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Debye potentials for monopoles in U(1) and SU(2): Identification of Higgs remnant in electrodynamics

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The Dirac vector potential and Banderet vector potential give the same monopole field. It is shown that, unlike the Dirac potential, the Banderet potential is not integrable to yield a Debye potential. Next, the coordinate-patch formalism is given for the Debye potentials with the usual trade off between two copies of regular potentials and one singular potential. Some simple examples of Debye potentials are given. For the spontaneously broken SU(2) gauge theory, the Higgs potential in the large-distance limit is seen to approach its U(1) analog in the form of the Debye potential within a factor of r.

I. INTRODUCTION

This paper reports the findings of three results in the study of Debye scalar potentials for the electromagnetic field and SU(2) gauge theory.

(1) In Ref. 1 a simple relation is found between the vector potential $\mathbf{A}$ and the Debye potential ψ [see Eq. (6) below]. While every Debye potential yields a vector potential, the converse is more subtle. Does every vector potential possess a Debye potential? Contrary to naive expectation, the answer turns out to be negative. We study the examples of the Dirac monopole vector potential and the Banderet monopole vector potential. They give the same monopole field and are related by a gauge transformation. Nevertheless, the Banderet vector potential is shown to be nonintegrable to yield a Debye scalar potential. Thus we demonstrate that not every vector potential can have a Debye scalar potential. This is the first result (Sec. II).

(2) Inasmuch as the singular Dirac monopole vector potential has associated with it a singular Debye scalar potential, and since, instead of dealing with the Dirac string and/or singular potential, the coordinate-patch formalism has been developed for the magnetic monopoles,[2] it is natural to ask whether the relationship between the vector potential and the Debye scalar potential can be extended in each region such that the corresponding quantities are indeed related by a gauge transformation. As might be expected intuitively, the answer is yes. This is the second result (Sec. III).

(3) In an effort toward a better understanding of the structural relationship between the vector potentials and the Debye potentials, a few simple examples are given in electrodynamics [Eqs. (13) and (14) below]. The following relation seems striking. When the vector potential is expressible as a vector product between a constant vector and the radial vector, the Debye potential is simply (apart from a minus sign) the scalar product involving the same quantity. This observation immediately reminds us of the similar situation between the vector potential and the Higgs potential for monopoles in the SU(2) gauge theory. A closer scrutiny of the latter case is undertaken. In particular, we study the asymptotic (large-distance) limit of the Higgs potential and find that when the symmetry is spontaneously broken from SU(2) to U(1), the remnant of the Higgs potential is simply the radial Hertz potential or, within a factor of r, the Debye scalar potential, in electrodynamics. This is the third result (Sec. V).

II. THE DIRAC AND BANDERET MONOPOLE POTENTIALS

Among the many possible vector potentials that describe a magnetic monopole field, two stand out. One is the well-known Dirac form[3] (purely azimuthal component)

$$\mathbf{A}^D = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\boldsymbol{\phi}} . \tag{1}$$

The other is the less familiar Banderet form[4] (purely polar-angle component)

$$\mathbf{A}^B = -\frac{g}{r}\phi\sin\theta\,\hat{\boldsymbol{\theta}} . \tag{2}$$

It is easily checked that both give the same *radial* monopole field

$$\nabla\times\mathbf{A}^D = \nabla\times\mathbf{A}^B = g\frac{\hat{\mathbf{r}}}{r^2} . \tag{3}$$

Much has been said concerning the Dirac potential.[2,5] The two vector potentials are in fact related by a gauge transformation. Explicitly, we have

$$A^D\hat{\boldsymbol{\phi}} = A^B\hat{\boldsymbol{\theta}} + \nabla\Lambda , \tag{4a}$$

$$\Lambda = g\phi(1-\cos\theta) . \tag{4b}$$

To escape from the usual predicament of vanishing divergence of curl $\mathbf{A}$, we are dealing with vector potentials that are inherently pathological, namely, the Dirac potential is singular at $\theta=\pi$, and the Banderet potential is not single valued in ϕ; more precisely, it does not respect the periodicity in ϕ, e.g., $A^B(\phi=2\pi) \neq A^B(\phi=0)$.

Is singular potential more tolerable than non-single-valued potential? We pose the question whether there exists any criterion by which one may make a judicious choice. We find one way to settle this question by invoking the Debye scalar potential.[1] For simplicity, we shall only consider the static magnetic case here. The magnetic Debye scalar potential corresponding to the Dirac vector potential is given by[1] (we shall hereafter suppress the subscript M for the magnetic mode)

$$\Psi^D = -\frac{2g}{r}\ln\cos\frac{\theta}{2} . \tag{5}$$

For the static case, $\mathbf{A}$ and Ψ are simply related as[1]

$$\mathbf{A} = -\mathbf{L}\Psi , \tag{6}$$

where

$$\mathbf{L} = -\mathbf{r}\times\nabla . \tag{7}$$

Equation (4) implies that, if there is a Ψ^B,

$$-L_\phi \Psi^D \hat{\phi} = -L_\theta \Psi^B \hat{\theta} + \nabla[g\phi(1-\cos\theta)] . \tag{8}$$

When integrated, the azimuthal component of this equation (recall that $L_\phi = -\partial/\partial\theta$) reproduces Ψ^D of Eq. (5). The polar-angle component of Eq. (8) yields [recall that $L_\theta = (1/\sin\theta)(\partial/\partial\phi)$]

$$\Psi^B = \frac{g}{2r}\phi^2\sin^2\theta + c(\theta,r) , \tag{9}$$

where $c(\theta,r)$ is independent of ϕ.

Such a Ψ^B is clearly unacceptable since it would give a spurious ϕ component of $\mathbf{A}^B$ and also a wrong radial magnetic field. We conclude that the Banderet potential (2) is not integrable to yield a Debye potential.

III. COORDINATE-PATCH FORMALISM FOR THE DEBYE SCALAR POTENTIALS

The coordinate-patch formalism which calls for two sets of nonsingular vector potentials $A^{(a)}, A^{(b)}$ that replace one singular Dirac potential A^D is presumably well known.[2] Here we simply point out that the corresponding entities exist for the Debye scalar potentials.

In region a

$$A^{(a)} = \frac{g(1-\cos\theta)}{r\sin\theta} , \tag{10a}$$

$$\Psi^{(a)} = -\frac{2g}{r}\ln\cos\frac{\theta}{2} \quad (\theta\neq\pi) . \tag{10b}$$

In region b

$$A^{(b)} = -\frac{g(1+\cos\theta)}{r\sin\theta} , \tag{11a}$$

$$\Psi^{(b)} = -\frac{2g}{r}\ln\sin\frac{\theta}{2} \quad (\theta\neq 0) . \tag{11b}$$

In the overlap region, the two sets are related by a gauge transformation,[2]

$$A_k^{(a)} = A_k^{(b)} + \partial_k\Phi, \quad \Phi = 2g\phi . \tag{12a}$$

For the Debye potentials, the corresponding relation reads

$$\Psi^{(a)} = \Psi^{(b)} + \frac{2g}{r}\ln\tan\frac{\theta}{2} . \tag{12b}$$

For this simple example, the results for the Debye potentials are just the θ integral of the corresponding relations for the Wu-Yang potentials.[2]

IV. SOME SIMPLE EXAMPLES OF DEBYE POTENTIALS

In electrodynamics, we have the following.

Example 1. Uniform B field

$$\begin{aligned} &\mathbf{A} = \tfrac{1}{2}\mathbf{B}\times\mathbf{r}, \quad \mathbf{B} = \text{constant vector} , \\ &\Psi = -\tfrac{1}{2}\mathbf{B}\cdot\mathbf{r} . \end{aligned} \tag{13}$$

Example 2. Multipole fields

$$\begin{aligned} &\mathbf{A} = \boldsymbol{\mu}\times\mathbf{r}f(r), \quad \boldsymbol{\mu} = \text{constant vector} , \\ &\Psi = -\boldsymbol{\mu}\cdot\mathbf{r}f(r) . \end{aligned} \tag{14}$$

Stretching beyond electrodynamics, we venture into the Wu-Yang ansatz[6] for the SU(2) gauge field:

$$A_i^a = -\epsilon_{aij}\frac{x_j}{r^2}f(r) . \tag{15}$$

When folded with the isospin Pauli matrices

$$A_i = A_i^a \frac{\tau^a}{2}$$

(15) reads

$$\mathbf{A} = \frac{\boldsymbol{\tau}\times\mathbf{r}}{2r^2}f(r) . \tag{15'}$$

The Debye potential reads

$$\Psi = -\frac{\boldsymbol{\tau}\cdot\mathbf{r}}{2r^2}f(r) . \tag{16}$$

It is known that in the asymptotic limit usually imposed for a monopole, $f(r)\to 1$ as $r\to\infty$, the Wu-Yang ansatz (15) is reducible to the Dirac form (1) by a *singular* gauge transformation.[2,7]

Apart from the multiplicative radial functions, the algebraic structure of the Debye potential here curiously resembles the expression for the Higgs potential in the 't Hooft–Polyakov ansatz.[8] This resemblance is further examined in the next section.

V. DESCENT FROM THE SU(2) GAUGE THEORY TO U(1) ELECTROMAGNETISM

Consider the *finite-energy* classical solution problem of the SU(2) Yang-Mills-Higgs system.[7] The asymptotic boundary conditions to be imposed are (we suppress the time component here)

$$D_k\Phi^a \equiv \partial_k\Phi^a - \epsilon^{abc}A_k^b\Phi^c = 0 , \tag{17}$$

$$(\Phi^a)^2 = 1 , \tag{18}$$

where the Higgs potential Φ^a is suitably normalized and we have set the gauge coupling constant to be unity.

The expression for the gauge vector potential A_k^a

satisfying Eq. (17) reads[9] (modulo a term parallel to Φ^a which we ignore here)

$$A_k^a=\epsilon^{abc}\Phi^b\partial_k\Phi^c \ . \tag{19}$$

Of all the known ansatz for the SU(2) or SO(3) theories, such as those of Wu and Yang,[6] 't Hooft and Polyakov,[8] and Bogomolny, Prasad, and Sommerfield,[10] the following algebraic structure is a common feature.

For the Higgs potentials, we have

$$\Phi^a=\frac{x^a}{r^2}H(r) \ . \tag{20}$$

The gauge potential reads

$$A_i^a=-\epsilon^{aij}\frac{x^j}{r^2}f(r) \ . \tag{21}$$

The boundary conditions are

$$H\to r, \quad f\to 1 \quad \text{as } r\to\infty \ . \tag{22}$$

The ansatz (20) and (21) is consistent with (19) in the asymptotic limit (22).

When the Higgs potential Φ^a picks a certain direction in isospin space [thereby breaking spontaneously the SU(2) symmetry down to U(1) electromagnetism], the mysterious mixing of the spatial and the internal (isospin) indices renders Φ^a to be radially directed (namely, parallel to $\hat{\mathbf{x}}^a$). In the asymptotic limit (22), the following two expressions coincide.

(a) From (19), we have

$$\begin{aligned} A_k^a&=\epsilon^{abc}\frac{x^b}{r^2}H(r)\partial_k\frac{x^c}{r^2}H(r) \\ &=\epsilon^{abc}x^b\delta_{kc}\frac{H^2(r)}{r^4} \\ &\underset{r\to\infty}{\sim}\epsilon^{kab}\frac{x^b}{r^2} \ . \end{aligned} \tag{23}$$

(b) On the other hand, we evaluate the analog of Eqs. (6) and (7). Take $\Psi=\Psi^a\tau^a/2$, $\Psi^a=(x^a/r^2)h(r)$ with $h(r)\to 1$ as $r\to\infty$.

We have

$$\begin{aligned} A_k^a&=-(\mathbf{r}\times\nabla)_k\Psi^a=-\epsilon_{kij}x_i\partial_j\frac{x^a}{r^2}h(r) \\ &\underset{r\to\infty}{\sim}\epsilon^{kai}\frac{x_i}{r^2} \ . \end{aligned} \tag{24}$$

This suggests that at the large distance when the broken-SU(2) symmetry retains the residual long-range U(1) electromagnetism, the Higgs potential Φ^a asymptotically becomes radially directed and approaches the radial Hertz potential, which is the Debye potential multiplied by a factor of r.

Whether this connection between the asymptotic Higgs potential and the Debye potential is fortuitous just for this monopole case remains to be seen.

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