be defined by its decomposition as a sum over partial amplitudes. In field theory, however, continuous paths are measure zero.[8] But one needs continuous paths to have a notion of homotopy. Are there then no homotopy effects in field theory?

The answer is that of course there are, but one must recognize how they show themselves. The specification of the homotopy factors corresponds to imposing boundary conditions on configuration space which are to be satisfied upon completing closed loops.[7,9] This is not immediately obvious from the original Laidlaw-DeWitt argument, but it is evident if one works in the universal covering space[7,9] where a modified proof of their result can be given (see the Appendix). The Laidlaw-DeWitt result thus gives all boundary conditions that can be imposed on configuration space consistent with unitarity. For the path integral to be well defined, these boundary conditions must be imposed before attempting the decomposition into partial amplitudes.

This can be done by using a differential expression[10] for the topological number (index) involved in the homotopy factor and including it in the Lagrangian. This is done in standard treatments[2,3] of the θ vacua where one adds θ times the Pontryagin density to the Yang-Mills Lagrangian. This has been recognized[11] as imposing boundary conditions on configuration space but its significance in terms of nontrivial homotopy has not been discussed. The inclusion of this boundary condition term in the Lagrangian gives an action which is correct on the subspace of continuous paths that see the topology of configuration space and it induces an effect on other paths through the modified Lagrangian.

Strictly speaking, in field theory there is no *a priori* decomposition of the exact full amplitude into partial amplitudes as in (1) because the path space does not decompose. But having specified the correct "in principle" path integral, when one performs a semiclassical (Gaussian, stationary phase) approximation to evaluate it, there will be a decomposition. One's first observation is that only finite action paths will contribute as stationary points to the path integral.[12] The finite-action paths are a subset of continuous paths, so there will be a homotopy classification of the stationary paths.

Deviations from these paths are expressed in terms of an expansion in eigenfunctions of the second variation with arbitrary coefficients and the Gaussian approximation involves integrating over these coefficients. This means that discontinuous paths are being included in the sum over paths. One might be concerned that these discontinuous paths might break the homotopy classification of the stationary paths. This does not happen for two reasons.

The first reason is the more important because it is so easily overlooked. In general the target space of a field theory with nontrivial homotopy is not a vector space. This means that one must define the operation + when one describes a field as a background plus a deviation: $\phi=\phi_0+\varphi$. This may be done[13] (if the target space is a manifold) by using Fermi normal coordinates and expressing the deviation as a displacement along a geodesic leaving normal to the stationary path. The trouble is that Fermi normal coordinates are local and cover at best a simply connected region. The same topology which induces nontrivial homotopy classes confines the range of deviations. One simply cannot express paths in a different homotopy class as deviations along a well-defined family of one-parameter curves.

This is supported by the second fact that the eigenfunctions of the second variation are continuous functions which satisfy boundary conditions compatible with the stationary path they are associated with. That is, they are in the same homotopy class as the stationary path. One can use them to construct paths which are discontinuous in the differential sense, but one cannot combine them to reach a path in a different homotopy class. The reason for this is that a path is described in terms of a one-parameter family of deviations from a stationary path, but one needs at least a two-parameter variation to make a discontinuous path and then use it to change homotopy classes.

Having justified the place of nontrivial homotopy in the path integral, it is necessary to determine which homotopy group is relevant in a given field theory. This has been done by Dowker[7] by directly applying the Laidlaw-DeWitt argument[6] in the configuration space C of the field theory to find that the homotopy group is $\pi_1(C)$. This result is, however, largely formal because the burden is then placed on properly identifying C. A more direct argument will be given with close attention to the role of compactification and the treatment of finite-time transitions in Minkowski time.

The idea is to generalize the Laidlaw-DeWitt argument[6] so that it applies to path histories which are not one dimensional. Suppose that spacetime has the topology $M\times R$. The image of M in the target space N under a field mapping ϕ is a field configuration. A one-parameter family of such configurations is a field history.

The first difference from quantum mechanics is that it is possible for there to be a homotopy classification of field configurations.[4] This happens when $[M,N]$, the homotopy group of mappings from M into N, is nontrivial. A simple example of this occurs when $M=S^1$ and $N=R^3\setminus R^1$ (three-space with a line removed). The configurations in which $\phi(M)$ encircles the line removed and in which it does not are homotopically inequivalent; one cannot be continuously deformed into the other without crossing the line removed from N. Homotopically inequivalent configurations lie in different topological sectors of the theory. There can be no continuous path connecting configurations in different homotopy classes; therefore there can be no finite-action path connecting them. (This statement is at odds with the conventional description[2-4] of the θ vacua in which the n vacua are claimed to be in different homotopy classes and to be connected by a finite-action Euclidean path. This seeming contradiction is one of the difficulties with the barrier-penetration description of the θ vacua and will be explained below.)

An additional subtlety of field theory is the possibility of twisted fields.[14] This arises because in actuality a field configuration is not the image of the manifold M in the target space N but is a cross section of a fiber bundle with base space M, fiber N, and bundle group G. It may happen that there are nontrivial fiber bundle structures and one must specify the bundle structure of the theory one is considering. This corresponds to specifying boundary conditions on field configurations. The homotopy of field histories is not sensitive to these boundary conditions and thus not to bundle structure, so without loss of generality, one can confine one's attention to trivial bundles in which one can work with field configurations as images in the target space.

As we will be interested in the semiclassical approximation, it is necessary to restrict our attention to paths associated with finite-action solutions to the equations of motion. In quantum mechanics we can restrict attention to finite-action paths because any path can be arbitrarily closely approximated by a smooth path while an infinite-action smooth path must have passed through a region of unbounded potential and cannot contribute to the path integral. In field theory, the situation is less clear. It is evident that we need not consider infinite-energy initial and final configurations because the path joining them will necessarily have infinite action. It is not certain however that smooth infinite-action paths joining finite-energy configurations are always associated to infinite-action stationary paths. Until this can be determined, it is prudent to include all continuous paths joining finite-energy configurations.

The importance of these remarks is that the assumption of finite-energy configurations imposes boundary conditions on fields which affect the topology of configuration space. For example, in a massive $(1+1)$-dimensional scalar field theory, the assumption of finite-energy configurations requires that $\phi \to 0$ as $|x| \to \infty$. This compactifies the configuration and may result in a homotopy classification of field configurations if $\pi_1(N) \neq 0$. In a massless $(1+1)$-dimensional scalar field theory, the condition is only that $\phi \to$ const as $|x| \to \infty$ and this does not compactify the configuration because the constant may be different at $\pm\infty$. In a massless $(2+1)$-dimensional scalar field theory, the configurations are again compactified, but the constant at ∞ may change as the configuration evolves (at least until one determines that all solutions of the equations of motion in which it does have infinite action).

To determine the homotopy class associated to a field history, one must consider both the evolution of the configurations in the target space and the evolution of the boundary in the subspace that it is confined to by the assumption of finite action. With this in mind, one can go on to generalize the Laidlaw-DeWitt argument. The original argument will go through unchanged once the construction identifying the homotopy group has been given (see the Appendix).

In the Laidlaw-DeWitt argument[6] one associates a path running from an initial configuration a to a final configuration b to an element of a homotopy group by completing the path into a closed loop based at an arbitrary (fixed) point $*$. This is done by attaching path segments connecting $*$ to a and b to $*$. The path segments are part of the homotopy mesh and it is the fact that the homotopy mesh is unphysical that allows one to determine the homotopy factors (see the Appendix). In the field theory case, one can complete a finite history to a closed "loop" by attaching cylinder path segments (of topology $M \times I$ where I is an interval) to the initial and final configurations joining them to a base configuration 0 in the same homotopy class. This gives a closed loop of topology $M \times S^1$.

The relevant homotopy group is then $[(M \times S^1, \partial M \times S^1),(N,F)]$ which is the group of maps which take $M \times S^1$ into the target space N while taking the boundary of M (and its evolution) into the space F that is defined by the constraint of finite action. It is amusing that the Laidlaw-DeWitt construction of completing the finite path between a and b into a closed loop is designed to avoid the notion of relative homotopy (homotopy between objects with fixed boundary) while the finite action constraint that the boundary of M lie in a particular subspace reintroduces it. It should be emphasized that in general this homotopy group is not one of the standard $\pi_n(N)$ homotopy groups which are written $[S^n, N]$ in this notation. The articles by Isham[4,14] describe how to calculate these homotopy groups (though not with the boundary constraint).

This result may now be applied to analyze Yang-Mills gauge theory. Consider a $(3+1)$-dimensional Yang-Mills gauge theory with potentials taking their values in the Lie algebra $\mathscr{g}$. The constraint that field configurations have finite energy requires that the potentials be pure gauge as $|x| \to \infty$:

$$A_\mu(x) \to g(x)\partial_\mu g^{-1}(x) \quad \text{as } |x| \to \infty \;. \tag{2}$$

This means that the surface at infinity is being mapped into the Lie group G. The homotopy group classifying paths is then $[(R^3 \times S^1, S^2 \times S^1),(\mathscr{g}/G, G)]$ where $\mathscr{g}/G$ is shorthand indicating that one is interested in images in $\mathscr{g}$ up to gauge transformations. If G is connected, then since $[R^3 \times S^1, \mathscr{g}/G] = \pi_1(\mathscr{g}/G) = \pi_0(G) = 0$, all of the topology is given by $[S^2 \times S^1, G]$. For simply connected simple Lie groups G, in particular for SU(n), this equals Z. In the standard derivation[1] of the θ vacua, the homotopy classification is given by $[S^3, G]$. This is because finite action is required for the full Euclidean history which requires the potential to be pure gauge on a three-sphere.[15]

It should be emphasized that the gauge has not yet been fixed. This is important because at this stage one is formulating the "in-principle" path integral and gauge fixing may affect the topology of the space of paths. It is correct that full account has not been explicitly taken of equivalence under gauge transformation. Isham and Kunstatter[16] show how holonomy effects connected with gauge transformations can modify the homotopy group—this is not a problem if G is simply connected.

If the nontrivial homotopy of field histories can be seen in Minkowski time, then what is the role of the Euclidean instantons? This is best understood by reconsidering the Belavin-Polyakov-Schwartz-Tyupkin (BPST) instanton.[15] The first step in the construction of this instanton is to write down a pure-gauge configuration on a three-sphere which has index 1. This is the image in the Lie algebra from a three-sphere in Euclidean space but it is just as well the image from a three-sphere in Minkowski space. One merely replaces x_4 by x_0. This may be taken as representing the boundary values of a gauge field which is pure gauge on hypersurfaces at times t_0 and t_1 and outside a sequence of two-spheres between t_0 and t_1. By filling in this sequence of two-spheres with nonsingular configurations one has an acceptable field history connecting two pure-gauge configurations.

This field history is in the winding-number-one homotopy sector by construction. It is not however a solution of the equations of motion and cannot be used as a stationary point for evaluating the path integral. The question one must ask if one wishes to evaluate the path integral semiclassically is the following: What is the field history in this homotopy class which does satisfy the equations of motion? The answer is that there is no such solution in Minkowski time, but there is a Euclidean-time solution, namely, the BPST instanton. If one is evaluating the path integral by stationary phase, one must deform the path integration contour to pass through this stationary point in order to account for the configurations in the winding-number-one homotopy class.

The reason that there is not a Minkowski-time solution has to do with the fact that one is imposing boundary data on a three-sphere in a hyperbolic equation and this is not a well-posed problem. The boundary value problem of finding a solution between a pure-gauge configuration on an initial Cauchy hypersurface and a pure-gauge configuration on a final Cauchy hypersurface is well posed and has a unique solution which is pure gauge everywhere. Being pure gauge everywhere, it is in the homotopically trivial class. There cannot be a Minkowski-time solution with nonzero winding number. The analytically continued problem however is elliptic and is well posed, having the instantons as solutions.

One can see clearly that the instanton does not connect homotopically distinct configurations. Both the initial and final configuration are pure gauge and are homotopic to the identity. One can give the impression that they are in different homotopy sectors by compactifying the configurations at infinity by imposing the boundary condition[2,3] $g(x) \rightarrow id$ as $|x| \rightarrow \infty$. With this boundary condition, the space of pure-gauge configurations is disconnected and there are no continuous zero action histories connecting configurations in different homotopy sectors.

One finds[17] however that by extending the space of configurations to all finite-energy configurations then there do exist continuous finite action histories between homotopically distinct zero-energy configurations. The dilemma about how it is possible to have a finite-action path between "homotopically distinct" configurations is thus resolved by observing that the configurations are not homotopically distinct in the larger space in which the path lies. It is not then surprising that instantons have nonzero action as this is precisely what allows them to tunnel between different (zero-energy) topological sectors.

One can go further to recall the origin of the barrier penetration description. As the pure-gauge configurations are energy minima, the finite-action history between two such homotopically distinct configurations will pass through configurations of greater energy. This is reminiscent of tunneling in quantum mechanics when a particle of a given energy tunnels through regions where the potential energy is greater. One is naturally drawn to a barrier-penetration analogy. However, since the "barrier" is not externally induced but is instead a property of the configurations, it is prudent to think in terms of the topology of configuration space and not to restrict oneself to images of potential barriers.

The propagator for the θ vacua with the compactification boundary condition $g(x) \rightarrow id$ as $|x| \rightarrow \infty$ can be derived in terms of nontrivial homotopy on configuration space. This was first done by Dowker[17] and as his argument has a slightly different perspective from the general one above and as it is not generally available, I will briefly summarize it. Working in the configuration space of finite-energy configurations without yet fixing the gauge, one first observes that the pure-gauge configurations decompose into homotopy sectors classified by $[S^3, G]$—maps of the compactified R^3 into the gauge group. One can think of configuration space as a fiber bundle in which the fibers are configurations related by "small" gauge transformations (gauge transformations that are homotopic to the identity). The large gauge transformations which take one from one pure-gauge homotopy sector to another may then be interpreted as a covering group on the (simply connected) space of finite-energy configurations.

There are distinct partial amplitudes for paths running between a chosen initial configuration fiber and each image of the final configuration fiber under large gauge transformations. These are the analogs of the homotopically distinct histories in the argument above. The physical space of finite-energy configurations is obtained by projection from this covering space and by the method of images one knows that one includes a phase factor with each partial amplitude to impose the boundary conditions of the physical space (cf. the Appendix). This phase factor is a one-dimensional unitary representation of the covering group, in this case, $\pi_3(G)$ which equals Z for $G = \mathrm{SU}(n)$. This leads to the standard result for the θ vacua with strong (compactification) boundary conditions.

There is a difficulty which arises in the compactified case which is not present in the more general one. This involves gauge fixing and the Gribov ambiguity.[18] As yet the gauge has not been fixed and there is the possibility that fixing the gauge could affect the topology of configuration space. The object of gauge fixing is to choose one member from each class of gauge-equivalent configurations so that there will be no double counting of configurations in the path integral.

In the fiber-bundle picture of Dowker,[17] this corresponds to finding a section in the bundle, reducing each configuration fiber to a particular configuration in the fiber. The difficulty is that Singer[19] has proven that there are no global gauges when space is compactified to S^3. That is, there is no global section of the fiber bundle that Dowker is using. This means that the space of gauge-fixed configurations is disconnected. But from above one knows that to get the full θ-vacuum result one must be able to reach all configurations—how is it possible for there to be a finite-action history between configurations in disconnected gauge-fixed sectors?

The answer is that the Gribov ambiguity allows for certain zero-action discontinuous evolutions. Because there are no global sections, the gauge fixing fails to always specify a unique configuration from each configuration fiber. When it does fail, there are two (or more) configurations which are related by a gauge transformation. Each of these lies on a different section of the fiber bundle and one's histories may move between the disconnected sections by way of these configurations.

Another subtlety confronting Dowker's description is that not all gauge choices preserve the distinction between small and large gauge transformations. The main significance of this is that the topological structure of the theory is more transparent in some gauges than in others. One finds that if only the small gauge transformations are fixed, then Dowker's covering space argument is unaffected. But if the large gauge transformations are also fixed, as they are in Couloumb gauge, then the topological structure indicated by the covering group is hidden. The result still holds, but the histories which were homotopically nontrivial are now discontinuous and are made possible by the Gribov ambiguity. This was explicitly demonstrated by Jackiw, Muzinich, and Rebbi.[20]

The analysis on the noncompactified space with finite-energy boundary conditions is somewhat simpler because Singer[19] mentions that there do exist global gauges. The description of the Yang-Mills θ vacua is thus cleaner if one does not invoke compactification to construct a barrier-penetration description. One does not have the disturbing circumstance of finite-action paths connecting apparently different topological sectors and one need not worry about the Gribov ambiguity. In addition it is seen that the θ vacua are a natural consequence of nontrivial homotopy of paths in field theory.

This description of tunneling in terms of nontrivial homotopy is valid for all problems in which the tunneling takes place by topological instantons. The topological classification relevant to a given theory is more general here than what is usually encountered because it allows for finite evolution times and takes more care with the topology associated with the boundary of configurations. The greatest difference would come in theories involving nonsimply connected target spaces for either the manifold or boundary, say in a Yang-Mills gauge theory with $G = \mathrm{SO}(n)$ (Refs. 4 and 14).

The main import of this discussion of nontrivial homotopy is the recognition of a tunneling mechanism other than barrier penetration. The question of the existence of a mechanism which does in fact tunnel between different topological sectors is made more tenable now that one need not conceive of a barrier to tunnel through. In a related vein, one should consider the effects of working with a base spacetime manifold whose topology is not $M \times R$ and whose sections can change topology. This would be of interest to both string theory and the question of the consistency of topology change[21] in quantum gravity. Work is in progress on these questions.

This work was supported in part by NSF Grant No. PHY-8515689.

APPENDIX

A modified version of the Laidlaw-DeWitt argument is given for completeness and to indicate the role of boundary conditions. The argument is similar to one given by Avis and Isham,[22] but the key step is a justification of the independence of the partial amplitudes which they assumed.

Having identified paths $p(a,b)$ with elements of a homotopy group H by attaching path segments $c(*,a)$ and $c(b,*)$ in the homotopy mesh to form a closed loop, one has the decomposition of the full amplitude into a sum over partial amplitudes as in (1). To derive the result, one then needs to know that the partial amplitudes are independent and that by a homotopy mesh change $c \to c'$, $K^{\alpha} \to K'^{\alpha} = K^{\lambda\alpha\mu}$, where $\lambda,\mu \in H$. The latter follows immediately from

$$c'(*,a)p(a,b)c'(b,*) = c'(*,a)c(a,*)c(*,a) \times p(a,b)c(b,*)c(*,b)c'(b,*) , \tag{A1}$$

where $c'(*,a)c(a,*)$ is the loop associated to λ, $c(*,b)c'(b,*)$ to μ, and $c(*,a)p(a,b)c(b,*)$ is the original identification to α.

To prove independence of the partial amplitudes, one needs to know that the short-time behavior of the partial amplitudes is such that, on the universal covering space,

$$K^{\alpha}(a,t;b,t') \to \delta(\alpha a,b) \quad \text{as } t \to t',\ t \neq t' , \tag{A2}$$

where αa is the α image of the point a on the universal covering space. (This is the new property used in the argument.) This is the boundary condition defining the partial amplitudes on the universal covering space. It is connected to the $t \to t'$ boundary condition for the full amplitude by the decomposition of the base manifold δ function on the universal covering space

$$\delta(a,b) = \sum_{\alpha \in H} \chi(\alpha)\delta(\alpha a,b) , \tag{A3}$$

where the $\chi(\alpha)$ impose boundary conditions on the base manifold after projecting down from the covering space. This is why determining the homotopy factors is equivalent to specifying the boundary conditions on configuration space. To guarantee the existence of the universal covering space, one must assume that configuration space C is locally simply connected. That is, there exists a neighborhood U of any point $a \in C$ such that for all $b \in U$, the paths from a to b are in the same homotopy class.

Independence of the partial amplitudes follows upon taking the short-time limit if all of the images of the point a are distinct on the covering space. Since there are no accumulation points of the images by the assumption that configuration spaced is locally simply connected, the only possibility is that two images associated to different homotopy classes agree. But this cannot happen. Suppose that two images were the same, $\alpha a = \beta a$, then a loop could be constructed running from this point to a and back. It would be in the homotopy class $\alpha\beta^{-1}$ since it runs between the α and β images of a via the identity, but all loops in the universal covering space are contractible so $\alpha\beta^{-1}=e$ which implies $\alpha=\beta$.

For a chosen homotopy mesh c, one has the full amplitude

$$K(a,t';b,t)=\sum_{\alpha\in H}\chi(\alpha)K^{\alpha}(a,t';b,t)\ . \qquad \text{(A4)}$$

Since the homotopy mesh is unphysical, if one changes it to c', the full amplitude can only change by a phase. If the change is made so that the mesh is only changed to one end point, then $K'^{\alpha}=K^{\beta\alpha}$, and labeling the phase change by β, we have

$$K'(a,t';b,t)=\exp(-i\phi_\beta)K(a,t';b,t) = \sum_{\alpha\in H}\chi(\alpha)K^{\beta\alpha}(a,t';b,t)\ . \qquad \text{(A5)}$$

Using (A4) in (A5) and linear independence of the partial amplitudes as seen in the short-time limit, one can equate the two series term by term. This gives

$$\exp(-i\phi_\beta)\chi(\beta\alpha)=\chi(\alpha)\ . \qquad \text{(A6)}$$

A choice of an overall phase can be made to set $\chi(e)=1$ where e is the identity element of the homotopy group. Then it is clear that

$$\chi(\beta)=\exp(i\phi_\beta) \qquad \text{(A7)}$$

and

$$\chi(\beta)\chi(\alpha)=\chi(\beta\alpha)\ . \qquad \text{(A8)}$$

The homotopy factors form a one-dimensional unitary representation of the homotopy group H.

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