Renormalization-group-improved Yennie-Frautschi-Suura theory

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We use the renormalization group to improve the ultraviolet aspect of the program of Yennie, Frautschi, and Suura for the cancellation and exponentiation of infrared divergences in Abelian gauge theories. Possible applications to high-precision Z_0 physics at the Stanford Linear Collider and CERN LEP are considered.

I. INTRODUCTION

There is currently some amount of interest in testing various aspects of the $SU(2)_L \times U(1)$ model of Glashow, Salam, and Weinberg to the level of $\sim 1\%$ at the Stanford Linear Collider (and CERN LEP) on or near the Z_0 resonance.[1] Such tests would require that the radiative corrections to $e^+e^- \to \gamma, Z_0 \to X$ be known to $\sim 0.3\%$ accuracy. In what follows, we wish to present a theoretical framework in which such an accuracy on the radiative corrections may be achieved.

Specifically, we will use the familiar method of Yennie, Frautschi, and Suura[2] (YFS) as the basis of our framework. It is well known that this theory implements the cancellation of infrared divergences to all orders in α in an arbitrary QED process; hence, an entirely analogous statement can be made for an arbitrary (unbroken) Abelian gauge theory. In what follows, we will illustrate our theoretical framework with pure QED since all of the infrared (IR) divergences in the theory of ultimate interest to us, the $SU(2)_L \times U(1)$ model, are in fact due to QED. It will be obvious how one applies the framework which we shall develop to an arbitrary process in an arbitrary theory containing an unbroken Abelian gauge symmetry.

At first sight, it may not be apparent that the YFS formalism by itself is not sufficient for our rather practical purposes. Thus, we will record, in what follows, our version of the relevant numerical estimates which make it clear that, in addition to the IR summation of the YFS theory, one also needs some kind of ultraviolet (UV) improvement of the respective perturbation series to obtain, in a practical way, the 0.3% accuracy desired at the SLC and LEP on the Z_0 resonance. This need for UV improvement has been realized by many authors.[3]

We will here find it convenient to use the renormalization-group equation,[4] as formulated by Weinberg, in arriving at the desired UV improvement of the YFS theory. In doing this, then, we shall arrive at a theoretical framework in which both the IR and the UV large logarithms are treated to all orders in perturbation theory. For practical applications, the various solutions to the respective renormalization-group equations will only be treated to the leading large logarithm behavior, since this will be sufficient for our purposes. A complete treatment of the $SU(2)_L \times U(1)$ theory with regard to $e^+e^- \to \gamma, Z_0 \to X$ at the SLC and LEP will be taken up elsewhere.[5]

We should emphasize that the framework which we shall develop is related to the works[3] of Tsai, Altarelli and Martinelli, and Kuraev and Fadin, for example. The work of Tsai is based on the Gell-Mann–Low formulation of the renormalization group and involves an approximate treatment of the mass effects in the respective cross sections, for example. The works of Altarelli and Martinelli and of Kuraev and Fadin use the partonic representation of the renormalization group provided by the Altarelli-Parisi equations[4] and do not effect the IR exponentiation of the YFS program. We emphasize that the physical idea to use the partial differential equations of quantum field theory to improve the UV aspect of the radiative corrections to e^+e^- annihilation processes is common to the works of Tsai, Altarelli and Martinelli, Kuraev and Fadin, and to the analysis presented herein.

Our work is organized as follows. In the next section we review the familiar problem of initial-state radiative corrections to $e^+e^- \to X$ in order to obtain an estimate of the order of perturbation theory which is required to obtain $\sim 0.3\%$ accurate radiative corrections to $e^+e^- \to \gamma, Z_0 \to X$ at $\sqrt{s} \sim M_{Z_0}$, where $\sqrt{s}$ is the e^+e^- center-of-momentum energy and M_{Z_0} is the Z_0 rest mass. In Sec. III we review the relevant aspects of the Yennie-Frautschi-Suura program. In Sec. IV we combine the renormalization-group program of Weinberg with the YFS theory to obtain a rigorous UV improvement of the YFS theory. We use QED to illustrate the formalism and show how one applies this formalism to the $SU(2)_L \times U(1)$ theory. Section V contains an example of such an application. Section VI contains some concluding remarks.

II. ESTIMATE OF IR AND UV EFFECTS IN RADIATIVE CORRECTIONS AT THE SLC AND LEP

In this section we wish to determine the generic size of the various radiative corrections to the Born processes in $e^+e^- \to \gamma, Z_0 \to X$ with an eye toward developing a 0.3% accurate radiative correction theoretical framework. Since the dominant IR effects and the dominant UV effects are already characterized by the pure QED part of the $SU(2)_L \times U(1)$ theory, we may consider the case of pure QED at $\sqrt{s} = M_{Z_0}$ for our purposes in this section.

Focusing then on the initial-state QED radiative corrections to $e^+e^- \to X$, we have the well-known diagrams in Fig. 1. Textbook formulas[6] allow us to write the cross sections associated with Fig. 1 as

$$d\sigma \simeq d\sigma_0(1+\delta) + \int_{k \geq k_0} \frac{dk}{k}\frac{2\alpha}{\pi}(1 - k/\sqrt{s} + 2k^2/s)[-1 + \ln(s/m_e^2)]d\sigma_0[2\sqrt{s}(\sqrt{s}/2 - k)] , \quad (1)$$

$$\delta = \frac{2\alpha}{\pi}\{[-1 + \ln(s/m_e^2)][\ln(2k_0/\sqrt{s}) + \tfrac{13}{12}] - \tfrac{17}{36} + \pi^2/6\} , \quad (2)$$

where we imagine that k_0 is the separation between soft and hard real photons and $d\sigma_0$ is the respective Born approximation differential cross section.

What we see in (1) and (2) are the following.

(a) The infrared effect involves radiative corrections of size

$$(2\alpha/\pi)[\ln(s/m_e^2) - 1]\ln(\sqrt{s}/2\bar{k}_0) ,$$

where $\bar{k}_0$ is some typical energy resolution-related photon energy. Thus, in a Monte Carlo simulation of the differential cross section, it may be desirable to take $\bar{k}_0 << \sqrt{s}/2$ so that, for $\sqrt{s} = M_{Z_0}$, such IR effects are $\gtrsim [\ln(\sqrt{s}/2\bar{k}_0)] \times 0.108$. Hence, we would need to sum up all such effects since they may be $\sim 100\%$ of the Born process in each order of the loop expansion in QED or the $SU(2)_L \times U(1)$ theory.

(b) The UV effects, which contribute to the factor of $\frac{13}{12}$ in (2) and to the hard bremsstrahlung, are also large, i.e., they are ~ 0.11 of the Born process; 0.3% accuracy would require computation of these effects to order $\{(2\alpha/\pi)[\ln(s/m_e^2) - 1]\}^n \lesssim 0.1\%$. This implies that $n \gtrsim 3$. The cross section would therefore, at the least, involve as many as 3 loops of perturbation theory. Point (a) clearly indicates that we need to use the YFS program, which rigorously sums all large IR effects to all orders of perturbation theory. Point (b), from a practical standpoint, requires a UV summation of large logarithms to three or more loops in perturbation theory and, hence, is most conveniently achieved by using the renormalization-group equation. We are thus led to consider the renormalization-group improvement of the YFS theory.

We shall begin our discussion of this renormalization-group-improved YFS theory by reviewing the relevant elements of the YFS theory itself. This we do in the next section.

III. YENNIE-FRAUTSCHI-SUURA THEORY

In this section, we wish to review the relevant aspects of the program of Yennie, Frautschi, and Suura as it relates to $e^+e^- \to Z_0 \to X$ at the SLC and LEP. Here, we shall have the full $SU(2)_L \times U(1)$ theory in mind.

Consider, then, the expansion of the full connected amplitude for $e^+e^- \to X$ at $\sqrt{s} = M_{Z_0}$ in terms of the number of virtual-photon loops. We illustrate a typical contribution to this amplitude $\mathcal{M}$ in Fig. 2. We may write

$$\mathcal{M}(p_e, p_{\bar{e}}) = \sum_{n=0}^{\infty} \mathcal{M}_n(p_e, p_{\bar{e}}) , \quad (3)$$

where $\mathcal{M}_n(p_e, p_{\bar{e}})$ is the contribution of all n virtual γ loop graphs to $\mathcal{M}$. The result of Yennie, Frautschi, and Suura is that

$$\mathcal{M}_n = \sum_{r=0}^{n} m_{n-r}(\alpha B)^r / r! , \quad (4)$$

where m_j do not have virtual infrared divergences and are of order α^j relative to $\mathcal{M}_0 \equiv m_0$. The famous virtual infrared function B is such that (here, we use the photon mass m_γ to cutoff the infrared divergence in B)

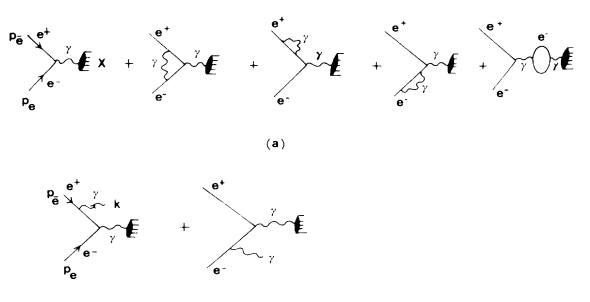

FIG. 1. Order-α radiative corrections to the initial state in $e^+e^- \to X$ in QED: (a) virtual effects; (b) bremsstrahlung.

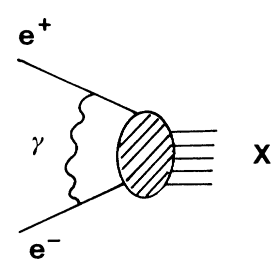

FIG. 2. Virtual-photon correction to $e^+e^- \to X$. This is a typical graph.

$$\alpha B(p_e,p_{\bar{e}})=\frac{i\alpha}{(2\pi)^3}\int\frac{d^4k}{k^2-m_\gamma^2+i\epsilon}\times\left[\frac{-2p_{\bar{e}\mu}-k_\mu}{-2p_{\bar{e}}\cdot k-k^2-i\epsilon}-\frac{2p_{e\mu}-k_\mu}{2p_e\cdot k-k^2-i\epsilon}\right]^2 . \quad (5)$$

Hence

$$\mathcal{M}(p_e,p_{\bar{e}})=\exp(\alpha B)\sum_{n=0}^{\infty} m_n . \quad (6)$$

This is the famous exponentiation of virtual infrared divergences of the YFS program.

To complete our review of the YFS theory, we consider next the differential cross sections for the processes $e^+e^-\to n(\gamma)+X'$, where $n(\gamma)$ represents the emission of n real photons of four-moment $k_1,\ldots,k_n$. For a given value of n, this differential cross section is (here, $E_{X'}=P^0_{X'}$, where $P_{X'}$ is the four-momentum of X')

$$d\sigma=\exp[2\,\mathrm{Re}(\alpha B)]\frac{1}{n!}\times\int\prod_{j=1}^{n}\frac{d^3k_j}{(k_j^2+m_\gamma^2)^{1/2}}\delta\left[\sqrt{s}-E_{X'}-\sum_{i=1}^{n}k_i\right]dE_{X'}\times\left|\sum_{n'=0}^{\infty} m_{n'}^{(n)}\right|^2 , \quad (7)$$

where $m_{n'}^{(n)}$ is now the special case of $m_{n'}$ in (6) in which X in Fig. 2 involves n real photons. See Fig. 3. The second theorem of the YFS program is that

$$\left|\sum_{n'=0}^{\infty} m_{n'}^{(n)}\right|^2=\tilde{S}(\mathbf{k}_1)\cdots\tilde{S}(\mathbf{k}_n)\bar{\beta}_0+\sum_{i=1}^{n}\tilde{S}(\mathbf{k}_1)\cdots\tilde{S}(\mathbf{k}_{i-1})\tilde{S}(\mathbf{k}_{i+1})\cdots\tilde{S}(\mathbf{k}_n)\bar{\beta}_1(\mathbf{k}_i)+\cdots+\sum_{i=1}^{n}\tilde{S}(\mathbf{k}_i)\bar{\beta}_{n-1}(\mathbf{k}_1,\ldots,\mathbf{k}_{i-1},\mathbf{k}_{i+1},\ldots,\mathbf{k}_n)+\bar{\beta}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n) , \quad (8)$$

where $\bar{\beta}_j$ is infrared divergence free and is of order α^j relative to $\bar{\beta}_0$. The real infrared divergence function $\tilde{S}$ is given by

$$\tilde{S}(\mathbf{k})=-\frac{\alpha}{4\pi^2}\left[\frac{p_{\bar{e}\mu}}{k\cdot p_{\bar{e}}}-\frac{p_{e\mu}}{k\cdot p_e}\right]^2 . \quad (9)$$

It follows that our cross section for the emission of an arbitrary number of real photons can be represented as

$$d\sigma=\exp[2\alpha(\mathrm{Re}B+\tilde{B})]\frac{1}{2\pi}\int_{-\infty}^{\infty}dy\,e^{iy(\sqrt{s}-E_{X'})+D}\times\left[\bar{\beta}_0+\sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{j=1}^{n}\frac{d^3k_j}{k_j}e^{-iyk_j}\bar{\beta}_n\right]dE_{X'} , \quad (10)$$

where we have defined

$$2\alpha\tilde{B}=\int^{k\le(\sqrt{s}-E_{X'})}\frac{d^3k}{(k^2+m_\gamma^2)^{1/2}}\tilde{S}(\mathbf{k}) , \quad (11)$$

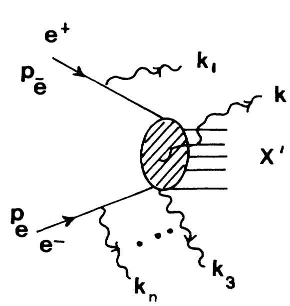

FIG. 3. Real-photon emission in $e^+e^-\to n(\gamma)+X'$. This is a typical graph.

$$D=\int^{k\le(\sqrt{s}-E_{X'})}\frac{d^3k}{k}\tilde{S}(e^{-iyk}-1) . \quad (12)$$

It may be verified that $\mathrm{Re}B+\tilde{B}$ is free of infrared divergences so that $d\sigma$ is indeed a physically meaningful quantity. The result (10), then, exhibits the cancellation of IR divergences to all orders in α.

As it stands, (10) is still not quite general enough for our purposes, for we generally wish to consider final states X' which involve charged particles: e^+e^-, $\mu^+\mu^-$, $q\bar{q}$, $\bar{q}'\bar{q}'^*$, etc. Thus, we wish to generalize (10) to the circumstance in which X' contains the charged-particle pair $f\bar{f}$ for charge e_f in units of the positron charge e. This case has also been discussed by Yennie, Frautschi, and Suura. In fact, Yennie, Frautschi, and Suura have also considered the scenario in which the volume in momentum space in which the detection of soft photons is not possible is bounded by a surface which depends on the spherical angles of the respective photon momenta. Hence, we would also like to record the generalization of (10):

$$d\sigma=\exp[2\alpha(\mathrm{Re}B+\tilde{B})]\frac{1}{(2\pi)^4}\int d^4y\,e^{iy\cdot(p_e+p_{\bar{e}}-P_{X'})+D}\times\left[\bar{\beta}_0+\sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{j=1}^{n}\frac{d^3k_j}{k_j}e^{-iy\cdot k_j}\bar{\beta}_n\right]\times dE_{X'}d^3P_{X'} , \quad (13)$$

where, now,

$$2\alpha\tilde{B}=\frac{\alpha}{4\pi^2}\int^{k\le K_{\max}}\frac{d^3k}{(k^2+m_\gamma{}^2)^{1/2}}\left[-\left[\frac{p_{\bar{e}\mu}}{p_{\bar{e}}\cdot k}-\frac{p_{e\mu}}{p_e\cdot k}\right]^2+e_f\left[\frac{p_{f\mu}}{p_f\cdot k}-\frac{p_{e\mu}}{p_e\cdot k}\right]^2-e_f\left[\frac{p_{\bar{f}\mu}}{p_{\bar{f}}\cdot k}-\frac{p_{e\mu}}{p_e\cdot k}\right]^2 -e_f\left[\frac{p_{f\mu}}{p_f\cdot k}-\frac{p_{\bar{e}\mu}}{p_{\bar{e}}\cdot k}\right]^2+e_f\left[\frac{p_{\bar{f}\mu}}{p_{\bar{f}}\cdot k}-\frac{p_{\bar{e}\mu}}{p_{\bar{e}}\cdot k}\right]^2-e_f{}^2\left[\frac{p_{\bar{f}\mu}}{p_{\bar{f}}\cdot k}-\frac{p_{f\mu}}{p_f\cdot k}\right]^2\right], \tag{14}$$

$$B=\frac{-i}{8\pi^3}\int\frac{d^4k}{k^2-m_\gamma{}^2+i\epsilon}\left[-\left[\frac{-2p_{e\mu}-k_\mu}{k^2+2k\cdot p_e+i\epsilon}+\frac{-2p_{\bar{e}\mu}+k_\mu}{k^2-2k\cdot p_{\bar{e}}+i\epsilon}\right]^2+e_f\left[\frac{-2p_{e\mu}-k_\mu}{k^2+2k\cdot p_e+i\epsilon}+\frac{2p_{f\mu}+k_\mu}{k^2+2k\cdot p_f+i\epsilon}\right]^2 -e_f\left[\frac{-2p_{e\mu}-k_\mu}{k^2+2k\cdot p_e+i\epsilon}+\frac{2p_{\bar{f}\mu}+k_\mu}{k^2+2k\cdot p_{\bar{f}}+i\epsilon}\right]^2-e_f\left[\frac{-2p_{\bar{e}\mu}-k_\mu}{k^2+2k\cdot p_{\bar{e}}+i\epsilon}+\frac{2p_{f\mu}+k_\mu}{k^2+2k\cdot p_f+i\epsilon}\right]^2 +e_f\left[\frac{-2p_{\bar{e}\mu}-k_\mu}{k^2+2k\cdot p_{\bar{e}}+i\epsilon}+\frac{2p_{\bar{f}\mu}+k_\mu}{k^2+2k\cdot p_{\bar{f}}+i\epsilon}\right]^2-e_f{}^2\left[\frac{2p_{f\mu}-k_\mu}{k^2-2k\cdot p_f+i\epsilon}+\frac{2p_{\bar{f}\mu}+k_\mu}{k^2+2k\cdot p_{\bar{f}}+i\epsilon}\right]^2\right], \tag{15}$$

and

$$D=\int\frac{d^3k}{k}\tilde{S}[e^{-iy\cdot k}-\theta(K_{\max}-k)] \tag{16}$$

with

$$2\alpha\tilde{B}\equiv\int^{k\le K_{\max}}\frac{d^3k}{(k^2+m_\gamma{}^2)^{1/2}}\tilde{S}\ . \tag{17}$$

Here, $K_{\max}$ may depend on the direction of **k**.

It is the result (13) that we will use in our study of $e^+e^-\to Z_0\to X$, where we assume that X contains the $f\bar{f}$ pair for charge e_f. In (13), all infrared divergences are canceled in the sum $\mathrm{Re}B+\tilde{B}$ to all orders in α.

As we illustrated with the QED part of the $SU(2)_L\times U(1)$ theory, there remain large UV effects in the $\bar{\beta}_n$ in (13). Thus, in the next section, we wish to use the partial differential equation of Weinberg to sum up such effects.

IV. RENORMALIZATION-GROUP-IMPROVED YFS THEORY

In order to address the physical consequences of the large UV effects in (13), we will appeal to the renormalization-group equation of Weinberg. We begin this section, then, by reviewing, briefly, the origin of this equation.

The basic idea of Weinberg is that multiplicatively renormalized Green's functions $\{\Gamma\}$ of a theory may be subtracted with the massless limits of the subtraction constants for the theory at a Euclidean scale μ. The fact that the unrenormalized theory is independent of μ then implies the equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\gamma_\Theta(g_R)m_R\frac{\partial}{\partial m_R}-\gamma_\Gamma(g_R)\right]\Gamma=0\ , \tag{18}$$

where, for simplicity, we imagine we have one renormalized coupling g_R and one renormalized mass m_R. In the $SU(2)_L\times U(1)$ theory, we would have two couplings, e_R and g_{WR}, where g_{WR} is the $SU(2)_L$ coupling and e_R is the electric charge of the positron, we would have renormalized mass parameters for the fermions in the presumed three families of quarks and leptons, we would have the mass parameter of the $W^\pm$ and Z_0 bosons, and the mass parameter of the physical Higgs particle (or the quartic coupling of the physical Higgs particle), as a minimal set of masses and couplings. The physics beyond the standard model would enlarge this set. The coefficient functions β, γ_Θ, and γ_Γ are computable in renormalized perturbation theory. The detailed application of (18) to (13) for the full $SU(2)_L\times U(1)$ theory will be presented elsewhere. Here, to illustrate how this application is effected, we will restrict ourselves to the QED part of this latter theory.

Specializing (18) to QED [note that (18) tacitly presumes the gauge of Landau], we can write (Q_ie_R is the electric charge of fermion i)

$$\beta(e_R)=\frac{1}{12\pi^2}\sum_i Q_i{}^2e_R{}^3+\cdots\ ,$$
$$\gamma_{\Theta_i}=\frac{3}{8\pi^2}Q_i{}^2e_R{}^2+\cdots\ , \tag{19}$$
$$\gamma_\Gamma=n_\gamma\beta(e_R)/e_R\ ,$$

where n_γ is the number of external photon lines in Γ. With regard to (13), we note that, if we write the respective amplitudes $\mathcal{M}^{(n)}$ as (we suppress the amputated photon labels)

$$\mathcal{M}^{(n)}=(\bar{u}_f)_{\alpha_1}(v_{\bar{f}})_{\alpha_2}(u_e)_{\nu_1}(\bar{v}_{\bar{e}})_{\nu_2}\mathcal{M}^{(n)\nu_2\nu_1}_{\alpha_1\alpha_2}\ , \tag{20}$$

then $\mathcal{M}^{(n)\nu_2\nu_1}_{\alpha_1\alpha_2}$ is an amputated connected *multiplicatively renormalized* (on-shell) Green's function. Thus, it satisfies the appropriate version of (18):

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(e_R)\frac{\partial}{\partial e_R}-\sum_i\gamma_{\Theta_i}(e_R)m_{iR}\frac{\partial}{\partial m_{iR}}-\gamma_{\mathcal{M}^{(n)}}(e_R)\right]\mathcal{M}^{(n)\nu_2\nu_1}_{\alpha_1\alpha_2}=0\ . \tag{21}$$

Further, it is convenient to write

$$p_e \equiv (\lambda\sqrt{s_0}/2, (\lambda^2 s_0/4 - m_e{}^2)^{1/2}\hat{\mathbf{z}}) ,$$
$$p_{\bar{e}} \equiv (\lambda\sqrt{s_0}/2, -(\lambda^2 s_0/4 - m_e{}^2)^{1/2}\hat{\mathbf{z}}) , \tag{22}$$

and, in $\mathcal{M}^{(n)}$,

$$p_f^0 + p_{\bar{f}}^0 = \lambda \left[\sqrt{s_0} - \sum_{i=1}^{n} k_{0i}^0\right], \quad k_i \equiv \lambda k_{0i} ,$$
$$\mathbf{p}_f + \mathbf{p}_{\bar{f}} = -\lambda \sum_{i=1}^{n} \mathbf{k}_{0i} . \tag{23}$$

We can always do this in the physical region provided that $\lambda\sqrt{s_0} > 2m_f$ and $\lambda\sqrt{s_0} > 2m_e$. We will always imagine, here, that $\sqrt{s_0} > 2m_e$ and that $\lambda > 1$. Then, since $\gamma_{\mathcal{M}^{(n)}} = n\beta(e_R)/e_R$,

$$\left[-\lambda\frac{\partial}{\partial\lambda} + \beta(e_R)\frac{\partial}{\partial e_R} - \sum_i [1+\gamma_{\Theta_i}(e_R)]m_{iR}\frac{\partial}{\partial m_{iR}} - n\beta(e_R)/e_R + \bar{D}_{\mathcal{M}^{(n)}}\right]\mathcal{M}^{(n)\nu_2\nu_1}_{\alpha_1\alpha_2} = 0 \tag{24}$$

so that, using

$$\mathcal{M}^{(n)\nu_2\nu_1}_{\alpha_1\alpha_2} = \exp(\alpha B) \sum_{n'=0}^{\infty} m^{(n)\nu_2\nu_1}_{n'\alpha_1\alpha_2} , \tag{25}$$

we find the solution

$$\mathcal{M}^{(n)\nu_2\nu_1}_{\alpha_1\alpha_2} = \exp[\alpha(1)B(s_0, m_{iR}(\lambda))] \sum_{n'=0}^{\infty} m^{(n)\nu_2\nu_1}_{n'\alpha_1\alpha_2}(p_i(1), m_{iR}(\lambda), \alpha(\lambda), \mu)\lambda^{\bar{D}_{\mathcal{M}^{(n)}}} \exp\left[-\int_1^\lambda \gamma_{\mathcal{M}^{(n)}}(e_R(\lambda'))d\lambda'/\lambda'\right]$$
$$= \exp[\alpha(1)B(s_0, m_{iR}(\lambda))] \sum_{n'=0}^{\infty} m^{(n)\nu_2\nu_1}_{n'\alpha_1\alpha_2}(p_i(1), m_{iR}(\lambda), \alpha(\lambda), \mu)\lambda^{\bar{D}_{\mathcal{M}^{(n)}}}[e_R(\lambda)/e_R(1)]^{-n} , \tag{26}$$

where $\bar{D}_{\mathcal{M}^{(n)}}$ is the engineering dimension of $\mathcal{M}^{(n)}$ and is given by

$$\bar{D}_{\mathcal{M}^{(n)}} = -2 - n , \tag{27}$$

$$\lambda\frac{d}{d\lambda}e_R(\lambda) = \beta(e_R(\lambda)) ,$$
$$\lambda\frac{d}{d\lambda}m_{iR}(\lambda) = -[1+\gamma_{\Theta_i}(e_R(\lambda))]m_{iR}(\lambda) , \tag{28}$$

and

$$\alpha(\lambda) \equiv e_R{}^2(\lambda)/4\pi . \tag{29}$$

We emphasize that (26) is a rigorous consequence of the renormalization-group equation.

The running masses $m_{iR}(\lambda)$ are

$$m_{iR}(\lambda) = [m_{iR}(1)/\lambda]\exp\left[-\int_1^\lambda \gamma_{\Theta_i}(e_R(\lambda'))d\lambda'/\lambda'\right]$$
$$\simeq [m_{iR}(1)/\lambda][e_R(\lambda)/e_R(1)]^{-c^0_{\Theta_i}/b_0} , \tag{30}$$

where

$$\beta(e_R) \equiv b_0 e_R{}^3 + \cdots , \quad \gamma_{\Theta_i}(e_R) \equiv c^0_{\Theta_i} e_R{}^2 + \cdots . \tag{31}$$

Note that, even though $f\bar{f}$ may be $e\bar{e}$, the error on the approximation in (30) is a factor of order

$$\exp[-\alpha(\lambda) + \alpha(1)] \simeq 1 - 6.4\times 10^{-4} . \tag{32}$$

Thus, one-loop coefficient functions should be adequate for our purposes, provided that n is not too big.

The important point to note about (26) is that it has the same algebraic structure as the original amplitude $\mathcal{M}$ in (6). Thus, the YFS program can be applied to (26). In this way we arrive at

$$d\sigma = \exp\{2\alpha(1)[\mathrm{Re}B(p_i(1), m_{iR}(\lambda)) + \tilde{B}(p_i(1), m_{iR}(\lambda), K_{\max}/\lambda)]\}$$
$$\times \frac{1}{(2\pi)^4}\int d^4y\, e^{iy\cdot(p_e + p_{\bar{e}} - P_{X'}) + D}\left[\bar{\bar{\beta}}_0(q_0) + \sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{l'=1}^{n}\frac{d^3k_{l'}}{k_{l'}}e^{-iy\cdot k_{l'}}\bar{\bar{\beta}}_n(q_n)\right]dE_{X'}d^3P_{X'} , \tag{33}$$

where

$$\bar{\bar{\beta}}_n(q_n) \equiv \lambda^{2\bar{D}_{\mathcal{M}^{(n)}}}\bar{\beta}_n(p_i(1), k_{0j}, m_{iR}(\lambda), \alpha(\lambda), \mu)$$
$$\times [e_R(\lambda)/e_R(1)]^{-2n} \tag{34}$$

and

$$D \equiv D(p_i(1), m_{iR}(\lambda), \alpha(\lambda), K_{\max}/\lambda) . \tag{35}$$

The result (33) is the basic result of this paper.

The physical interpretation of (33) in comparison with (13) is the following. The exponentiation of the infrared effects occurs now with the running coupling $\alpha(1)$. The $\bar{\beta}_n$ are scaled by $[e_R(\lambda)/e_R(1)]^{-2n}\lambda^{2\bar{D}_{\mathcal{M}^{(n)}}}$. It is interesting that the invariance of the physical masses $m_{i,\mathrm{phys}}(m_{iR}, e_R, \mu)$ under the renormalization-group operator means that, if the amplitudes $\mathcal{M}^{(n)}$ are on shell, the net effect of the running masses $m_{iR}(\lambda)$ is to keep the renormalization-group-improved amplitudes on shell. We emphasize that (33) is a rigorous consequence of the renormalization-group equation.

In the practical applications, we have in mind to work to the value $n=2$ in (33). The error on this real hard two-photon emission term due to the factor of $[e_R(\lambda)/e_R(1)]^{-4}$ in (33), which we approximate to the leading order in $\beta(e_R(\lambda))$ in (28) and (31) is $\sim 0.02\%$ and this hard term is itself $\sim 0.85\times 10^{-2}$ relative to the Born cross section in $\bar{\bar{\beta}}_0$. Thus, we see that the one-loop values of the coefficient functions are indeed adequate. Thus, the result (33), taken together with the one-loop expressions (30) and

$$e_R{}^2(\lambda)\simeq e_R{}^2(1)/[1-2b_0e_R{}^2(1)\ln\lambda]\ , \tag{36}$$

provides a basis for the Monte Carlo simulation of $e^+e^-\to f\bar{f}, f\bar{f}+\gamma, f\bar{f}+\gamma_1\gamma_2$ where γ, γ_1, and γ_2 are hard photons and arbitrary numbers of soft phonons are understood, and where $\sqrt{s}\simeq M_{Z_0}$, to the level of $\sim 0.3\%$ of the basic Born cross section in $\bar{\bar{\beta}}_0$.

V. EXPONENTIATION OF MONTE CARLO EVENT GENERATORS

Thus, a primary use of a formula such as (33) would be in exponentiating large infrared effects and summing large UV effects in a way which allows an event generator, such as MMG1 in Ref. 7, to reflect the respective net effects in $e^+e^-\to X$ near the Z_0 resonance, for example. Accordingly, in this section, we wish to show how (33) would be applied to the results in Ref. 8 for $e^+e^-\to\mu^+\mu^-(\gamma)$, which are the basis of the event generator MMG1. [The application of (33) to the general one-loop calculation of $e^+e^-\to X$ from the standpoint of event generators will be taken up elsewhere by Jadach and the author[5] and by the Mark II SLC Z_0 Mass and Width Physics Working Group.[9]] In this way, we hope to clarify the relationship between (33) and the results in Ref. 3, for example, and to illustrate the type of applications we have in mind for (33).

More precisely, in specializing (33) to the results in Ref. 8, we may identify $\bar{\beta}_0(q_0)$ as

$$\bar{\beta}_0(q_0)=\frac{d\sigma}{d\Omega_\mu}(1\text{ loop})-2\,\mathrm{Re}[\alpha(1)B]\frac{d\sigma_0}{d\Omega_\mu}\ , \tag{37}$$

where $d\sigma(1\text{ loop})/d\Omega_\mu$ is the one-loop cross section in Eq. (2.27) of Ref. 8 and $d\sigma_0/d\Omega_\mu$ is the lowest-order cross section in Eq. (2.2) in Ref. 8. $\alpha(1)$ is the fine-structure constant at $\sqrt{s}=2m_{\mu,\text{phys}}$, for example.

Similarly, the cross section $\bar{\beta}_1$ is identified as $(k=k_1)$

$$\bar{\beta}_1=\frac{d\sigma^{B1}}{d\Omega_\mu d\Omega_\gamma k\,dk}-\tilde{S}(k)\frac{d\sigma_0}{d\Omega_\mu}\ , \tag{38}$$

where $d\sigma^{B1}$ is given by Eq. (3.13) of Ref. 8 and $\tilde{S}(k)$ is given by (17).

Clearly, the virtual infrared function B should be computed in a complete way in order to make (37) as precise as it is desired. We find

$$\begin{aligned}B&=B_1(p_e,p_{\bar e})+B_2(p_e,p_f)-B_2(p_f\to p_{\bar f})\\&\quad -B_2(p_e\to p_{\bar e})+B_2(p_e\to p_{\bar e},p_f\to p_{\bar f})\\&\quad +e_f{}^2B_1(p_e\to p_f,p_{\bar e}\to p_{\bar f},m_e\to m_f)\ ,\end{aligned} \tag{39}$$

where

$$\begin{aligned}B_1=&-\frac{1}{2\pi}+\frac{1}{4\pi}(1-4m_e{}^2/s)^{1/2}\ln\left[\frac{1+\beta_e}{1-\beta_e}\right]-\tfrac{1}{4}i\beta_e\theta(s-4m_e{}^2)-\frac{1}{2\pi}\ln(m_\gamma{}^2/m_e{}^2)\\&+\frac{s-2m_e{}^2}{\pi s\beta_e}\left\{\tfrac{1}{2}\ln\left[\frac{\beta_e{}^2}{m_\gamma{}^2/s}\right]\ln\left[\frac{1+\beta_e}{1-\beta_e}\right]+\tfrac{1}{2}Li_2[-(1-\beta_e)/(1+\beta_e)]\right.\\&\qquad+\frac{1}{4}\left[\ln^2[(1-\beta_e)/(1+\beta_e)]-\ln^2\left[\frac{(m_\gamma{}^2/s)(1-\beta_e)}{\beta_e{}^2(1+\beta_e)}\right]\right]\\&\qquad+\tfrac{1}{2}Li_2[-(1-\beta_e)/(1+\beta_e)]+\tfrac{1}{4}\ln^2\left[\frac{(1+\beta_e)(m_\gamma{}^2/s)}{(1-\beta_e)\beta_e{}^2}\right]+\tfrac{1}{2}\ln^2(1+\beta_e)+\frac{\pi^2}{4}\\&\qquad\left.+Li_2[1/(1+\beta_e)]+Li_2(1-\beta_e)\right\}+i\frac{s-2m_e{}^2}{s\beta_e}\theta(s-4m_e{}^2)[\tfrac{1}{2}\ln\beta_e{}^2-\tfrac{1}{2}\ln(m_\gamma{}^2/s)]\end{aligned} \tag{40}$$

and

$$\begin{aligned}B_2=&\frac{3e_f}{4\pi}+\frac{e_f}{4\pi}[\ln(m_\gamma{}^2/m_e{}^2)+\ln(m_\gamma{}^2/m_f{}^2)]+\frac{e_f}{8\pi}[\ln(s/m_e{}^2)+\ln(s/m_f{}^2)]\\&+\frac{e_f(s_{ef}-m_e{}^2-m_f{}^2)}{2\pi[-t_{ef}s_{ef}+(m_e{}^2-m_f{}^2)^2]^{1/2}}\left\{-\tfrac{1}{2}\ln\left[\frac{(1+\Sigma_+)(1+\Sigma_-)}{(1-\Sigma_-)(1-\Sigma_+)}\right]\ln\left[\frac{m_\gamma{}^2t_{ef}}{s_{ef}t_{ef}-(m_e{}^2-m_f{}^2)^2}\right]\right.\\&\qquad-\tfrac{1}{2}Li_2[-2\Sigma_-/(1-\Sigma_-)]+\tfrac{1}{2}Li_2[2\Sigma_+/(1+\Sigma_+)]\\&\qquad+Li_2[-\Sigma_-/(1-\Sigma_-)]+Li_2[-\Sigma_+/(1-\Sigma_+)]\end{aligned}$$

$$-Li_2[\Sigma_-/(1+\Sigma_-)]-Li_2[\Sigma_+/(1+\Sigma_+)]$$

$$+\ln 2\ln\left[\frac{(1-\Sigma_-)(1-\Sigma_+)}{(1+\Sigma_-)(1+\Sigma_+)}\right]-\tfrac{1}{2}Li_2[-2\Sigma_+/(1-\Sigma_+)]$$

$$\left.+\tfrac{1}{2}Li_2[2\Sigma_-/(1+\Sigma_-)]\right\}$$

$$-\frac{e_f}{4\pi}[\ln(-t_{ef}/m_f{}^2)+\ln(-t_{ef}/m_e{}^2)+b_{21}(p_e,p_f,m_e,m_f)+b_{21}(p_f,p_e,m_f,m_e)]-\frac{e_f}{2\pi}b_{22}(p_e,p_f,m_e,m_f)\ , \qquad (41)$$

where

$$\Sigma_{\mp}=\sqrt{-t_{ef}}[1\mp(m_e{}^2-m_f{}^2)/(-t_{ef})]/[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}\ , \qquad (42)$$

$$\begin{aligned}b_{21}(p_e,p_f,m_e,m_f)=&\{1-(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})-[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}\}\\&\times\ln\{(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})+[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}-1\}\\&+\{(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})+[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}\}\\&\times\ln\{(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})+[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}\}\\&+\{1-(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})+[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}\}\\&\times\ln\{1-(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})+[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}\}\\&+\{(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})-[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}\}\\&\times\ln\{-(m_e{}^2-m_f{}^2-t_{ef})/(-2t_{ef})+[(m_e{}^2-m_f{}^2-t_{ef})^2/4t_{ef}{}^2-m_f{}^2/t_{ef}]^{1/2}\}\ ,\end{aligned} \qquad (43)$$

and

$$\begin{aligned}b_{22}(p_e,p_f,m_e,m_f)=&1+\tfrac{1}{2}\ln 2+\frac{\sqrt{s}}{4\sqrt{-t_{ef}}}\\&\times\left\{\frac{1}{\sqrt{s}}\{[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}-\sqrt{-t_{ef}}[1+(m_e{}^2-m_f{}^2)/t_{ef}]\}\right.\\&\times\left[-\tfrac{1}{2}(1+\ln 2)+\ln\left[\frac{1}{\sqrt{s}}[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}-\frac{\sqrt{-t_{ef}}}{\sqrt{s}}[1+(m_e{}^2-m_f{}^2)/t_{ef}]\right]\right]\\&-\frac{1}{\sqrt{s}}\{[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}+\sqrt{-t_{ef}}[1-(m_e{}^2-m_f{}^2)/t_{ef}]\}\\&\times\left[-\tfrac{1}{2}(1+\ln 2)+\ln\left[\frac{1}{\sqrt{s}}[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}+\frac{\sqrt{-t_{ef}}}{\sqrt{s}}[1-(m_e{}^2-m_f{}^2)/t_{ef}]\right]\right]\\&-\frac{1}{\sqrt{s}}\{[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}+\sqrt{-t_{ef}}[1+(m_e{}^2-m_f{}^2)/t_{ef}]\}\\&\times\left[-\tfrac{1}{2}(1+\ln 2)+\ln\left[\frac{1}{\sqrt{s}}[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}+\frac{\sqrt{-t_{ef}}}{\sqrt{s}}[1+(m_e{}^2-m_f{}^2)/t_{ef}]\right]\right]\\&+\frac{1}{\sqrt{s}}\{[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}-\sqrt{-t_{ef}}[1-(m_e{}^2-m_f{}^2)/t_{ef}]\}\\&\left.\times\left[-\tfrac{1}{2}(1+\ln 2)+\ln\left[\frac{1}{\sqrt{s}}[s_{ef}-(m_e{}^2-m_f{}^2)^2/t_{ef}]^{1/2}-\frac{\sqrt{-t_{ef}}}{\sqrt{s}}[1-(m_e{}^2-m_f{}^2)/t_{ef}]\right]\right]\right\}\end{aligned} \qquad (44)$$

with

$$s=(p_e+p_{\bar{e}})^2,\quad s_{ef}\equiv(p_e+p_f)^2,\quad t_{ef}\equiv(p_e-p_f)^2,\quad \beta_e=(1-4m_e{}^2/s)^{1/2}\ . \qquad (45)$$

e_f is the electric charge of fermion f.

Similarly, we note that the real infrared function $\tilde{B}$ which cancels the infrared singularities in ReB may be represented as

$$\tilde{B}(p_e,p_{\bar{e}},p_f,p_{\bar{f}})=\tilde{B}_1(p_e,p_{\bar{e}},m_e)+\tilde{B}_2(p_e,p_f,m_e,m_f)-\tilde{B}_2(p_f\to p_{\bar{f}})-\tilde{B}_2(p_e\to p_{\bar{e}})$$
$$+\tilde{B}_2(p_e\to p_{\bar{e}},p_f\to p_{\bar{f}})+e_f{}^2\tilde{B}_1(p_e\to p_f,p_{\bar{e}}\to p_{\bar{f}},m_e\to m_f)\ , \tag{46}$$

where, for a spherical cutoff $K_{\max}$ for the photon momentum magnitude,

$$\tilde{B}_1(p_e,p_{\bar{e}},m_e)=\frac{-2m_e{}^2\ln(2K_{\max}/m_\gamma)}{\pi s\beta_e}\left[\frac{1}{1-\beta_e}-\frac{1}{1+\beta_e}\right]-\frac{s-2m_e{}^2}{\pi s\beta_e}\ln(2K_{\max}/m_\gamma)\ln\left[\frac{1-\beta_e}{1+\beta_e}\right]\ , \tag{47}$$

$$\tilde{B}_2(p_e,p_f,m_e,m_f)$$
$$=\frac{e_f m_f{}^2}{\pi s\beta_f}\ln(2K_{\max}/m_\gamma)\left[\frac{1}{1-\beta_f}-\frac{1}{1+\beta_f}\right]+\frac{e_f m_e{}^2}{\pi s\beta_e}\ln(2K_{\max}/m_\gamma)\left[\frac{1}{1-\beta_e}-\frac{1}{1+\beta_e}\right]$$
$$+\frac{(s_{ef}-m_e{}^2-m_f{}^2)e_f\ln(2K_{\max}/m_\gamma)}{4\pi\{(m_f{}^2-m_e{}^2-t_{ef})^2/4+m_e{}^2[s_{ef}-2(m_e{}^2+m_f{}^2)]\}^{1/2}}$$
$$\times\left[\ln\left|\frac{s_{ef}-2(m_e{}^2+m_f{}^2)-\left[\frac{m_f{}^2-m_e{}^2-t_{ef}}{2}\right]-\left[\left[\frac{(m_f{}^2-m_e{}^2-t_{ef})^2}{4}\right]+m_e{}^2[s_{ef}-2(m_e{}^2+m_f{}^2)]\right]^{1/2}}{(m_f{}^2-m_e{}^2-t_{ef})/2+\left[\left[\frac{(m_f{}^2-m_e{}^2-t_{ef})^2}{4}\right]+m_e{}^2[s_{ef}-2(m_e{}^2+m_f{}^2)]\right]^{1/2}}\right|\right.$$
$$\left.-\ln\left|\frac{s_{ef}-2(m_e{}^2+m_f{}^2)-\left[\frac{m_f{}^2-m_e{}^2-t_{ef}}{2}\right]+\left[\left[\frac{(m_f{}^2-m_e{}^2-t_{ef})^2}{4}\right]+m_e{}^2[s_{ef}-2(m_e{}^2+m_f{}^2)]\right]^{1/2}}{(m_f{}^2-m_e{}^2-t_{ef})/2-\left[\left[\frac{(m_f{}^2-m_e{}^2-t_{ef})^2}{4}\right]+m_e{}^2[s_{ef}-2(m_e{}^2+m_f{}^2)]\right]^{1/2}}\right|\right]\ , \tag{48}$$

where

$$\beta_f\equiv(1-4m_f{}^2/s)^{1/2}\ . \tag{49}$$

Hence, we have completely specified $\bar{\beta}_0$ and $\bar{\beta}_1$; we now turn to $\bar{\bar{\beta}}_0$ and $\bar{\bar{\beta}}_1$.

Considering first $\bar{\bar{\beta}}_0$, we have [the $\bar{\beta}_i$ in (37) and (38) contain a standard phase-space factor relative to those in (33)]

$$\bar{\bar{\beta}}_0=\lambda^{-2}\bar{\beta}_0(p_e(1),p_{\bar{e}}(1),p_\mu(1),p_{\bar{\mu}}(1),m_{e,\text{phys}}/\lambda,m_{\mu,\text{phys}}/\lambda,\alpha(\lambda))\ , \tag{50}$$

where $\sqrt{s_0}\equiv 2m_{\mu,\text{phys}}$ and

$$p_f(1)\equiv\left[\frac{\sqrt{s_0}}{2},\hat{\mathbf{z}}_f\left[\frac{s_0}{4}-\frac{m_{f,\text{phys}}{}^2}{\lambda^2}\right]^{1/2}\right]\ ,\qquad f=e,\ \bar{e},\ \mu,\ \text{and}\ \bar{\mu} \tag{51}$$

with $\hat{\mathbf{z}}_e=-\hat{\mathbf{z}}_{\bar{e}}\equiv\hat{\mathbf{z}}$ and $\hat{\mathbf{z}}_\mu=-\hat{\mathbf{z}}_{\bar{\mu}}$. Here, $\lambda=M_{Z_0}/2m_{\mu,\text{phys}}$. For the constant b_0 in (36) we may take $8/12\pi^2$ with $\alpha(1)\simeq\frac{1}{137}$.

Similarly, for $\bar{\bar{\beta}}_1$, we have

$$\bar{\bar{\beta}}_1=\lambda^{-4}\bar{\beta}_1(p_i(1),m_{i,\text{phys}}/\lambda,k_1/\lambda,\alpha^3\to\alpha(1)\alpha^2(\lambda))\ . \tag{52}$$

This, then, completely specifies $\bar{\bar{\beta}}_0$ and $\bar{\bar{\beta}}_1$.

Thus, in our example $E_{X'}=p_f^0+p_{\bar{f}}^0$, $\mathbf{P}_{X'}=\mathbf{p}_f+\mathbf{p}_{\bar{f}}$, and we have

$$d\sigma=\exp[2\alpha(1)(\text{Re}B+\tilde{B})]\frac{1}{(2\pi)^4}\times\int d^4y\,\exp[iy\cdot(p_e+p_{\bar{e}}-P_{X'})+D]\times\left[\bar{\bar{\beta}}_0(q_0)+\int\frac{d^3k_1}{k_1}e^{-iy\cdot k_1}\bar{\bar{\beta}}_1(q_1)\right]\times dE_{X'}d^3P_{X'}\ , \tag{53}$$

where

$$D=\int\frac{d^3k}{k}[e^{-iy\cdot k}-\theta(K_{\max}-k)]\tilde{S}\ . \tag{54}$$

We note that, as one may check from (39)–(49), Re$B+\tilde{B}$ does not contain infrared singularities.

The effect of e^D in (53) has been discussed in detail by Jadach in Ref. 10. The basic result is that, for Monte Carlo simulation, one should write (53) as

$$d\sigma=\exp\{2\alpha(1)[\mathrm{Re}B+\tilde{B}(p_i(1),m_{iR}(\lambda),E_{\gamma,\max}/\lambda)]\}$$
$$\times\left[\delta(\sqrt{s}-E_{X'})\tilde{\bar{\beta}}_0(\sqrt{s})\int_0^{K_{\max}}\rho(\epsilon')d\epsilon'+\theta(\epsilon-K_{\max})\tilde{\bar{\beta}}_0(\sqrt{s})\frac{\alpha(1)A}{\epsilon}(\epsilon/E_{\gamma,\max})^{\alpha(1)A}\right]dE_{X'}$$
$$+\exp\{2\alpha(1)[\mathrm{Re}B+\tilde{B}(p_i(1),m_{iR}(\lambda),E'_{\gamma,\max}/\lambda)]\}$$
$$\times\tilde{\bar{\beta}}_1(k')\frac{d^3k'}{k'}\left[\delta(\epsilon-k')\int_0^{K_{\max}}\rho'(\epsilon'-k')d(\epsilon'-k')+\theta(\epsilon-k'-K_{\max})\frac{\alpha(1)A}{\epsilon-k'}\left[\frac{\epsilon-k'}{E'_{\gamma,\max}}\right]^{\alpha(1)A}\right]dE_{X'}\ , \quad (55)$$

where we have introduced

$$\alpha(1)A\equiv2\alpha(1)\tilde{B}(p_i(1),m_{iR}(\lambda),K_{\max}/\lambda)\times[\ln(2K_{\max}/\lambda m_\gamma)]^{-1}\ , \quad (56)$$

$$\epsilon=\sqrt{s}-E_{X'}=\sqrt{s}-E_f-E_{\bar{f}}\ , \quad (57)$$

and

$$\rho(\epsilon)=\frac{\alpha(1)A}{\epsilon}\left[\frac{\epsilon}{E_{\gamma,\max}}\right]^{\alpha(1)A}\ ,\qquad \rho'(\epsilon)=\frac{\alpha(1)A}{\epsilon}\left[\frac{\epsilon}{E'_{\gamma,\max}}\right]^{\alpha(1)A}\ , \quad (58)$$

with

$$E_{\gamma,\max}=\sqrt{s}/2-2m_f^2/\sqrt{s}\ ,\qquad E'_{\gamma,\max}=\frac{s-2k'\sqrt{s}-4m_f^2}{2(\sqrt{s}-2k')}\ . \quad (59)$$

[Note that $f=\mu$ in (53).] Hence, here $K_{\max}$ is the maximum energy of a photon which cannot be detected by the respective detector. In order to implement (55), one proceeds as follows. One uses $\rho(\epsilon)$ $[\rho'(\epsilon-k')]$ to choose a value for ϵ $[\epsilon-k']$ by standard Monte Carlo methods. One sets the number n of Yennie-Frautschi-Suura "soft" photons equal to 0 if $\epsilon\le K_{\max}$ $[\epsilon-k'\le K_{\max}]$. One picks n according to the Poisson distribution

$$P_{n-1}=\frac{e^{-\bar{n}}(\bar{n})^{n-1}}{(n-1)!}\ ,$$
$$\bar{n}=\alpha(1)A\ \ln(\epsilon/K_{\max})\ \text{for}\ \epsilon>K_{\max}$$
$$\left[\bar{n}=\alpha(1)A\ \ln\left[\frac{\epsilon-k'}{K_{\max}}\right]\ \text{for}\ \epsilon-k'>K_{\max}\right]\ , \quad (60)$$

where the $n-1$ variables used to generate P_{n-1} in Ref. 10 may be used to choose the photon energies $k_1,\ldots,k_n$ such that $\sum_i k_i=\epsilon(\sum_i k_i=\epsilon-k')$. The angular distribution of the n photons is then chosen, by the standard Monte Carlo methods, according to $\tilde{S}(k)d^3k/k$. In this way, a one-loop event generator based on results such as those in Ref. 8 may be rigorously exponentiated.

As a reminder, we have used Ref. 8 as a pedagogical example. The method illustrated by (37)–(60) applies to any electroweak Monte Carlo event generator.

Currently, there is an effort by Berends's group[11] to create an order-α^4 event generator for e^+e^- annihilation into $\mu^+\mu^-(\gamma,\gamma\gamma)$. Thus, it is of some interest to record the analog of (37) and (38) at order α^4.

In (37), we would use

$$\bar{\beta}_0(q_0)=\frac{d\sigma(\alpha^4)}{d\Omega_\mu}-2\,\mathrm{Re}[\alpha(1)B]\frac{d\sigma(\alpha^3)}{d\Omega_\mu}+\frac{\{2\,\mathrm{Re}[\alpha(1)B]\}^2}{2}\frac{d\sigma_0}{d\Omega_\mu} \quad (61)$$

and, in (38), we would use

$$\bar{\beta}_1(k_1)=\frac{d\sigma^{B1}(\alpha^4)}{d\Omega_\mu d\Omega_\gamma k_1dk_1}-2\,\mathrm{Re}[\alpha(1)B]\frac{d\sigma^{B1}(\alpha^3)}{d\Omega_\mu d\Omega_\gamma k_1dk_1}-\tilde{S}(k_1)\bar{\beta}_0\ , \quad (62)$$

where $d\sigma^{B1}(\alpha^4)$ is the cross section for single bremsstrahlung through order α^4; the analogous definition holds for $d\sigma(\alpha^n)$. In addition, to order α^4, the cross section $\bar{\beta}_2(k_1,k_2)$ may be identified as

$$\bar{\beta}_2=\frac{d\sigma^{B2}}{d\Omega_\mu\frac{d^3k_1}{k_1}\frac{d^3k_2}{k_2}}-\tilde{S}(k_1)\tilde{S}(k_2)\frac{d\sigma_0}{d\Omega_\mu}-\tilde{S}(k_1)\bar{\beta}_1(k_2)-\tilde{S}(k_2)\bar{\beta}_1(k_1)\ , \quad (63)$$

where $d\sigma^{B2}$ is the respective order-α^4 double bremsstrahlung cross section. Formulas analogous to (50)–(52) may then be used to obtain $\tilde{\bar{\beta}}_0$, $\tilde{\bar{\beta}}_1$, and $\tilde{\bar{\beta}}_2$. The steps leading from (53) to (55) may then be repeated. The net result is to add to (55) the term

$$\exp\{2\alpha(1)[\mathrm{Re}B+\tilde{B}(p_i(1),m_{iR}(\lambda),E''_{\gamma,\max}/\lambda)]\}\frac{\tilde{\bar{\beta}}_2(k',k'')}{2}\frac{d^3k'}{k'}\frac{d^3k''}{k''}$$
$$\times\left[\delta(\epsilon-k'-k'')\int_0^{K_{\max}}\rho''(\epsilon'-k'-k'')d(\epsilon'-k'-k'')+\theta(\epsilon-k'-k''-K_{\max})\frac{\alpha(1)A}{\epsilon-k'-k''}\left[\frac{\epsilon-k'-k''}{E''_{\gamma,\max}}\right]^{\alpha(1)A}\right]dE_{X'}\ , \quad (64)$$

where (here $k'\cdot k''=k'k''-\mathbf{k}'\cdot\mathbf{k}''$ so that $k^0=|\mathbf{k}|\equiv k$ for all k)

$$E''_{\gamma,\max}=\frac{(\sqrt{s}-k'-k''+|\mathbf{k}'+\mathbf{k}''|)[s-2\sqrt{s}(k'+k'')+2k'\cdot k''-4m_f{}^2]}{2[s-2\sqrt{s}(k'+k'')+2k'\cdot k'']},\rho''(\epsilon)=\rho(\epsilon)\Big|_{E_{\gamma,\max}=E''_{\gamma,\max}} . \tag{65}$$

Thus, it is clear how to extend (55) to order-α^4 input. (We ignore, here, the processes $e^+e^-\to\mu^+\mu^-+f\bar{f}$, $f=e,\mu$, for pedagogical reasons; they pose no particular problem but are expected to be insignificant at the level of accuracy of interest to us here.)

Several comments are in order. First, the use of $E'_{\gamma,\max}$ and $E''_{\gamma,\max}$ for the respective upper limits of the radiated photon energy is a refinement; these two can both be replaced by their maximum value, which is just $E_{\gamma,\max}$. Second, we have not allowed $K_{\max}$ to depend on the spherical angles (θ,ϕ) of the respective photons. This we have done for simplicity. The expression (55) is flexible enough to allow one to include a possible angular dependence of $K_{\max}$. Indeed, let $\bar{K}_{\max}=$minimum value of $K_{\max}(\theta,\phi)$ for the respective detector. Then, if we set $K_{\max}=\bar{K}_{\max}$ in (55), we have a correct formula. We can then include the effect of $K_{\max}(\theta,\phi)$ by amending our prescription for choosing ϵ $(\epsilon-k')$ and n: if $\epsilon>\bar{K}_{\max}(\epsilon-k'>\bar{K}_{\max})$ and $n>0$, use the bremsstrahlung distribution $\tilde{S}(k)d^3k/k$ to pick the respective angles (θ_i,ϕ_i) of the n photons with energies $\{k_i\}$ as determined by the procedure in Ref. 10. Because of the angular dependence of $K_{\max}(\theta,\phi)$, some subject of the n photons with energies $\{k_{i_1},\ldots,k_{i_j}\}$ may not be detected. Let the energies of the detected photons be $\{k_{i_{j+1}},\ldots,k_{i_n}\}$. Then, treat the event as an event with n-j detected photons with $\sqrt{s}-E_{X'}=\epsilon$ where only $\sum_{l=j+1}^{n}k_{i_l}$ of $\epsilon(\epsilon-k')$ is detected. In this way, we maintain a realistic description of the cross section in (55).

Third, we would like to emphasize that the value

$$b_0=8/12\pi^2$$

in (36) represents the Weinberg prescription for three families of quarks and leptons when one ignores the W^+, W^-, and Z_0. If we include the latter bosons, we would have $b_0=11/48\pi^2$, which reflects the non-Abelian character of the $SU(2)_L$ symmetry group.

Finally, in the interest of completeness, we would like to describe the procedure[10] which one uses to choose the photon energies associated with (60). Specifically, these energies are generated as

$$k_i=\epsilon e^{z_i}\Big/\left[\sum_{j=1}^{n}e^{z_j}\right]$$
$$\left[k_i=(\epsilon-k')e^{z_i}\Big/\left[\sum_{j=1}^{n}e^{z_j}\right]\right] , \tag{66}$$

where the z_i are such that $z_i=\ln k_i+Y$. Here, for $i=2,\ldots,n$, we take

$$z_i=\ln\epsilon+(\ln K_{\max}-\ln\epsilon)(R_i/\bar{n})$$
$$(z_i=\ln(\epsilon-k')+[\ln K_{\max}-\ln(\epsilon-k')](R_i/\bar{n})) , \tag{67}$$

where we recall that $\bar{n}$ is defined in (60) and we note that the R_{i+1} are generated from a series of uniformly distributed random numbers $r_i\in(0,1)$ with $R_{N+1}=-\sum_{i=1}^{N}\ln r_i$, $1\le N\le n-1$, where $(n-1)$ is the value of N for which R_{N+2} first exceeds $\bar{n}$. Then, Y is fixed so that $k_1=\epsilon-\sum_{i=1}^{n-1}k_{i+1}$ $(k_1=\epsilon-k'-\sum_{i=1}^{n-1}k_{i+1})$ and

$$z_1=\ln\epsilon \quad [z_1=\ln(\epsilon-k')] , \tag{68}$$

which means that, at the end of the process, we must reject the entire event if $z_n-Y\le\ln K_{\max}$, i.e., if $k_n\le K_{\max}$. The prescription represented by (37)–(68) is now a practical way to implement (33).

What we see is that (37)–(68) afford one a method for summing the large IR and UV effects in $e^+e^-\to\mu\bar{\mu}(\gamma)$ without encountering mass singularity problems and without presuming the parton model, at the level of a realistic Monte Carlo event generator. To repeat, the general application of such "exponentiated" event generators will be taken up elsewhere.[5,9]

VI. CONCLUSION

We have derived a rigorous renormalization-group-improved version of the Yennie-Frautschi-Suura program using the renormalization-group equation of Weinberg. The detailed application of our formalism to the $SU(2)_L\times U(1)$ theory for the processes $e^+e^-\to Z_0\to X$ will be discussed elsewhere.[5,9] We have, however, illustrated how one would use our formalism by giving an explicit recipe for the renormalization-group-improved exponentiation of the popular Monte Carlo event generator MMG1 in Ref. 7.

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