Numerical simulation of cosmic-string evolution in flat spacetime

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We present a numerical simulation of the evolution of a system of cosmic strings in flat spacetime. Our algorithm for the string dynamics exactly integrates the Nambu equation of motion. The strings are represented by discrete points on a spatial lattice, allowing an intercommuting algorithm which exactly preserves all relevant quantities. We find that the system evolves to a steady state in which most of the energy is in loops of the smallest size allowed in the simulation. "Infinite" strings present in the initial state fragment into a distribution of small loops. In the absence of an energy cutoff there is no equilibrium, and the steady state artificially imposed by the cutoff bears no resemblance to the initial state.

I. INTRODUCTION

Despite much recent work,[1–5] the evolution of cosmic strings is not yet fully understood. The most attractive possibility is the scale-invariant evolution scenario[2] in which the typical scale of the system of strings at any time t is comparable to the horizon scale t. This picture is supported by numerical simulations performed by Albrecht and Turok,[3] but, because of the small size of the simulations, some doubt still remains.

An alternative approach to the problem has been suggested by Kibble,[4] who derived equations for the formation and absorption of closed loops by long strings in order to study the problem analytically. His equations include several unknown parameters, the most important of which is the loop production function describing the rate at which closed loops of various sizes are chopped off the infinite strings. Kibble attempted to determine this function by requiring that the equilibrium solution of his equations in a nonexpanding universe should resemble the random configuration in which the strings are formed. With this assumption, he found that the system of strings evolves either to a scale-invariant regime or else toward a string-dominated universe, depending on the initial conditions.

Kibble's approach was later extended by Bennett,[5] who also pointed out an inconsistency between his analytic results and the numerical results by Albrecht and Turok. He raised the possibility that what Albrecht and Turok see in their simulations is not a scale-invariant evolution, but rather a transient behavior which is an artifact of a special choice of initial conditions.

Thus, none of the approaches to the evolution of cosmic strings has yet given conclusive results. Numerical simulations suffer mainly from their limited size, while the main drawback of the analytic approach is a large number of unknown parameters.

In this paper we introduce a new numerical method to study the evolution of a system of strings in flat spacetime. An attractive feature of this method is that it represents the string dynamics exactly, in the sense explained below. Our model system can be used as a laboratory for numerical experiments with strings and as a testing ground for the analytic approaches to string evolution.

In this paper we apply our model to investigate the existence of an equilibrium state of a system of cosmic strings in flat spacetime. We find that the system evolves to a steady state which has no resemblance to the tangled configuration of strings at formation. In the steady state, most of the energy is in small loops of strings whose energy is comparable to the minimum loop allowed in our discrete model. "Infinite" strings which initially extend across the simulation completely break up. We conclude that no equilibrium state exists, in the absence of a minimum energy cutoff.

II. DESCRIPTION OF THE MODEL

Before we discuss the details of our discrete model, let us briefly review the dynamics of continuous strings. We shall describe a string in the usual way[6] by a vector function $\mathbf{x}(\sigma,t)$ satisfying two conditions:

$$\mathbf{x}'\cdot\dot{\mathbf{x}}=0 \tag{1a}$$

and

$$\mathbf{x}'^2+\dot{\mathbf{x}}^2=1 \ . \tag{1b}$$

Here σ is a parameter along the string; primes and overdots denote derivatives with respect to σ and t, respectively. Mathematically, these are gauge conditions on the parametrization of the string's world sheet which make the Nambu equation of motion reduce to the simple wave equation

$$\ddot{\mathbf{x}}-\mathbf{x}''=0 \tag{2}$$

in flat spacetime. Physically, the first gauge condition means that each segment of string moves perpendicularly to itself; the second means that the energy in a segment $d\sigma$ is just $\mu\, d\sigma$. (Here μ is the mass per unit length of the string, a constant determined by the scale of symmetry breaking for the string's Higgs field.)

The wave equation (2) has the general solution

$$\mathbf{x}(\sigma,t)=\tfrac{1}{2}[\mathbf{a}(\sigma-t)+\mathbf{b}(\sigma+t)] \ . \tag{3}$$

The functions **a** and **b** are not arbitrary, because of the gauge conditions. However, it is easy to show that the equation of motion preserves conditions (1), so one needs only to require that the initial state satisfies them.

Moving strings can intercommute (or change partners) at intersection points with a certain probability p. In fact, the intercommuting process is totally deterministic, and the word "probability" refers to averaging over the relative orientations and velocities of the intersecting strings. Numerical calculations performed by Shellard[7] for a particular model of strings indicate that intercommuting occurs in practically all configurations, and so p is very close to one. However, these results can be model dependent, and it is quite possible that in other models p can be substantially different from unity.

The discrete model of strings in flat spacetime that we have developed permits us to investigate the motion and intercommuting of strings with a computer. At any instant we describe a string by a finite number of points on it which are equally spaced, in the parameter σ, by some separation δ. The segments of string between these points all have the same energy.

To evolve these points in time, we can exploit a surprising consequence of Eq. (3). Any function satisfying the wave equation (2) satisfies a finite-difference equation as well:

$$\mathbf{x}(\sigma,t+\delta)=\mathbf{x}(\sigma+\delta,t)+\mathbf{x}(\sigma-\delta,t) -\mathbf{x}(\sigma,t-\delta) \ . \tag{4}$$

Although this appears to be only a finite-difference approximation to (2) obtained by introducing

$$\ddot{\mathbf{x}}(\sigma,t)\simeq[\mathbf{x}(\sigma,t+\delta)-2\mathbf{x}(\sigma,t)+\mathbf{x}(\sigma,t-\delta)]/\delta^2 \tag{5a}$$

and

$$\mathbf{x}''(\sigma,t)\simeq[\mathbf{x}(\sigma+\delta,t)-2\mathbf{x}(\sigma,t)+\mathbf{x}(\sigma-\delta,t)]/\delta^2 \ , \tag{5b}$$

it is in fact an exact relation, as the substitution of (3) into it will reveal. It relates the values of **x** on a diamond lattice in the (σ,t) plane, as indicated in Fig. 1. If we know the **x**'s at time $t-\delta$ and at time t, then we can calculate them at all future times.

Actually, rather than using the second-order difference equation (4), it is more intuitive to define

$$\mathbf{v}(\sigma,t)\equiv\{\mathbf{x}(\sigma,t+\delta)-\tfrac{1}{2}[\mathbf{x}(\sigma+\delta,t)+\mathbf{x}(\sigma-\delta,t)]\}/\delta \tag{6}$$

so that (4) becomes two first-order difference equations:

$$\mathbf{x}(\sigma,t+\delta)=\tfrac{1}{2}[\mathbf{x}(\sigma+\delta,t)+\mathbf{x}(\sigma-\delta,t)] +\mathbf{v}(\sigma,t)\delta \tag{7a}$$

and

$$\mathbf{v}(\sigma,t+\delta)=\tfrac{1}{2}[\mathbf{v}(\sigma+\delta,t)+\mathbf{v}(\sigma-\delta,t)] +[\mathbf{x}(\sigma+2\delta,t)-2\mathbf{x}(\sigma,t) +\mathbf{x}(\sigma-2\delta,t)]/4\delta \ . \tag{7b}$$

Clearly, $\mathbf{v}(\sigma,t)$ is a discrete version of the string velocity $\dot{\mathbf{x}}(\sigma,t)$ which becomes exact as $\delta\rightarrow 0$. These difference equations allow us to compute **x** and **v** exactly at all times if we know them at one instant, as indicated in Fig. 2.

Note that the **x**'s and the **v**'s are determined on two interlocking but nonoverlapping diamond lattices in the (σ,t) plane. One can visualize the **v**'s as the velocities of the links of string between the points that we are tracking. (Note that each of these links has the same energy, namely, $2\mu\delta$.) The points have position but not velocity, while the links have velocity but not position. (Note also that at time $t+\delta$ we obtain the **x**'s for a different set of values of σ than at time t, although it is the same set as at $t-\delta$.) The difference equations (7) generate an exact solution of the wave equation (2). As $\delta\rightarrow 0$, this solution becomes the one for which the initial position is $\mathbf{x}(\sigma,0)$ and the initial velocity is $\dot{\mathbf{x}}(\sigma,0)$.

As discrete versions of the gauge conditions (1) we adopt

$$\mathbf{u}\cdot\mathbf{v}=0 \tag{8a}$$

and

$$\mathbf{u}^2+\mathbf{v}^2=1 \ . \tag{8b}$$

Here **v**, which was defined in (6), is the discrete counterpart of $\dot{\mathbf{x}}$, and

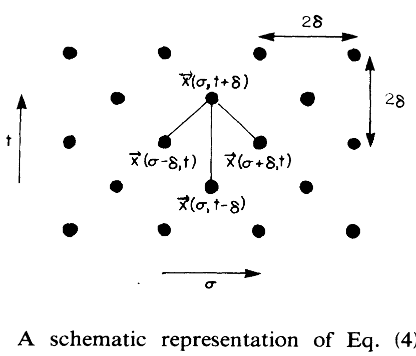

FIG. 1. A schematic representation of Eq. (4), by which $\mathbf{x}(\sigma,t)$ can be computed on a diamond lattice in the (σ,t) plane.

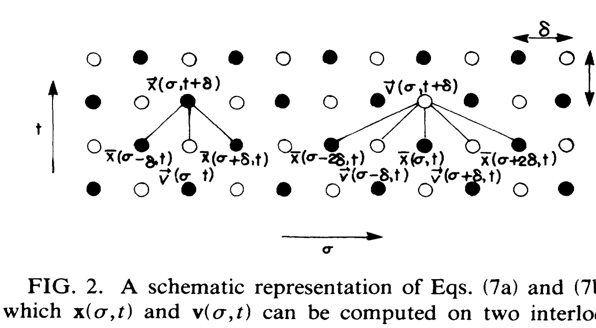

FIG. 2. A schematic representation of Eqs. (7a) and (7b), by which $\mathbf{x}(\sigma,t)$ and $\mathbf{v}(\sigma,t)$ can be computed on two interlocking diamond lattices in the (σ,t) plane.

$$\mathbf{u}(\sigma,t) \equiv [\mathbf{x}(\sigma+\delta,t) - \mathbf{x}(\sigma-\delta,t)]/2\delta \qquad (9)$$

is the discrete version of $\mathbf{x}'$. (It should be visualized as belonging to the links, like $\mathbf{v}$.) Obviously, as $\delta \to 0$ these become Eqs. (1). It is less obvious, although easy to check, that these discrete gauge conditions are preserved by the discrete evolution equations (7), just as the continuous gauge conditions (1) are preserved by the wave equation (2). We must ensure that our choice of initial state satisfies them.

A simple way to do this is to restrict the possible $\mathbf{u}$'s and $\mathbf{v}$'s to a set of discrete values. Up to this point, our model has been discrete in the sense that we determine the $\mathbf{x}$'s and $\mathbf{v}$'s on a lattice in (σ,t) space. We can go further and make the model spatially discrete as well. From (4) it is clear that if the components of all the $\mathbf{x}$'s at two successive time steps are integral multiples of δ, then they remain so forever. By (6) and (9) this means that the components of the $\mathbf{u}$'s and $\mathbf{v}$'s will be integral or half-integral. The discrete gauge conditions (8) then allow only three types of links, as shown in Fig. 3.

The first type has $|\mathbf{u}|=1$ and $\mathbf{v}=0$. Its end points are separated by $|\Delta\mathbf{x}|=2\delta$, which implies that one component of $\Delta\mathbf{x}$ is $\pm 2\delta$ and the other two are zero. Such links are "fully stretched" and are at rest.

The second type has $|\mathbf{u}|=|\mathbf{v}|=\sqrt{2}/2$. Its end points are separated by $\sqrt{2}\delta$, which implies that two components of $\Delta\mathbf{x}$ are $\pm\delta$ and the other is zero. Similarly, two components of $\mathbf{v}$ are $\pm\frac{1}{2}$ and one is zero. Such partially contracted links move at 0.707 light speed perpendicularly to themselves.

The third type has $\mathbf{u}=0$ and $|\mathbf{v}|=1$. Its end points are degenerate, with $\Delta\mathbf{x}=0$. One component of $\mathbf{v}$ is ± 1 and the other two are zero. Such links are fully contracted and move at light speed parallel to the x, y, or z axis. They are the discrete versions of the cusps that exist in continuous strings.[8]

A sample loop, lying in a plane for ease of drawing, that is constructed from such links is shown in Fig. 4. With a little thought one can see that in three dimensions the string points at any instant will all lie on a face-centered-cubic lattice. We can construct initial states satisfying the discrete gauge conditions (8) by using just these types of links to make strings on such a lattice.

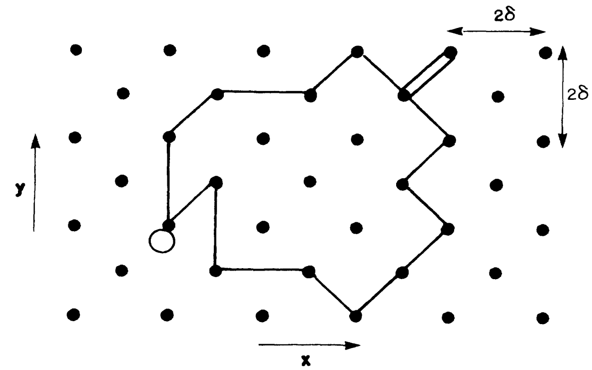

FIG. 4. An example of a planar loop of string in our model. The end points of the links of a three-dimensional loop would lie on a face-centered-cubic spatial lattice.

By imposing spatial discreteness we reap a computational benefit: we can use integer arithmetic to avoid any round-off errors in the evolution. This means, for example, that if the loops are not allowed to intercommute, then they will execute exactly periodic motion indefinitely.[9]

After evolving all the strings ahead one time step δ, we allow them to intercommute before proceeding with the next time step. (All allowable intercommutings are performed "simultaneously" before the next evolution time step.) We use a very simple but attractive intercommuting algorithm: two strings[10] which pass through the same lattice site simply reconnect as in Fig. 5 with probability p. Neither the positions of any points nor the velocities of any links are altered by this process—only the "connections" between the points and the links. Our model of intercommuting thus exactly preserves the gauge. More importantly, it exactly preserves the energy, momentum, and angular momentum of the strings. This is the principal benefit gained by imposing spatial discreteness.

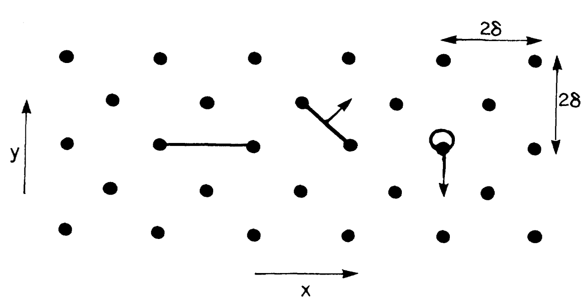

FIG. 3. The three types of links of string allowed by the discrete gauge conditions (8) in our spatially discrete model. The components of the end points of the links are all integral multiples of δ. All links have energy $2\mu\delta$.

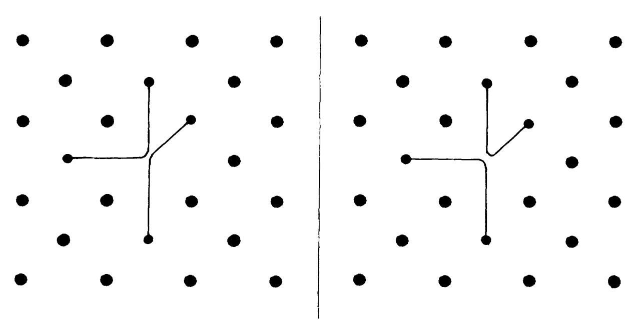

FIG. 5. An example of intercommuting strings. In our model, strings are allowed to commute only if they pass through the same lattice point. This ensures that the gauge conditions are preserved and that the energy, momentum, and angular momentum of the system stay constant.

An additional point concerning intercommuting is important. We do not allow two points to intercommute if they are on the same loop and have fewer than some critical number N_c of links between them. This prevents the formation of loops which have fewer than N_c points and therefore an energy less than $2N_c\mu\delta$. The reason for doing this is that our model, which represents strings by discrete points, artificially imposes some minimum energy for a loop. Thus it is important to study how this energy cutoff influences physical processes. When we present our results in the next section we shall discuss their dependence on N_c.

III. RESULTS OF THE SIMULATION

In order to evolve a system of strings according to the algorithm presented in the previous section, we require an initial state at $t=0$. Such a state should satisfy the discrete gauge conditions (8), and should have the spatial discreteness necessary for our intercommuting algorithm.

Cosmic strings which may have formed during a phase transition in the early Universe would have been created in a complicated tangle, thanks to the random phases that the Higgs field would have assumed in causally disconnected domains as the Universe cooled.[2] A discrete Monte Carlo simulation of this process has been developed in Ref. 11.

The cubic lattice employed in this simulation produces spatially discrete strings which are perfectly suited for our use. They are composed of segments in three orthogonal directions whose length is the correlation length ξ of the Higgs field at the time of formation. These strings are trivially discretized by dividing each segment into some fixed number $\xi/2\delta$ of the "fully stretched" links discussed in the previous section, thereby producing an initial state which meets all our requirements. (Note that all the strings are at rest initially.) We have used two links per segment (i.e., $\delta=\xi/4$) in the simulations reported here; with this choice the smallest loop at formation is the 8-link square of energy $4\mu\xi$ or $16\mu\delta$ shown in Fig. 6.

A typical initial state produced by the Monte Carlo algorithm is shown in Fig. 7. The volume of the simulation region is $(8\xi)^3$ or $(32\delta)^3$. Periodic boundary conditions have been used so that strings leaving the box reenter on the opposite side. Most of the string energy—about 80% in the infinite-volume limit[11]—condenses into a few long, contorted loops which extend across the entire simulation region. (In an infinite, expanding universe, these would be the "infinite" strings which span the horizon.) The remaining 20% of the energy condenses into numerous loops which are much smaller than the size of the simulation.

We come now to the central question: What happens to such an initial state as the strings move, intersect, and intercommute? The qualitative answer is evident in Fig. 8. The system fragments into many small loops, most of which have an energy comparable to the cutoff energy imposed by the discreteness of the model. (For this simulation, the cutoff parameter N_c was 2, so the smallest loops have two links.)

Despite the tendency toward fragmentation, there are some "infinite" strings remaining. Those that survive become shorter and straighter as loops break off from them. They cannot completely disappear, however. Our intercommuting algorithm simulates directed strings, and the net flux associated with these directed strings is a conserved quantity. The initial state in Fig. 8 has a nonzero net flux due to strings that "wind around" the 3-torus that the periodic boundary conditions create. The system cannot completely fragment without losing this net flux.

It appears that the system is trying to fragment completely but is frustrated by flux conservation. To test this hypothesis, the boundary conditions in the Monte Carlo algorithm were changed to prevent any strings from crossing the boundary in the initial state. The system then has no net flux and should be able to fragment. An example of an initial state formed with this "reflecting" boundary condition is show in Fig. 9. This particular configuration consists of 18 loops, ranging from 8 to 220 links long, in a Universe of volume $(8\xi)^3$ or $(32\delta)^3$. The total number of links is 700.

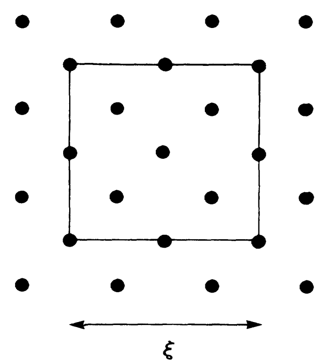

FIG. 6. The smallest loop present at formation in our simulations.

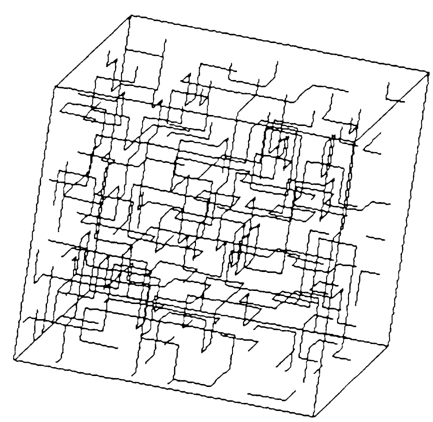

FIG. 7. An example of an initial state formed by the Monte Carlo simulation of string formation by a phase transition. The periodic boundary conditions allow the system to have a nonzero net flux associated with the strings.

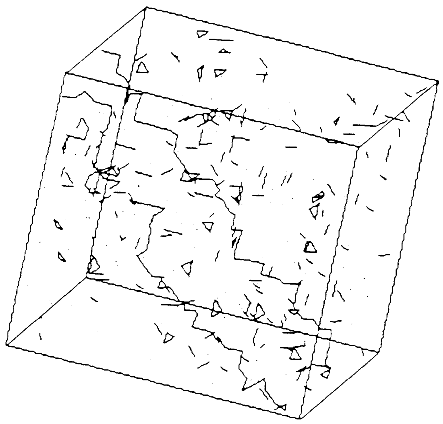

FIG. 8. The result of evolving the strings in Fig. 7 for 250 time steps. The smallest loop that was allowed to form had two links. An "infinite" string remains, due to the conservation of flux.

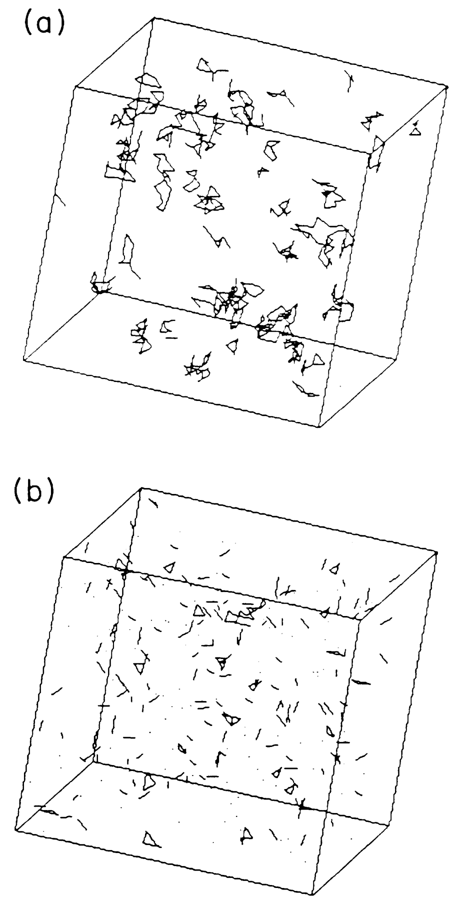

FIG. 10. The result of evolving the strings in Fig. 9 for 500 time steps. The strings fragment into loops on the smallest allowed scale, which is 8 links for (a) and 2 links for (b). No "infinite" strings remain in either case.

Figure 10(a) is the result of evolving the strings in Fig. 7 for 500 time steps with $p=1$. The cutoff parameter N_c for this simulation was 8, meaning that no loops were allowed to be produced that were smaller than the smallest loops at formation. The "infinite" strings initially present have broken up. The largest loop that exists at $t=500\delta$ has only 45 links; over half the energy is in loops with 8, 9, or 10 links.

Figure 10(b) is the result of evolving the same initial state for the same amount of time with the same intercommuting probability, but with a cutoff parameter of 2. It is obvious that the strings fragmented even further. The largest loop has only 12 links, and over half the energy is in the smallest loops with just 2 links.

These results at $t=500\delta$ are typical of the system's appearance after the first 200 time steps or so. By that time the strings have completely fragmented onto the smallest scale allowed by the cutoff, and a steady state ensues. This steady state is an artifact of discretizing strings into a finite number of links. A system of real, continuous strings would continue to fragment into smaller and smaller loops. No true, cutoff-independent equilibrium state would exist.

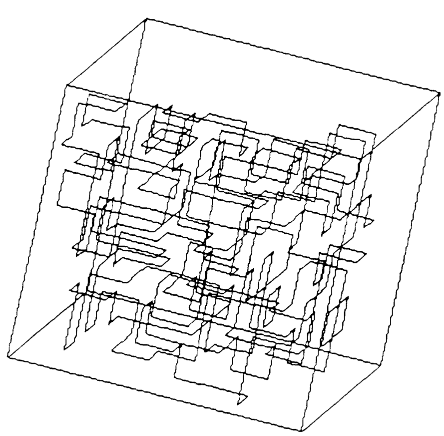

FIG. 9. An example of a 700-link initial state formed by the Monte Carlo simulation with "reflecting" boundary conditions. The strings have no net flux in any direction.

For the purpose of gathering quantitative results with better statistics, we evolved a similar but larger system of strings with no net flux. The initial state, formed by the Monte Carlo algorithm, contained 110 loops, ranging in size from 8 to 2188 links, in a volume of size $(16\xi)^3$ or $(64\delta)^3$. The total number of links was 6492. We evolved this state for 1000 time steps with $p=1$, for both cutoffs 8 and 2 as before.

The two lower curves in Fig. 11 show the number of loops in these two simulations as a function of time. By about $t=200\delta$ in both cases, the system has reached a steady state and the number of loops has become constant. (Of course, there are more loops when the cutoff is 2 than when it is 8.)

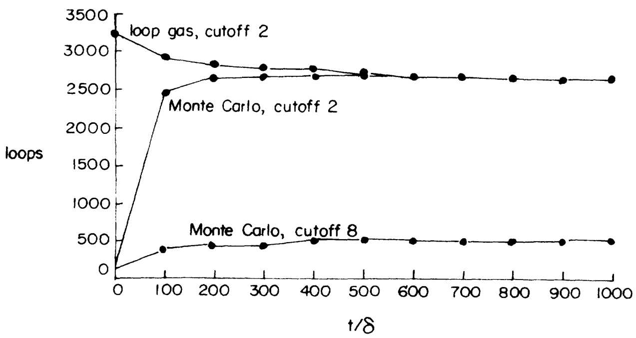

FIG. 11. The number of loops as a function of time during the evolution of two different initial states with 6492 links. The evolution of the Monte Carlo initial state is shown for two different cutoffs.

The steady-state distribution of loop energies is shown in Fig. 12. The number of loops whose energy is greater than or equal to E (averaged over time steps $t = 500$, 600, 700, 800, 900, and 1000δ) has been plotted against E. (Note that $E/2\mu\delta$ is the number of links in a loop.) For cutoff 8, over half the energy (53%) of the system is in loops with 8–12 links; for cutoff 2, over half (61%) is in loops with just 2 links. As in the smaller simulation, the strings have fragmented into loops on the smallest allowed scale. In both cases, the "infinite" strings (such as the longest one in the initial state, which had 2188 links) have disappeared; there are essentially none left that are longer than about 10 links for cutoff 2, or about 60 links for cutoff 8.

To test the hypothesis that the system reaches a cutoff-imposed steady state which is independent of the initial conditions, we simulated a system with the same amount of string arranged in a completely different initial configuration. We generated a "loop gas" containing 3246 loops, each consisting of just two "degenerate" links. They were distributed at random throughout a volume of $(64\delta)^3$. Such links must move at light speed, but the direction of motion of each one was chosen randomly (two links belonging to the same loop were allowed to have different velocities).

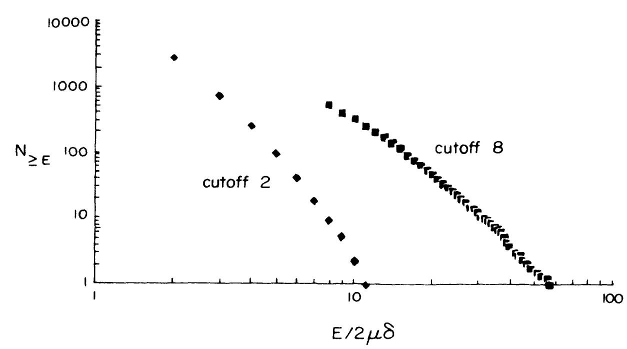

FIG. 12. The energy distribution of the loops in the steady state for the system with 6492 links.

This loop gas was evolved for 1000 time steps with $p = 1$ and cutoff 2. In this simulation, all the loops had the minimum allowed energy to begin with, so the tendency was to build a small number of larger loops in loop collisions. The resulting number of loops as a function of time is shown in the top curve in Fig. 11. The agreement between the two curves with cutoff 2 indicates that the steady state is indeed independent of the initial conditions. Furthermore, the steady-state distribution of loop energies for the loop gas is found to be the same—within statistical fluctuations—as the distribution evolved (with cutoff 2) from the Monte Carlo initial state with the same density. In view of the extreme qualitative difference between the two initial states, these distributions are striking evidence that the strings forget their initial state.

The steady-state distribution of loops depends on the density of string in the simulation, n, which can be defined as the ratio of the total number of links to the number of points in the lattice. Our discrete model can be expected to give a realistic representation of string evolution only for $n \ll 1$. However, we could not resist the temptation to study the nature of the steady state for $n \sim 1$ as well. When n approaches 1, the spatial lattice "fills up" with links, they cannot avoid each other, and the consequent intercommuting produces structure on large scales. We have found that for $n > 0.2$ the steady state has "infinite" strings which include a substantial fraction of the total number of links. It would be interesting to study the character of the phase transition at the critical density $n_c \sim 0.2$.

Finally, our model suggests that individual loops tend to fragment by self-intersection. We observe that loops on the smallest scales frequently break off from larger ones, in regions near cusplike structures. Whether this is an artifact of our model or not is an unresolved question at this time.

IV. CONCLUSIONS

Our simulations strongly indicate that fragmentation is the dominant process in a system of strings in flat spacetime. Small loops break off from large ones much more readily than they become reattached. There appears to be no equilibrium state in flat spacetime, contrary to previous assumptions. In particular, the "infinite" strings present at formation do not persist, but get broken into smaller and smaller loops. Current analytic models that have assumed or predicted the existence of an equilibrium due to reconnection need modification. The distribution of loop energies we have found in the cutoff-imposed steady state should be useful for testing future analytic models.

ACKNOWLEDGMENTS

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