PHYSICAL REVIEW D VOLUME 37, NUMBER 4 15 FEBRUARY 1988

Lattice QCD at finite density

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(Received 17 August 1987)

I discuss various aspects of finite-density QCD on the lattice. At the heart of the paper lies a discussion of the similarity of simulations in the quenched approximation and those of a certain eight-flavor theory with the fermionic determinant taken into account. The results of some numerical simulations on small lattices are presented. Also, I comment on various methods that aim at properly including the phase of the determinant, in the case the latter is non-positive-definite or complex.

I. INTRODUCTION

A satisfactory way to simulate lattice QCD at nonzero baryon number (i.e., finite density) has at this point in time not yet been found. A solution to this problem would lead to important predictions about the fundamental properties of the "QCD plasma," soon to be probed in the collisions of ultrarelativistic heavy ions. Clearly, in these experiments we not only look at a system at a finite temperature, but also one at finite density or equivalently, when the grand canonical ensemble is used to describe it, at finite chemical potential. This important application alone should suffice to "put the problem on the map," but it is likely that also other areas of research would benefit from a solution. This remark will become clearer later on.

Let me explain what I mean by a satisfactory solution to the problem. It is well known that the inclusion of the fermionic determinant in numerical simulations of lattice QCD presents us with enormous difficulties. But the development of clever algorithms has brought us to a point where with the use of hundreds of hours of supercomputer time we can obtain qualitatively reliable results on reasonably large lattice. Once a chemical potential is added the situation becomes worse. For example, when one uses the most popular way of including a chemical potential in the action,[1] the fermionic determinant becomes complex and the usual stochastic methods fail due to the absence of a probabilistic interpretation.

A solution to the problem would therefore entail either finding a new way of adding a chemical potential to the action (or for that matter an entirely new approach to finite-density physics, e.g., microcanonical) or coming up with an algorithm that can efficiently deal with the complex phase in the determined. The solution would be satisfactory, if it reduces the amount of work to about the same level already necessary to simulate QCD at zero baryon density.

In this paper, to be sure, I do not offer a solution to the problem. I rather point out which attempts at a solution are *unlikely* to succeed. In the next section, I will review what is known about finite-density QCD on the lattice. In Sec. III, I discuss shortly a one-dimensional model which was studied in this context by Gibbs.[2] Section IV contains the results of a simulation of an eight-flavor theory with dynamical fermions at finite density. The last section is a short summary.

II. LATTICE QCD AT FINITE DENSITY

Imagine a system of free electrons and positrons at thermal equilibrium. The Hamiltonian of the system is

$$H=\sum_{p,s}E(p)[N_e(p,s)+N_p(p,s)]\ , \tag{2.1}$$

where $N_{e,p}(p,s)$ is the number operator for electrons and positrons of momentum p and spin s, respectively. Clearly, electron and positron number are separately conserved. As is obvious from the basic QED vertex, however, when electromagnetic interactions are added, only the *difference* N_e-N_p is conserved. Hence it is this quantity which will be held fixed up to fluctuations by the introduction of a chemical potential. The grand partition function reads

$$Z(\beta,\mu)=\mathrm{Tr}\exp\left[-\beta\left[\sum_{p,s}E(p)(N_e+N_p)-\mu(N_e-N_p)\right]\right]. \tag{2.2}$$

Here the trace is in Fock space and μ is the chemical potential appropriately coupled to the fourth component of a conserved current $j_4=:\psi^\dagger\psi:$. The signs in (2.2) are such that μ "favors" the appearance of electrons. As a matter of act, the total pressure $P=(1/\beta V)\ln Z$ is the sum of electron and positron pressure:

$$P=\frac{2}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1+e^{-\beta(E_e-\mu)})+(\mu\rightarrow-\mu)\ . \tag{2.3}$$

Hence, at zero temperature, for $\mu<m$ there is no pressure and therefore all levels are unoccupied, whereas for $\mu>m$ all electronic levels up to $E=\sqrt{m^2+p^2}=\mu$ are occupied by spin-up and spin-down electrons. There are no additional positrons created at zero temperature.

The lesson to be learned here is that, for $m<\mu$ at zero temperature, things are independent of μ. For larger values of μ we begin to produce additional electrons. For a system of bosons, the plus sign in front of the ex-

ponential in (2.3) becomes a minus sign, so that at $\mu=m$ we produce an infinite number of particles.

In QCD the situation is similar. Here the conserved charge is baryon minus antibaryon number and, in analogy to the example above, we would expect that above $\mu=m_B/3$ one starts to produce additional baryons. (There are three quarks in a baryon.) To see whether this is indeed so in the interacting theory, we have to study lattice QCD at finite density.

Given the naïve lattice action for fermions (details of the fermionic action are irrelevant here) one possibility to include a chemical potential is to proceed in analogy with the continuum expression:

$$S_F=\tfrac{1}{2}\sum_{n,\mu=1}^{3}[\bar{\psi}(n)U_\mu\gamma_\mu\psi(n+\mu)-\bar{\psi}(n)U_\mu^\dagger\gamma_\mu\psi(n-\mu)]$$
$$+\frac{\mu}{2}\sum_n[\bar{\psi}(n)U_0\gamma_0\psi(n+0)+\bar{\psi}(n)U_0^\dagger\gamma_0\psi(n-0)+2m\bar{\psi}(n)\psi(n)]\ . \tag{2.4}$$

The piece involving μ is symmetrized so that it is proportional to the fourth component of a conserved current on the lattice. A major difficulty with this formulation is that the energy density for the free theory contains a term that diverges in the continuum limit: $\epsilon a^4\sim(\mu a)^4+(\mu a)^2$ (Refs. 1 and 3). A formulation which avoids the divergence (there are others[3]) is due to Hasenfratz and Karsch.[1] It is obtained by making the following replacement in the naive fermionic action:

$$U_0(n)\to e^{\mu}U_0(n),\quad U_0^\dagger(n)\to e^{-\mu}U_0^\dagger(n)\ . \tag{2.5}$$

Note that, as in the continuum, μ formally appears as the fourth component of an imaginary, constant external vector potential. Despite the fact that the substitution (2.5) solves the problem of unwanted divergences in the continuum limit there remains a major obstacle to using it in numerical simulations: The kinetic term in the fermionic action ceases to be anti-Hermitian. The fermionic determinant is therefore complex and traditional Monte Carlo methods are no longer applicable.

The easiest way to avoid this problem is to neglect the effect of the determinant altogether, i.e., to use the quenched approximation. It is becoming increasingly clearer that for many important properties of QCD the approximation is a rather good one. One seems to be able to absorb most of the effects of the determinant into a simple renormalization of parameters. Hence one might suspect that this might also be a sensible thing to try in the case of QCD at finite chemical potential. As a matter of fact, an early investigation of the subject[4] presented arguments in favor of this idea. More recently, however, the state of affairs has changed. In a detailed study[5] some serious problems with finite-density simulations were unearthed. Recall that our naïve expectation was that as long as $\mu<m_B/3$ there should be no effect of the chemical potential on observables. Above the threshold we expect chiral symmetry to be restored at some point and the emergence of, as it now seems to be the case,[6] a parity-doubled spectrum of color-singlet excitations of the QCD plasma. The authors of Ref. 5 did indeed find a threshold, but unfortunately in the wrong place: $\mu_c=m_\pi/2$. It is as though there exists a baryonic excitation at $m_B=3m_\pi/2$, whose mass goes to zero as the chemical potential goes to zero. Obviously, there is no such "Goldstone baryon" in nature. (Note that according to our picture only particles made of quarks, not antiquarks, can be produced above threshold. This is due to the factor $e^{-\mu}$ suppressing antiparticles running "backwards" in time.) It is interesting to note that the same result was also obtained by the authors of Ref. 5 when the effect of the determinant was included by using a complex Langevin equation.[7] This is presumably the reason for why these strange results were not immediately blamed on a failure of the quenched approximation.

The use of the quenched approximation in finite-density simulations was first challenged by Gibbs in Ref. 2. He gave arguments for why one should expect a threshold associated with the pion mass rather than the baryon mass. The most interesting argument indicating that the quenched approximation should not be applied to finite density simulations is based on Gibb's discussion of a simple toy model. I will discuss this model from a somewhat different point of view in the next section.

Another interesting work, Ref. 8, also indicates that the results of Ref. 5 do not represent "physics," but rather are due to numerical problems. There the authors compared the numerical simulation of SU(4) lattice gauge theory at infinite gauge coupling, using a high-precision "Dimer Monte Carlo" calculation[9] to the analytical formulas of a simple mean-field theory based on the $1/d$ expansion.[10] The numerical data agreed very well with the theory and indicate the following. (One should keep in mind though that this is strong coupling.) Chiral symmetry is restored abruptly at a critical value of μ that lies *in between* the quenched value related to the pion mass and the expected value related to the strong-coupling baryon mass. This means that the baryon mass may not play the important role that we had anticipated on the basis of our naïve agreement. Rather, a phase "full of quarks" becomes energetically more favorable in a first-order fashion. It would be extremely interesting to see whether or not the same hold true for [SU(3)] QCD at weak coupling. All of this indicates that it is a severe mutilation of the theory to neglect the phase of the determinant when the latter is complex. We will see later on that instead one should consider the quenched approximation as the limit of

another theory. But first let us discuss the previously mentioned toy model.

III. A TOY MODEL: U(1) IN ONE DIMENSION

The content of the model we are going to discuss can be simply stated as U(1) gauge theory in one dimension using staggered fermions:

$$S=\sum_{n,m}\bar{\chi}(n)(e^{\mu+i\theta}\delta_{m,n+1}-e^{-\mu-i\theta}\delta_{m,n-1})\chi(m) +m\sum_{n}\bar{\chi}(n)\chi(n)\ . \tag{3.1}$$

Here we have chosen a "gauge" in which the "gauge field" θ is a constant. The partition function is defined, however, by integrating e^{-S} over *both* the fermions and θ. Equation (3.1) has the usual *continuous* U(1)×U(1) symmetry of staggered fermions:

$$\chi(n)\to e^{i\phi}\chi(n),\quad \bar{\chi}(n)\to e^{-i\phi'},\quad n \text{ even}\ , \qquad \chi(n)\to e^{i\phi'}\chi(n),\quad \bar{\chi}(n)\to e^{-i\phi},\quad n \text{ odd}\ . \tag{3.2}$$

The mass term breaks this symmetry down to diagonal U(1) ($\phi=\phi'$). It is not hard to compute the partition function which is obtained by integrating a simple determinant over θ. The interesting result is that the chiral symmetry (3.2) is "spontaneously broken" since $\bar{\chi}\chi=\frac{1}{2}$ *independent* of μ in the limit $m\to 0$. In the quenched approximation, in which one computes $\bar{\chi}\chi$ *before* averaging over θ, one obtains $\bar{\chi}\chi=\frac{1}{2}$ for $\mu<m'/2$ and $\bar{\chi}\chi=0$ otherwise. (Here $m'\to m$ in the continuum.) Hence in this very simple example the quenched approximation differs from the full theory for $\mu>m'/2$. It is crucial to include the phase of the determinant.

Anticipating the discussion in the next section, let us now consider what would happen if we were to use $|\det|^2$ instead of the determinant. This corresponds to adding to (3.1) another "flavor" of fermions, but one that interacts with the Hermitian conjugate of the operator in which the propagation of antiparticles is favored [essentially $\mu\to-\mu$ in (3.1)]. Note that we can look upon the quenched approximation just as well as an approximation to *this* theory, i.e., the formal $n_f\to 0$ limit. In the presence of a chemical potential it is actually likely to be a *good* approximation, since we do not neglect any complex phase. Again, the computation is very simple and as expected we obtain, for the *full* theory,

$$\bar{\chi}\chi=\begin{cases}1 & \text{if } \mu<\dfrac{m'}{2}\ ,\\ 0 & \text{otherwise}\ .\end{cases} \tag{3.3}$$

At the same time the number density jumps from 0 to 2. Hence the theory behaves the same way as in the quenched approximation.

Next, let us look at the free theory. The partition function obtained by integrating (3.1) over the fermions defines the statistical mechanics of a one-dimensional system of anticommuting objects, or equivalently a one-dimensional field theory. As such it is equivalent via the transfer-matrix formalism to the quantum mechanics of a finite number of anticommuting operators in the Hamiltonian formalism [as opposed to a path integral over the coordinate $x(t)$, which leads to canonical quantization with commutators]. It is rather surprising that such a system should suffer a spontaneous breakdown of a continuous symmetry, since, according to a well-known theorem,[11] the lower critical dimension for such a breaking is two. In particular, quantum mechanics of finitely many degrees of freedom always allows for a unique symmetric ground state. To investigate this point further let us look at the behavior of the ground state in the Hamiltonian formulation. To derive the Hamiltonian, I followed the transfer-matrix formalism as explained in the article by Creutz.[12] The partition function can be written as $\mathrm{Tr}\hat{T}^N$ where $\hat{T}$ is the transfer operator and N the number of sites. We have $\hat{T}=e^{-\epsilon\hat{H}}$ where ϵ is the spacing in the time direction. After some algebra one obtains the following explicit expression for $\hat{T}$:

$$\hat{T}=\tfrac{1}{2}(a^\dagger b^\dagger+ba+a^\dagger a-b^\dagger b)-m\epsilon a^\dagger a b^\dagger b\ . \tag{3.4}$$

In (3.4) the operators a and b obey, together with their adjoints, a set of anticommutation relations. The mass m is now the dimensionful mass. The operator in (3.4) is not positive definite and therefore does not define a Hamiltonian. Instead the Hamiltonian stepping the system 2ϵ forward in time can be obtained from the square of (3.4). For the Hamiltonian we obtain

$$\hat{H}=m(ba+a^\dagger b^\dagger)\ . \tag{3.5}$$

To make the discussion of symmetry breaking more transparent, let us rename $a=\psi_L$ and $b^\dagger=\psi_R$, in which case (3.5) can be written as

$$\hat{H}=m\bar{\psi}\psi=\psi^\dagger\gamma_0\psi\ , \tag{3.6}$$

where $\psi=\begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix}$, $\gamma_0=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, and $\gamma_5=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. We can think of the Hamiltonian as breaking *explicitly* a γ_5 chiral symmetry. It is clear though that $\bar{\psi}\psi$, being proportional to the Hamiltonian, will be independent of m, so that in the limit $m\to 0$ a "condensate" remains. The symmetric states will, however, become degenerate with the finite-m ground state in this limit, as can be seen by diagonalizing (3.6). Defining $u=(1/\sqrt{2})(\psi_L+\psi_R)$ and $v=(1/\sqrt{2})(\psi_L-\psi_R)$ we get

$$\hat{H}=m(u^\dagger u-v^\dagger v)\ . \tag{3.7}$$

At finite m the ground state $|01\rangle$ has one v particle present, the excited states are ($|00\rangle$, $|11\rangle$) and $|10\rangle$ in this order. In the $m\to 0$ limit the symmetric states $|00\rangle$ and $|11\rangle$ (they are annihilated by the chiral charge $\bar{\psi}\gamma_5\psi$) become degenerate with the state $|01\rangle$. A chemical potential in the action leads to the Hamiltonian

$$\hat{H}=[-mv^\dagger v+(m-2\mu)u^\dagger u]\ . \tag{3.8}$$

Hence for $\mu>m/2$ the ground state is the chirally symmetry state $|11\rangle$. To conclude this section let me summarize. In the quantum-mechanical system derived

from (3.1) we can define a continuous chiral symmetry which is explicitly broken by the Hamiltonian and is restored at some critical chemical potential. The formal $m \to 0$ limit leaves a nonvanishing $\bar{\psi}\psi$; the symmetric states, however, become degenerate with the nonsymmetric finite-m ground state. The fact that the one-dimensional field-theory version of the model leads to a magnetization at zero mass, might be due to the fact that the equivalent bosonic field theory is nonlocal in the zero-mass limit.

IV. MONTE CARLO SIMULATION OF AN 8-FLAVOR THEORY

In the preceding section we have seen that one can look upon the simulation using the absolute square of the determinant as corresponding to adding additional flavors that interact with the conjugate interaction. We also saw that, as far as the dependence on the chemical potential is concerned, the quenched approximation was a good one. As we will see now, the same holds true also in QCD in four dimensions. To be explicit, I have simulated the following theory:

$$Z = \int [dU] e^{-S_G + \bar{\chi}_1 M \chi_1 + \bar{\chi}_2 M^\dagger \chi_2} . \tag{4.1}$$

M is the Dirac operator for staggered fermions, and the chemical potential has been included via the prescription (2.5). Upon integration over the fermions one obtains $|\det M|^2$ in the measure and in the $\mu \to 0$ limit (4.1) reduces to eight-flavor QCD. What do we expect the dependence on the chemical potential to be like in this theory? First of all, it is clear that there will be a threshold at half the pion mass below which we expect everything to be independent of μ. This is because propagation of "type 2" antiparticles is favored, just as propagation of type 1 *particles* is favored by the exponential factor e^{μ}. Because of confinement in this theory we can, above threshold, only produce additional pions, i.e., $q_1 \bar{q}_2$ bound states.

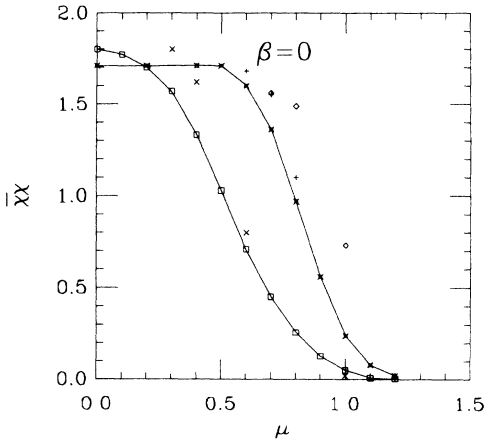

FIG. 1. The chiral order parameter $\bar{\chi}\chi$ as a function of μ at infinite gauge coupling. The solid lines are drawn through the $n_{\rm pf}=50-10$ (10 iterations discarded for thermalization) data at $m=0.2$ (squares) and $m=0.5$ (fancy squares). $\times$ and $+$ denote the same quantities for $n_{\rm pf}=240-40$. The $\Diamond$ denotes $\bar{\chi}\chi$ at $m=0.75$, $n_{\rm pf}=50-10$.

But there is also the possibility, that as we increase μ we will see a difference from the quenched approximation: pions are bosons and, as was mentioned in the introduction, if it were not for the fermionic nature of their constituents we would expect to immediately produce infinitely many of them above threshold. But because of the hard-core repulsion there is a chance for the dependence on μ to be smooth above $m_\pi/2$. In this case it is not inconceivable that once we reach $\mu = m_B/3$ we see a *second* threshold effect due to the production of type-1 baryons and type-2 antibaryons. If this were the case in our simulation (which does include the effect of dynamical fermions) it would be likely that in QCD with dynamical fermions the naïve threshold $m_B/3$ applies. To look at the effects of baryons I have in addition to the SU(3) theory also simulated the U(3) theory, which has no baryons.

Most of the simulations were done at infinite gauge coupling. Because of my limited resources I ran on small 4^4 lattices. The boundary conditions in the time direction were antiperiodic and periodic in the spatial directions. I used the pseudofermion method[13] to take into account the effect of dynamical fermions, and also to compute $\bar{\chi}\chi$ and the number density. The program passed all the checks I performed; in particular, it agrees with the analytical formulas for $\bar{\chi}\chi$ and the number density as a function of μ of the free theory. In Fig. 1 the results for $\bar{\chi}\chi$ are shown for three different values of the mass. I used the same number $n_{\rm pf}$ of pseudofermionic iterations both for the update and the measurements. Shown in the figure are the results for $n_{\rm pf}=50-10$ and $n_{\rm pf}=240-40$. There is a pronounced shift in the location of μ_c when one increases the number of iterations. In Fig. 2 the number density ρ is plotted for two values of the mass. It is remarkable that at $n_{\rm pf}=50-10$ the Maxwell relation[4]

$$\left.\frac{\partial \rho}{\partial m}\right|_{\mu} = \left.\frac{\partial \bar{\chi}\chi}{\partial \mu}\right|_{m}$$

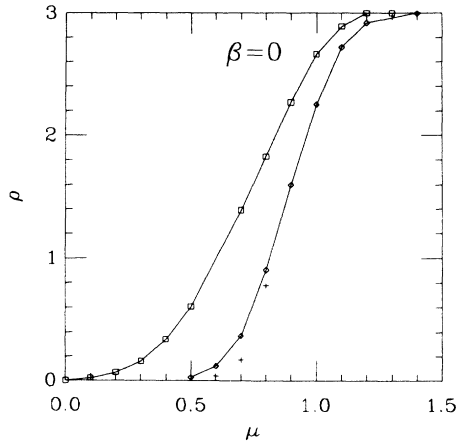

FIG. 2. The number density ρ at infinite gauge coupling. The solid lines correspond to the ones in Fig. 1. The data marked by a $+$ are taken at $m=0.5$ and $n_{\rm pf}=240-10$. The shift of the critical chemical potential as the number of pseudofermion iterations is increased is evident.

already holds to good accuracy. It is apparent that $\mu_c \to 0$ as the mass is decreased. There is absolutely *no* indication of a second threshold. As a matter of fact both observables change quite rapidly above threshold and the change becomes steeper with increasing number of pseudofermion iterations. Just as in Ref. 8, the number density quickly reaches its maximum allowed value of $\rho=3$. (Note that I measure type-1 observables only, i.e., $\bar{\chi}_1\chi_1$ and ρ_1.) I have repeated the simulations also for the U(3) theory. The results are *identical* to the ones shown in Figs. 1 and 2. Hence we conclude that there is no evidence for baryon production in this theory, but similar to SU(4) QCD (Ref. 8) there is a transition to a quark-rich phase with three quarks/flavor. [In a large-μ evaluation of the determinant, such a contribution arises in U(3) from the six-link integral.]

I have also redone the simulations in the quenched approximation. Again for *both* SU(3) and U(3) the results are within the errors the same as those shown in Fig. 1 and 2. This is not surprising, since the effects of dynamical fermions are expected to be small at infinite coupling. Hence, the quenched approximation "works" as an approximation to the eight-flavor theory. From this point of view the fact that we find a threshold for the U(3) theory is not surprising. In Ref. 5 it as claimed, however, that due to the absence of baryons we do not expect such a threshold in U(3). This is not so. First of all, there is no connection to baryons in the quenched approximation as far as the dependence on μ is concerned. In addition, the simplest example that in U(N) the dependence on μ only integrates out in the *partition function* is the U(1) model of the previous section. There the quenched approximation did show a threshold where as the full theory was independent of μ.

To see what happens at finite values of the gauge coupling I have performed the same Monte Carlo simulations at $\beta=4.5$. This value is below the critical value for the first-order transition known to occur when the number of flavors is eight.[14] The results are similar to the strong-coupling ones. I used 24–4 pseudofermionic iterations; the acceptance was 0.82. Note, that in the presence of a chemical potential a change in a timelike link is multiplied by an exponential of the chemical potential. This has to be taken into account when deciding on a reasonable acceptance rate The threshold again decreases with the quark mass m. The result for the chiral order parameter and the Wilson line are shown in Fig. 3. Note the curious behavior of the Wilson line ($m=0.1$). First it rises, consistent with our expectation that the system should in some sense "deconfine," but then it drops again. The U(3) results are identical to the SU(3) ones, except for the Wilson line which vanishes. In the quenched approximation both SU(3) and U(3) show the threshold except that the Wilson line does not show any μ dependence, which is expected since there is no fermion feedback.

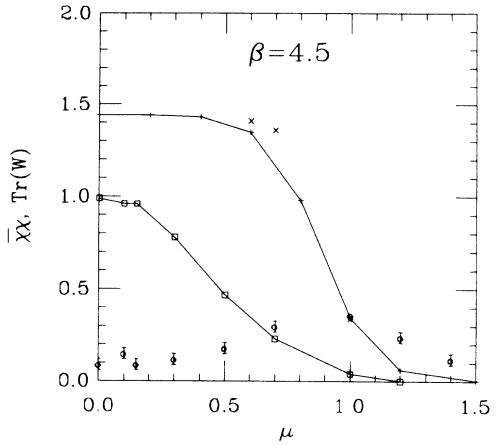

FIG. 3. Data at a finite value of β. The Wilson line is shown with errors, all other errors are of the order of the size of the points. Solid lines are drawn through the data for $\bar{\chi}\chi$ at $m=0.5$ (+) and $m=0.2$ (squares) at $n_{\rm pf}=24$–4. The three points denoted by a $\times$ are $m=0.5$, $n_{\rm pf}=200$–40 data.

It is clear from the above discussion that the phase of the determinant must somehow be included in the simulation. Without it the results will be similar to the eight-flavor theory (or for that matter any theory defined by a power of the absolute value of the determinant). Although it is quite clear that the procedure is doomed to fail, I have tried to measure the phase separately and included it in the expectation values. What I mean is that the expectation value of a physical observable O can be written as

$$\langle O \rangle = \frac{\langle\langle O e^{i\theta} \rangle\rangle}{\langle\langle e^{i\theta} \rangle\rangle} , \tag{4.2}$$

where $\langle\langle\ \rangle\rangle$ denotes an expectation value with the measure $|\det M|\, e^{-S_G}$. Note that in the pseudofermion method, one only computes the *change* in the determinant, so that one has to start with a configuration of known determinant (phase) and determine subsequent values of the determinant by adding up the changes. Such a procedure can of course lead to a large accumulation of error. As expected, the method fails miserably. The phase oscillates wildly between configurations, and at the end one obtains the same values for the observables but with huge errors. Partly, importance sampling is to blame here. In Ref. 15 we have tried to remedy this part by using the microcanonical methods of Ref. 16 in some simple models. We have found that these methods do improve on Monte Carlo sampling, but a relatively small amount of noise in the phase leads to a rapid deterioration of the quality of the results. Noise in the phase will of course be inherent to any stochastic method that one can conceive of. One possibility for including the phase of the determinant is the complex Langevin equation.[7] But since it seems to give the same results as the quenched approximation,[5] this is probably not a good way to go. Hence we are faced with serious difficulties.

V. CONCLUSIONS

In this paper I have discussed a variety of issues concerned with simulations of lattice QCD at finite density. We have argued that the quenched approximation

should be considered as a good approximation to a class of theories that make use of the absolute value of the determinant in the presence of a chemical potential. This has been shown to be the case explicitly for the eight-flavor theory. The behavior of these theories as a function of the chemical potential is drastically different from what we expect to be that of QCD, although the high-density phase should be very similar. It might actually be a good idea to study this phase in more detailed in the spirit of Ref. 6. An interesting quantity that can be studied with presently available methods is the quark-number susceptibility at finite temperature and *zero* chemical potential.[17] In this last reference the authors introduced different chemical potentials for "up" and "down" quarks, leading to what they call flavor-singlet and flavor-nonsinglet susceptibilities. The eight-flavor model I discussed is a special case of degenerate flavors of equal and opposite chemical potential and corresponds to the flavor-singlet sector.

I believe that it is not clear at all that the naïve idea of a threshold associated with the baryon mass is realized in nature. Before we can answer this question and many others concerned with the nature of the dense phase new ideas are needed. Traditional Monte Carlo methods are doomed to fail.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy under Grant No. DE-AT03-81-ER40029. I would like to thank the SDSC allocation committee for computer time on the Cray XMP-48 and R. Dashen, P. Rossi, D. Toussaint, and P. Wills for helpful conversations.

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