CP violation in $K \to 3\pi$

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For the weak nonleptonic decay $K \to 3\pi$, we analyze the amount of *CP* violation expected to arise in the Kobayashi-Maskawa model using two recently uncovered aspects: (1) quadratic terms in the $K \to 3\pi$ amplitude (found in the *CP*-conserving data) and (2) electromagnetic and isospin-breaking effects in the relation of the *CP*-violating parameters of $K \to 2\pi$ and $K \to 3\pi$. Previous to the inclusion of these effects the non-ϵ signal in η_{+-0} ($\eta_{+-0}=\epsilon+\epsilon'_{+-0}$) required interference with the highly suppressed $\Delta I=\frac{3}{2}$ amplitude; however in the present analysis the ϵ'_{+-0} signal is not suppressed. We estimate values for ϵ'_{+-0} about an order of magnitude larger than previously anticipated, although still smaller than ϵ.

I. INTRODUCTION

Attempts to understand the origin of *CP* violation are hampered by the scarcity of experimental information on the subject. All of the many models of *CP* nonconservation can fit the one observed nonzero signal (i.e., ϵ measured in $K_L \to 2\pi$). It is important that *CP* violation be observed in other systems in order to be able to distinguish the various models.[1] In this paper we reanalyze the decays of $K_S \to 3\pi$ within the Kobayashi-Maskawa (KM) model of *CP* violation.[2] This is the system which is closest to the observed *CP* violation in $K_L \to 2\pi$ and there is presently an experimental program attempting to find evidence of *CP* nonconservation in $K \to 3\pi$ transitions.

There has been considerable previous theoretical work on the $K \to 3\pi$ decays.[3,4] The dominant nonleptonic amplitudes can be related to those in $K \to 2\pi$ by use of current algebra. However, there are two new ingredients in our analysis which lead to a significant modification of previous results. (1) In the PCAC (partial conservation of axial-vector current) connection of $K \to 3\pi$ and $K \to 2\pi$ the presence of higher-order momentum dependence,[5] visible in the known $K \to 3\pi$ amplitudes,[6] removes the $\Delta I=\frac{3}{2}$ suppression of the *CP*-odd interference, producing a larger signal. (2) In $K \to 2\pi$, the importance of electromagnetic and isospin-breaking effects has only recently been understood.[7,8] We include these effects in our analysis of $K \to 3\pi$. The combined result of these two contributions allows the direct, "non-ϵ," portion of the signal to be nearly an order of magnitude larger than previously estimated.

In order to see how this result obtains we shall in the following section define the model within which we work and review its impact in the $K \to 2\pi$ sector. In Sec. III we define notation and undertake the theoretical calculation of *CP*-violating effects within the $K \to 3\pi$ system. Finally we summarize our findings in Sec. IV.

II. *CP*-VIOLATING FORMALISM AND $K_L \to 2\pi$

We shall work here with the so-called "standard" model of the weak interaction, wherein the charged weak current takes the form

$$J_\mu=\bar{q}U_{\rm KM}\gamma_\mu(1+\gamma_5)q \tag{1}$$

with $U_{\rm KM}$ being the unitary generation-mixing matrix written down originally by Kobayashi and Maskawa,[2] whose notation we shall employ.

For our application, we require the weak interaction responsible for $\Delta S=1$ nonleptonic decay. The form of this operator is now well established and is generally written in terms of a series of local four-quark operators O_i:

$$H_w^{\rm NL}\equiv\frac{G}{2\sqrt{2}}\cos\theta_1\sin\theta_1\sum_{i=1}^{6}c_iO_i \ . \tag{2}$$

Here the expansion coefficients c_i are complex numbers, with the dominant *CP*-violating contribution arising from the penguin diagram:[9]

$$c_5O_5\simeq(-0.047-i0.097\tau)\bar{d}\gamma_\mu(1+\gamma_5)\lambda^a s \times\sum_{i=u,d\ldots}\bar{q}_i\gamma^\mu(1-\gamma_5)\lambda^a q_i \ , \tag{3}$$

where

$$\tau=\sin\theta_2\sin\theta_3\sin\delta \ .$$

For our purposes, we shall use only the two features that (i) all operators θ_i are right-handed singlets, and transform as 8- or 27-dimensional representations of SU(3) under left-handed rotations, and (ii) all *CP* violation is in the $\Delta I=\frac{1}{2}$ octet sector.

We also shall employ the so-called "electromagnetic penguin" (EMP) diagram,[7] which involves the replacement of a gluon in the usual penguin term by a virtual photon. The resulting interaction has both purely left-handed terms and also those with a nontrivial transformation property in the right-handed sector. The former are small $O(\alpha_{\rm EM}/\alpha_s)$ corrections to the usual weak Hamiltonian and are generally neglected. However, the latter play a more important role, as we shall demonstrate. These terms have the form[7]

$$H_{\rm EMP}=-\frac{G_F}{\sqrt{2}}\cos\theta_1\cos\theta_3\sin\theta_1(c_7O_7+c_8O_8) \tag{4}$$

with

$$O_7=\bar{s}_a\gamma_\mu(1+\gamma_5)d_a[\bar{u}_b\gamma^\mu(1-\gamma_5)u_b-\tfrac{1}{2}\bar{d}_b\gamma^\mu(1-\gamma_5)d_b-\tfrac{1}{2}\bar{s}_b\gamma^\mu(1-\gamma_5)s_b]\ ,$$
$$O_8=\bar{s}_a\gamma_\mu(1+\gamma_5)d_b[u_b\gamma^\mu(1-\gamma_5)u_a-\tfrac{1}{2}\bar{d}_b\gamma^\mu(1-\gamma_5)d_a-\tfrac{1}{2}\bar{s}_b\gamma^\mu(1-\gamma_5)s_a]\ , \quad (5)$$

and

$$c_7=(0.037-0.067\tau)\alpha_{\rm EM}\ ,$$
$$c_8=(0.008-0.011\tau)\alpha_{\rm EM}\ . \quad (6)$$

The key feature here is that both the left- and right-handed currents are in SU(3)-octet representations. It will turn out that it is the $\Delta I=\frac{3}{2}$ piece of this operator which will be most important to our own work.

Finally, we shall require the inclusion of isospin breaking (ISB) via mass terms in the QCD Lagrangian

$$L_m=-m_u\bar{u}u-m_d\bar{d}d-m_s\bar{s}s\ , \quad (7)$$

where here the isospin-breaking term

$$L_m^{\rm ISB}=-\tfrac{1}{2}(m_u-m_d)(\bar{u}u-\bar{d}d) \quad (8)$$

is proportional to the light-quark mass difference. (Electromagnetic isospin-breaking effects are less important and will be neglected here.) The role of this piece of the Lagrangian is to mix a portion of the (large) $\Delta I=\frac{1}{2}$ amplitude into matrix elements which transform as $\Delta I=\frac{3}{2}$. Because direct contributions to the latter are generally small, even a modest amount of mixing can generate a significant impact.

Within this framework, we can now review the treatment of $K_L\rightarrow 2\pi$ in order to define notation. The $K^0\rightarrow 2\pi$ matrix elements can be decomposed in terms of their final-state isospin structure

$$A(K^0\rightarrow\pi^+\pi^-)\equiv A_0e^{i\delta_0}+\frac{1}{\sqrt{2}}A_2e^{i\delta_2}\ ,$$
$$A(K^0\rightarrow\pi^0\pi^0)\equiv A_0e^{i\delta_0}-\sqrt{2}A_2e^{i\delta_2}\ , \quad (9)$$

where δ_I are the phase shifts for elastic scattering by two pions in an isospin I configuration and A_I are the associated weak decay amplitudes. Experimentally we have[1]

$${\rm Re}A_0\simeq 5.5\times 10^{-7}m_K\ ,$$
$$\delta_2-\delta_0=-42^\circ\pm 8^\circ\ , \quad (10)$$
$$\omega\equiv\frac{{\rm Re}A_2}{{\rm Re}A_0}\simeq 0.045\ .$$

Of course, in a theory with CP violation A_0, A_2 are, in general, complex numbers and thus we define

$$A_0\equiv|A_0|e^{i\xi},\quad A_2\equiv|A_2|e^{i\xi'}\ . \quad (11)$$

The physical eigenstates constructed from K^0 and $\bar{K}^0$ can be written as

$$|K_S^0\rangle\simeq\left[\tfrac{1}{2}\right]^{1/2}[(1+\bar{\epsilon})|K^0\rangle+(1-\bar{\epsilon})|K^0\rangle]\ ,$$
$$|K_L^0\rangle\simeq\left[\tfrac{1}{2}\right]^{1/2}[(1+\bar{\epsilon})|K^0\rangle-(1-\bar{\epsilon})|K^0\rangle]\ , \quad (12)$$

where here and in the remainder of our work we shall drop quantities that are second order in CP violation, such as $\bar{\epsilon}^2$.

Since $\Delta S=+1$ and $\Delta S=-1$ transitions cannot interfere with one another, there exists the freedom to make a strangeness gauge transformation

$$|K^0\rangle\rightarrow e^{i\phi}|K^0\rangle\ ,$$
$$|\bar{K}^0\rangle\rightarrow e^{-i\phi}|\bar{K}^0\rangle\ , \quad (13)$$

without affecting the physics. The value of the mixing parameter $\bar{\epsilon}$ will depend on this phase

$$\bar{\epsilon}\rightarrow\bar{\epsilon}+i\phi \quad (14)$$

and hence is not directly measurable. However, physical quantities are phase invariant. It is conventional to define

$$\eta_{\pm}=\frac{\langle\pi^+\pi^-|H_w|K_L\rangle}{\langle\pi^+\pi^-|H_w|K_S\rangle}=\epsilon+\epsilon'\ , \quad (15)$$

$$\eta_{00}=\frac{\langle\pi^0\pi^0|H_w|K_L\rangle}{\langle\pi^0\pi^0|H_w|K_S\rangle}=\epsilon-2\epsilon'\ . \quad (16)$$

Here ϵ is the value of $\bar{\epsilon}$ in the gauge where ${\rm Im}A_0=0$. More generally, we have

$$\epsilon\simeq\frac{e^{i\pi/4}}{2\sqrt{2}}\left[\frac{{\rm Im}M_{12}}{{\rm Re}M_{12}}+2\frac{{\rm Im}A_0}{{\rm Re}A_0}\right]\ ,$$
$$\epsilon'\simeq\frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\omega\left[\frac{{\rm Im}A_2}{{\rm Re}A_2}-\frac{{\rm Im}A_0}{{\rm Re}A_0}\right]\ , \quad (17)$$

where M_{ij} is the mass matrix which connects $K^0,\bar{K}^0$. Since under a gauge change

$$\frac{{\rm Im}A_I}{{\rm Re}A_I}\rightarrow\frac{{\rm Im}A_I}{{\rm Re}A_I}+\phi\ ,$$
$$\frac{{\rm Im}M_{12}}{{\rm Re}M_{12}}\rightarrow\frac{{\rm Im}M_{12}}{{\rm Re}M_{12}}-2\phi\ , \quad (18)$$

we see that these definitions of ϵ, ϵ' are unchanged under a strangeness gauge change.

In applying the previously described interactions to this process, we shall emphasize their different chiral transformation properties. Thus, for the conventional weak Hamiltonian, which has the behavior $(8_L,1_R)$ and $(27_L,1_R)$ under chiral rotations, we write

$$L_{(2)}^{\rm NL}=g_8{\rm Tr}\lambda_6\partial_\mu U\partial^\mu U^\dagger+g_{27}C_i^{8}{}_j^{8}{}_6^{27}{\rm Tr}(\lambda_i\partial_\mu UU^\dagger\lambda_jU\partial_\mu U^\dagger)\ , \quad (19)$$

where

$$U=\exp\left[\frac{i}{F_\pi}\sum_{j=1}^{8}\lambda_j\phi_j\right] \quad (20)$$

is the usual nonlinear matrix describing the pseudoscalar mesons, $F_\pi \simeq 94$ MeV is the pion decay constant, and the C symbol represents an SU(3) Clebsch-Gordan coefficient. We require $g_{27} \ll g_8$ according to the validity of the $\Delta I = \frac{1}{2}$ rule.

On the other hand, the dominant part of the electromagnetic penguin operator has an $(8_L, 8_R)$ chiral property and can be represented by the effective Langrangian

$$L^{\rm EMP} = g_{\rm EMP} {\rm Tr}\{\lambda_6 U[\lambda_3 + \left[\tfrac{1}{3}\right]^{1/2}\lambda_8]U^\dagger\} \ . \qquad (21)$$

Finally, isospin-breaking π^0-η^0 mixing effects utilize the usual nonleptonic weak interaction but are generated by the mass-mixing Lagrangian

$$L^{\rm ISB} = \mu\, {\rm Tr}\left[\begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}(U+U^\dagger)\right] \ . \qquad (22)$$

In our discussion, we shall also include π^0-$\eta^{0'}$ mixing, whose treatment is outside the traditional chiral-Lagrangian framework. For calculations of this mixing, we utilize the quark model as a guide. Such a procedure should be adequate within the considerable uncertainty of the final result.

Applying these interactions to $K \to 2\pi$, one finds, as described more fully in Ref. 8,

$$\begin{aligned} A(K^0 \to \pi^+\pi^-) &= A_0^{\rm NL} + \frac{1}{\sqrt{2}} A_2^{\rm NL} + A^{\rm EMP} \ , \\ A(K^0 \to \pi^0\pi^0) &= A_0^{\rm NL} - \sqrt{2} A_2^{\rm NL} + A^{\rm ISB} \ , \end{aligned} \qquad (23)$$

where $A_0^{\rm NL}$, $A_2^{\rm NL}$, and $A^{\rm EMP}$ receive contributions from $g_8 - g_{27}, g_{27}$, and $g_{\rm EMP}$, respectively, while $A^{\rm ISB}$ arises from g_8 and the π^0-η^0-$\eta^{0'}$ mixing discussed above. Solving for ϵ' we find

$$\begin{aligned} |\epsilon'| &\simeq \frac{\omega}{\sqrt{2}} \frac{{\rm Im} A_0^{\rm NL}}{{\rm Re} A_0^{\rm NL}} \left[1 - \frac{\sqrt{2}}{3\omega}(1-\sqrt{2}\omega)\frac{{\rm Im} A^{\rm EMP}}{{\rm Im} A_0^{\rm NL}} + \frac{2\sqrt{2}}{3\omega}(1+\sqrt{2}\omega)\frac{{\rm Im} A^{\rm ISB}}{{\rm Im} A_0^{\rm NL}}\right] \\ &= \frac{\omega}{\sqrt{2}}(1-\Omega_{\rm EMP}-\Omega_{\rm ISB})\frac{{\rm Im} A_0^{\rm NL}}{{\rm Re} A_0^{\rm NL}} = \frac{\omega}{\sqrt{2}}(1-\Omega_{\rm EMP}-\Omega_{\rm ISB})\xi \ . \end{aligned} \qquad (24)$$

We note here that both $\Omega_{\rm EMP}$ and $\Omega_{\rm ISB}$ contain the factor

$$\frac{\sqrt{2}}{\omega} \approx 31 \qquad (25)$$

which enhances these contributions considerably over the $O(\alpha) \approx 1\%$ effects which might be naively expected. The reason for this enhancement is that while the EMP and ISB terms are indeed $O(\alpha)$ relative to the leading $\Delta I = \frac{1}{2}$ piece, these corrections are sizable, of $O(\alpha/\omega)$, relative to the $\Delta I = \frac{3}{2}$ component to which ϵ' is sensitive.

Numerical estimates of $\Omega_{\rm EMP}$, $\Omega_{\rm ISB}$ in Ref. 8 yielded

$$\begin{aligned} \Omega_{\rm EMP} &\simeq -0.38(1 \pm 50\%) \ , \\ \Omega_{\rm ISB} &= \Omega_\eta + \Omega_{\eta}' = 0.34 \to 0.46 \ , \end{aligned} \qquad (26)$$

where the specific size of the latter number depends upon the analysis of η-η' mixing. Overall then these factors lead to a modification of the simple prediction

$$\epsilon'^{(0)} = \frac{\omega}{\sqrt{2}} \frac{{\rm Im} A_0^{\rm NL}}{{\rm Re} A_0^{\rm NL}} \equiv \frac{\omega}{\sqrt{2}}\xi \qquad (27)$$

by the factor

$$0.7 \le (1-\Omega_{\rm EMP}-\Omega_{\rm ISB}) < 1.3 \ . \qquad (28)$$

Reversing this calculation, we can relate the CP-violating phase ξ of the $A_0^{\rm NL}$ amplitude to the experimental value of ϵ'

$$\xi = \frac{\sqrt{2}\,|\epsilon'|}{\omega(1-\Omega_{\rm EMP}-\Omega_{\rm ISB})} \approx (24 \to 40)\,|\epsilon'| \ . \qquad (29)$$

We now move to consider the $K \to 3\pi$.

III. *CP* VIOLATION IN $K \to 3\pi$

A number of authors have previously treated the $K \to 3\pi$ system, both CP conserving and violating, by relating $K \to 3\pi$ to $K \to 2\pi$ via current-algebra PCAC methods. This relationship is possible because of the left-handed transformation property of the weak Hamiltonian, which yields

$$[F_5^i, H_w^{\rm NL}] = [F^i, H_w^{\rm NL}] \ , \qquad (30)$$

where F_5^i and F^i are the axial-vector and vector charges, respectively. Thus, for example, defining

$$\eta_{+-0} = \frac{\langle \pi^+\pi^-\pi^0 | H_w^{(-)} | K_S \rangle}{\langle \pi^+\pi^-\pi^0 | H_w^{(+)} | K_L \rangle} = \epsilon + \epsilon'_{+-0} \ , \qquad (31)$$

where $H_w^{(+)}$ ($H_w^{(-)}$) represents the CP-even (-odd) component of the weak Hamiltonian, Li and Wolfenstein have derived the result[3,10]

$$\epsilon'_{+-0} = -2\epsilon' \qquad (32)$$

which relates the direct $\Delta S = 1$ CP-violating transition in $K_S \to 3\pi$ to that found in $K_L \to 2\pi$. We shall see, however, that one should expect considerable modification of this result from the electromagnetic penguin and mixing effects discussed in the previous section and, in addition, from a new feature which is relevant for $K \to 3\pi$ (but not for $K \to 2\pi$)—the possibility of higher-order terms in the effective weak Lagrangian. Thus, the lowest-order effective Lagrangian [Eq. (19)] involves two derivatives. One anticipates, however, that there will exist higher-order contributions involving four or more derivatives:

$$L^{NL}=L^{NL}_{(2)}+\frac{g'}{\Lambda_1^2}\mathrm{Tr}\lambda_6\partial_\mu U\partial^\mu U^\dagger\partial_\nu U\partial^\nu U^\dagger+\frac{g''}{\Lambda_2^2}\mathrm{Tr}\lambda_6\partial_\mu U\partial_\nu U^\dagger\partial^\mu U\partial^\nu U^\dagger+\cdots \quad (33)$$

Here the leading two-derivative piece reproduces all the standard current-algebra PCAC analysis of the *CP*-even and -odd contributions to $K\to 3\pi$, wherein the decay amplitude is expanded to first order in the momentum squared. The terms in g', g'' are two of the many possible Lagrangians with four derivatives. Contributions from these pieces of the Lagrangian are of order q^2/Λ^2 with respect to those from $L^{NL}_{(2)}$, where Λ is a chiral scale parameter, expected to have a value of about one GeV. Taking $q^2\sim m_K{}^2$ we find that these higher-order terms contribute effects only at the $m_K{}^2/\Lambda^2\sim 25\%$ level and hence they are usually neglected. We note also that since they contribute terms of the form

$$p_K\cdot p_{\pi_1}p_{\pi_2}\cdot p_{\pi_3}\ ,$$

etc., which vanish as any of the four-momenta become soft, there is no restriction on such terms arising from the $K\to 2\pi$ sector. Nevertheless, one anticipates their presence both on theoretical grounds—the chiral Lagrangian method is really a low-energy expansion with additional contributions becoming relevant at higher energy—and based upon phenomenological considerations, wherein such terms are necessary to explain the size of the quadratic terms in the $K\to 3\pi$ decay amplitude.[5]

Thus, in the case of $K^0\to\pi^+\pi^-\pi^0$ it is traditional to write the decay amplitude as an expansion about the center of the Dalitz plot:

$$A(K^0\to\pi^+\pi^-\pi^0)=M(0)[1+\sigma(s_3-s_0)+\beta(s_+-s_-)+\gamma(s_+-s_-)^2+\delta(s_3-s_0)^2+\cdots]\ . \quad (34)$$

Here σ is the slope, β describes the charge asymmetry, and γ,δ are called the "quadratic" terms.[11] We employ the notation

$$s_i=(k-p_i)^2,\quad s_0\equiv\tfrac{1}{3}(s_++s_-+s_3)\ . \quad (35)$$

The available experimental data have been analyzed by Devlin and Dickey[6] yielding a fitted form to the (*CP*-even) decay amplitude:

$$\sqrt{2}A(K^0\to\pi^+\pi^-\pi^0)\simeq 9.15-0.71+(14.1+1.31)\frac{s_3-s_0}{s_0}-4.85\left[\frac{s_3-s_0}{s_0}\right]^2+0.88\left[\frac{s_+-s_-}{s_0}\right]^2+\cdots \quad (36)$$

in units of 10^{-7}. In the above the first number refers to the $\Delta I=\frac{1}{2}$ component of the transition and the second to that with $\Delta I=\frac{3}{2}$. The associated uncertainties are at the 0.2% level for $M(0)$, σ and about 20% for γ,δ. We see then that

$$\frac{g_{27}}{g_8}\sim\frac{\Delta I=\frac{3}{2}}{\Delta I=\frac{1}{2}}\sim 10\% \quad (37)$$

as expected and that the quadratic terms, which can only arise from g',g'', are definitely present. A detailed fit to this momentum dependence in fact yields

$$\Lambda_1,\Lambda_2\sim 1\ \mathrm{GeV}$$

in agreement with our theoretical prejudice.[5] The results of the fit are given below.

Now examine the decay $K^0\to\pi^+\pi^-\pi^0$ including all relevant features of our analysis: (i) lowest-order terms[12] g_8,g_{27}, (ii) higher-order chiral pieces g',g'', (iii) electromagnetic penguin g_8, and (iv) π^0-η^0-$\eta^{0'}$ mixing. We find then the contribution from terms (i) and (ii) to be

$$A(K^0\to\pi^+\pi^-\pi^0)=\frac{1}{6F_\pi}\frac{m_K{}^2}{m_K{}^2-m_\pi{}^2}\left[A_0^{NL}-\sqrt{2}A_2^{NL}+\left[A_0^{NL}+\frac{5}{2\sqrt{2}}A_2^{NL}\right]3\frac{s_3-s_0}{m_K{}^2}\right]+\frac{1}{6F_\pi}\frac{m_K{}^2}{m_K{}^2-m_\pi{}^2}[b\,k\cdot p_0p_+\cdot p_-+c\,(k\cdot p_+p_-\cdot p_0+k\cdot p_-p_+\cdot p_0)]\ , \quad (38)$$

where here A_0^{NL}, A_2^{NL} are the $K\to 2\pi$ decay amplitudes defined in the previous section and b,c are unknown constants related to g',g''. (The analysis of Ref. 5 has shown that these two combinations of four-momenta give the most general form of the quadratic terms, aside from factors which are absorbed into the definition of A_0.) The forms arising from mechanisms (iii) and (iv) are somewhat involved. We quote here only the *CP*-violating contribution at the center of the Dalitz plot, for which we find

$$A(K^0\to\pi^+\pi^-\pi^0)=0\times g_{\mathrm{EMP}}-i\frac{1.15\to 0.03}{6F_\pi}\mathrm{Im}A_0^{NL}\frac{m_d-m_u}{m_s}\ , \quad (39)$$

where the range is between full η-η' mixing ($\theta\approx 20°$)$\to$ no mixing ($\theta=0$). We note that the electromagnetic penguin does not contribute to this channel. In the case of the isospin-breaking terms, the full contribution appears in Appendix A.

As mentioned previously, the terms involving b,c above are not restricted by $K_L\to 2\pi$ data since both kinematical quantities vanish in all soft-pion limits. Instead the coefficients b,c must be determined empirically, by fitting the experimental data. The results obtained in such a fit are[5]

$$s_0{}^2\frac{\mathrm{Re}b}{\mathrm{Re}A_0}\approx 1.5,\quad s_0{}^2\frac{\mathrm{Re}c}{\mathrm{Re}A_0}=-0.4\ . \tag{40}$$

(Of course only the real portion of the amplitude is determined.) In order to see the impact that such terms deliver, we note that the contribution of the leading (two-derivative) portion of the Lagrangian to this process is (all amplitudes are quoted in units of 10^{-7})

$$\sqrt{2}A_{(2)}(K^0\to\pi^+\pi^-\pi^0)\approx(7.5-0.5)+(9.1+0.7)\frac{s_3-s_0}{s_0} \tag{41}$$

while the higher-order (four-derivative) terms in b,c yield

$$\sqrt{2}A_{(4)}(K^0\to\pi^+\pi^-\pi^0)\approx 1.9+5.5\frac{s_3-s_0}{s_0}-3.7\left[\frac{s_3-s_0}{s_0}\right]^2+0.6\left[\frac{s_+-s_-}{s_0}\right]^2 . \tag{42}$$

The total amplitude

$$\sqrt{2}A_{\mathrm{tot}}(K^0\to\pi^+\pi^-\pi^0)\approx(9.4-0.5)+(14.6+0.7)\left[\frac{s_3-s_0}{s_0}\right]-3.7\left[\frac{s_3-s_0}{s_0}\right]^2+0.6\left[\frac{s_+-s_-}{s_0}\right]^2 \tag{43}$$

provides a good picture of the experimental data [cf. Eq. (36)].

By looking at the imaginary piece of the $K^0\to\pi^+\pi^-\pi^0$ amplitude in Eqs. (38) and (39), we can now proceed to calculate ϵ'_{+-0}. The only problem here is that the phase of the higher-order parameters b,c is unknown, so we parametrize these by

$$b=|b|e^{i\phi_b},\quad c=|c|e^{i\phi_c}\ . \tag{44}$$

Since both b,c are generated by $\Delta I=\frac{1}{2}$ Hamiltonians there is no reason to expect the vanishing of either phase, as will be explicitly demonstrated in a particular model in Sec. IV.

Working in the "natural" phase convention in which

$$K^0\sim\bar{s}d \quad\text{and}\quad \bar{K}^0\sim\bar{d}s$$

the $\Delta I=\frac{3}{2}$ amplitude A_2^{NL} has no CP-violating phase ($\xi'=0$) but A_0^{NL} does:

$$A_0^{\mathrm{NL}}=|A_0^{\mathrm{NL}}|e^{i\xi}\ . \tag{45}$$

Then we find, at the center of the Dalitz plot,

$$\epsilon'_{+-0}=i\left[\frac{\mathrm{Im}M(0)}{\mathrm{Re}M(0)}-\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right]=i\sqrt{2}\omega\xi(1-\bar{\Omega}_{\mathrm{EMP}}-\bar{\Omega}_{\mathrm{ISB}})+i[0.45(\phi_b-\xi)-0.24(\phi_c-\xi)] \tag{46}$$

with

$$\bar{\Omega}_{\mathrm{EMP}}=0\ , \tag{47}$$

$$\bar{\Omega}_{\mathrm{ISB}}=(-\tfrac{1}{2}\to+1)\Omega_{\mathrm{ISB}}\ , \tag{48}$$

where the range in the latter corresponds to differing values of η-η' mixing, with the first value being no mixing and the latter full mixing. Combining this with the previously derived expression for ϵ we find

$$\begin{aligned}\epsilon'_{+-0}&\simeq-\epsilon'\left[2\left[\frac{1-\bar{\Omega}_{\mathrm{EMP}}-\bar{\Omega}_{\mathrm{ISB}}}{1-\Omega_{\mathrm{EMP}}-\Omega_{\mathrm{ISB}}}\right]-\frac{\sqrt{2}}{\omega\xi}\frac{0.45(\phi_b-\xi)-0.24(\phi_c-\xi)}{1-\Omega_{\mathrm{EMP}}-\Omega_{\mathrm{ISB}}}\right]\\&\approx-\epsilon'\left[(2\leftrightarrow 4)-(12\to 20)\frac{\xi-\phi_b}{\xi}+(6\to 10)\frac{\xi-\phi_c}{\xi}\right],\end{aligned} \tag{49}$$

where we have used Eq. (28).

Of course, due to the unknown phases ϕ_b,ϕ_c no firm prediction can be made, but the order of magnitude of each term is considerably larger than the result

$$\epsilon'_{+-0}\approx-2\epsilon' \tag{50}$$

obtained previously, so that barring strong accidental cancellations, one expects a very much larger value for ϵ'_{+-0} than given in earlier analyses.

Of course, the above analysis is purely phenomenological. In order to verify that such an enhancement in the ϵ'_{+-0} signal can actually occur, we have in Appendix B examined, within the framework of the vacuum-saturation approximation, the origin of these higher-order terms. We find that these terms do indeed have a different *CP*-violating phase from that found for the lowest-order Lagrangian and that even though these higher-order components are suppressed by a factor $m_K^2/\Lambda^2 \sim 25\%$, they produce a value for ϵ'_{+-0}:

$$\epsilon'_{+-0} \sim 10\epsilon' \tag{51}$$

which is an order of magnitude above that found in $K \to 2\pi$. We stress, however, that this result should not be used as a definitive prediction—the vacuum-saturation method is *not* presented here as a reliable approach in this regard, and the results are not to be trusted in detail. Rather, our calculation is simply meant as an illustration that large values of $\epsilon'_{+-0}/\epsilon'$ do really arise.

IV. SUMMARY

The new features contained in the present analysis are the inclusion of electromagnetic and isospin-breaking effects in both $K \to 2\pi$ and $K \to 3\pi$ amplitude, and also the awareness of the effects caused by quadratic terms present in the $K \to 3\pi$ amplitude. The former (EMP and ISB) can modify the prediction for ϵ' in $K \to 2\pi$ as was previously pointed out in Ref. 8. In η_{+-0} their effect is not in general identical. However, this situation requires that the relation between $\epsilon'_{+-0} \equiv \eta_{+-0} - \epsilon$ and ϵ' be reviewed. Also, quadratic terms are seen to be present in the *CP*-conserving decay amplitude. These play a major role in the analysis of *CP* violation because when they are included, the interference in the decay amplitude can take place between larger terms both of which have $\Delta I = \frac{1}{2}$, rather than requiring interference with the suppressed $\Delta I = \frac{3}{2}$ amplitude.

The combination of these effects increases the predicted value of $\epsilon'_{+-0}/\epsilon'$, with the overall effect expected to be an enhancement by up to an order of magnitude. The quadratic terms unfortunately have a weak phase which cannot be reliably determined from theory, so that firm predictions cannot be made. However, their effect on ϵ'_{+-0} was calculated in Eq. (46), and their size was estimated in a particular model in Eq. (51). The effect is large primarily because of the lack of $\Delta I = \frac{3}{2}$ suppression.

Similar considerations will modify the analysis of the rate asymmetry in charged kaon decays, $K^{\pm} \to \pi^{\pm}\pi^{\pm}\pi^{\mp}$. The electromagnetic penguin was included in the analysis of Grinstein, Rey, and Wise, and provides a small suppression (their parameter ξ would be $+0.4$ according to Ref. 4). Isospin-breaking mixing does not affect $K^+ \to \pi^+\pi^+\pi^-$, but can modify the relation of the asymmetry to ϵ'. Again quadratic pieces will enter in an unpredictable manner because of their unknown weak phase. However, they should remove the $\Delta I = \frac{3}{2}$ suppression because they can contribute to both the $I = 1$ symmetric and mixed symmetry states. Again we would expect roughly an order of magnitude increase due to these effects, so that

$$\frac{\Gamma(K^+ \to \pi^+\pi^+\pi^-) - \Gamma(K^- \to \pi^-\pi^-\pi^+)}{\Gamma(K^+ \to \pi^+\pi^+\pi^-) + \Gamma(K^- \to \pi^-\pi^-\pi^+)} \sim (0.2\sqrt{2}\,\mathrm{Re}\epsilon')_{\text{complete Dalitz plot}} \sim (-\tfrac{3}{2}\sqrt{2}\,\mathrm{Re}\epsilon')_{\text{restricted Dalitz plot}}\,, \tag{52}$$

where the last number incorporates a cut to the relatively small portion of the Dalitz plot with $s_3/4m_\pi^2 < 1.1$ while

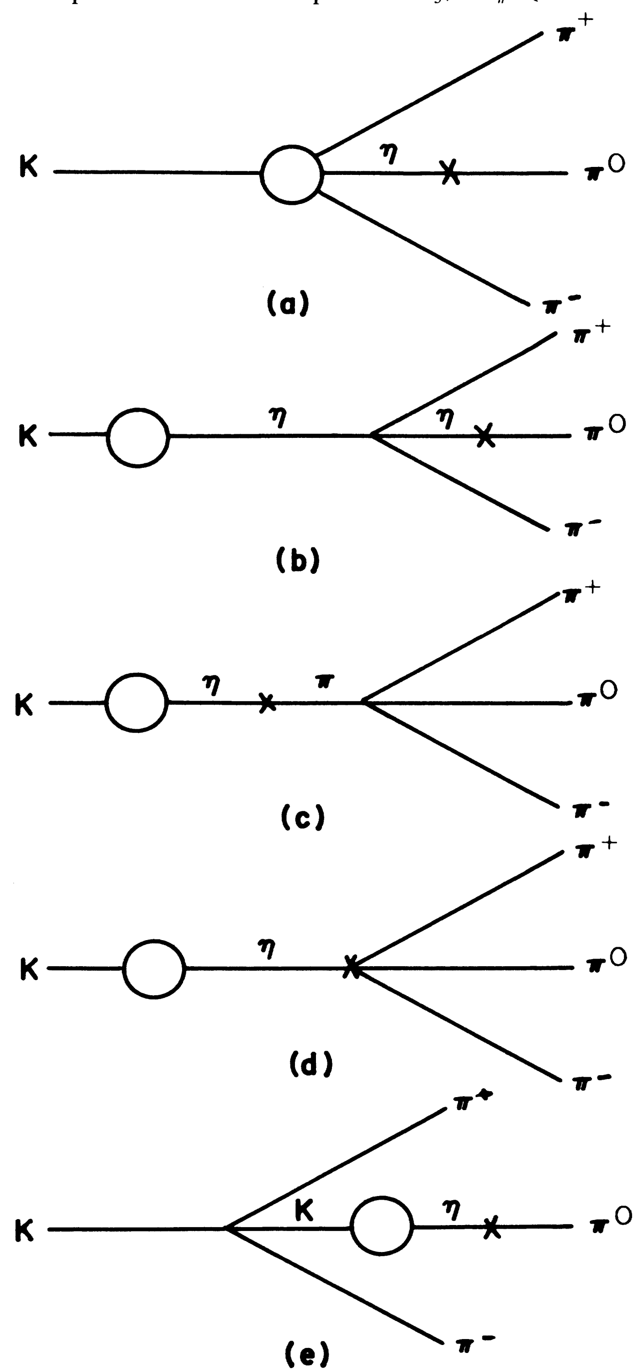

FIG. 1. The diagrams which contribute to the isospin-breaking effect in $K^0 \to \pi^+\pi^-\pi^0$. The notation is such that a circle represents the weak interaction, an $\times$ represents the isospin-breaking interaction, and all remaining vertices are the usual strong interaction.

the former number integrates over the whole Dalitz plot.

The direct CP-violating effects studied in this paper are not large. Presently operating experiments measuring η_{+-0} hope to have a sensitivity of order ϵ, and could not yet see the deviations of η_{+-0} from ϵ which are predicted here. In the more general class of CP-violating theories the measurement of η_{+-0} to order ϵ is, however, significant. For example, a theory of CP violation which is purely parity conserving could generate ϵ in the $K^0\bar{K}^0$ mass matrix and could have η_{+-0} as large as several times ϵ while generating no signal in ϵ' because the latter occurs in a parity-violating process. The only constraint on η_{+-0} would arise indirectly from the strength of the parity-conserving dispersive contributions to ϵ such as $K^0\to\pi^0\to\bar{K}^0$. However, the KM model contains both parity-conserving and parity-violating interactions. We have estimated the relation of these in $K\to 2\pi$ and $K\to 3\pi$, and found a larger ϵ'_{+-0} signal than previously thought. If future experiments can push the sensitivity in η_{+-0} down, our estimates suggest that ϵ'_{+-0} could be as large as 0.1ϵ if $\epsilon'/\epsilon\approx 0.005$.

APPENDIX A: THE ISOSPIN-BREAKING CORRECTION

The dominant effect of isospin breaking comes from the inequality of the up- and down-quark masses in

$$H_m = m_u \bar{u}u + m_d \bar{d}d \ . \tag{A1}$$

For example, this interaction induces π^0-η mixing:

$$\begin{aligned} A_{\pi\eta_8} &\equiv \langle \eta_8 | H_m | \pi^0 \rangle = -\frac{\sqrt{3}}{4}\left[\frac{m_d-m_u}{m_s-m_u}\right](m_\eta^{\;2}-m_\pi^{\;2}) \\ &\approx -\frac{\sqrt{3}}{4}\left[\frac{m_d-m_u}{m_s}\right]m_{\eta'}^{\;2} \ , \\ A_{\pi\eta_0'} &\equiv \langle \eta_0 | H_m | \pi^0\rangle = \sqrt{2}\langle \eta_8 | H_m | \pi^0\rangle \ , \end{aligned} \tag{A2}$$

with the first result following from SU(3) and the second using the quark model. At this stage we neglect $\eta\eta'$ mixing but will add it later. In $K\to 2\pi$, this mixing generates contributions to A_0 and A_2:

$$\begin{aligned} A(K^0\to\pi^+\pi^-) &= A_0^{\rm NL} + A_2^{\rm NL}/\sqrt{2} \equiv A_0 + A_2/\sqrt{2} \ , \\ A(K^0\to\pi^0\pi^0) &= A_0^{\rm NL} - \sqrt{2}A_2^{\rm NL} \\ &\quad + 2A(K^0\to\pi^0\eta)\frac{1}{m_\pi^{\;2}-m_\eta^{\;2}}A_{\pi\eta} \\ &\quad + 2A(K^0\to\pi^0\eta')\frac{1}{m_\pi^{\;2}-m_{\eta'}^{\;2}}A_{\pi\eta'} \\ &\equiv A_0 - \sqrt{2}A_2 \end{aligned} \tag{A3a}$$

or

$$\begin{aligned} A_0 &= A_0^{\rm NL} + \tfrac{2}{3}A(K^0\to\pi^0\eta)\frac{1}{m_\pi^{\;2}-m_\eta^{\;2}}A_{\eta\pi} \\ &\quad + \tfrac{2}{3}A(K^0\to\pi^0\eta')\frac{1}{m_\pi^{\;2}-m_{\eta'}^{\;2}}A_{\eta'\pi} \ , \\ A_2 &= A_2^{\rm NL} - \frac{2\sqrt{2}}{3}A(K^0\to\pi^0\eta)\frac{1}{m_\pi^{\;2}-m_\eta^{\;2}}A_{\pi\eta} \\ &\quad - \frac{2\sqrt{2}}{3}A(K^0\to\pi^0\eta')\frac{1}{m_\pi^{\;2}-m_{\eta'}^{\;2}}A_{\eta'\pi} \ . \end{aligned} \tag{A3b}$$

The imaginary parts of these amplitudes satisfy

$$\begin{aligned} \frac{{\rm Im}\,A(K^0\to\pi^0\eta_0)}{{\rm Im}\,A_0^{\rm NL}} &= -\frac{1}{\sqrt{3}} \ , \\ \frac{{\rm Im}\,A(K^0\to\pi^0\eta_0')}{{\rm Im}\,A_0^{\rm NL}} &= -\frac{2}{\sqrt{3}} \ . \end{aligned} \tag{A4}$$

Again the first result is due to SU(3), while the second follows in a quark model. To be specific, the important quark-model features are (1) ${\rm Im}\ A(K^0\to\eta')=\sqrt{2}\,{\rm Im}\,A(K^0\to\eta)$, which follows in all quark models, and (2) the momentum dependence of the $K\to\pi^0\eta'$ amplitude is such that the amplitude vanishes in the soft η' limit, as would be required by nonet symmetry or the vacuum-saturation method.

In $K\to 3\pi$, we must consider all the diagrams of Fig. 1. For η_8 all can be calculated using chiral SU(3). We find (setting $m_\pi^{\;2}=0$)

$$\begin{aligned} A(K^0\to\pi^+\pi^-\pi^0)\Big|_{\eta_8} &= -\frac{A_0^{\rm NL}A_{\eta_8\pi}}{6\sqrt{3}F_\pi m_{\eta_8}^{\;2}}\left[5+2\frac{s_3-s_0}{s_0}\right] && \text{(a+b)} \\ &\quad + \frac{A_0^{\rm NL}}{6\sqrt{3}F_\pi}\frac{A_{\eta_8\pi}}{m_{\eta_8}^{\;2}-m_K^{\;2}}\left[2+2\frac{s_3-s_0}{s_0}\right] && \text{(c+d)} \\ &\quad +0 && \text{(e)} \\ &= \frac{A_0^{\rm NL}A_{\eta_8\pi}}{\sqrt{3}F_\pi m_{\eta_8}^{\;2}}\left[1+2\frac{s_3-s_0}{s_0}\right] \ . \end{aligned} \tag{A5}$$

In the case of η', the above quark-model assumptions plus the vanishing of the $K_L\to\pi^+\pi^-\eta'$ amplitude in the soft η' limit, are sufficient to determine the form of the amplitudes. We find

$$A(K^0\to\pi^+\pi^-\pi^0)\Big|_{\eta_0'}=-\frac{\overline{A}A_{\eta_8'\pi}}{6F_\pi\sqrt{6}m_{\eta_0'}^2}\left[2-\frac{s_3-s_0}{s_0}\right]-\frac{\overline{A}A_{\eta_0'\pi}}{6F_\pi\sqrt{6}(m_{\eta_0'}^2-m_K^2)}\left[2+2\frac{s_3-s_0}{s_0}\right]+0 \tag{A6}$$

with $\mathrm{Im}\,\overline{A}=-2\,\mathrm{Im}\,A_0^{\mathrm{NL}}$.

At this stage we consider η-η' mixing using

$$\eta=\cos\theta\,\eta_8-\sin\theta\,\eta_0\,,\qquad \eta'=\sin\theta\,\eta_8+\cos\theta\,\eta_0 \tag{A7}$$

with $\theta\approx-20°$. Using (A2) and (A4), the $K\to 2\pi$ amplitudes become

$$\begin{aligned}\mathrm{Im}\,A_0&=\mathrm{Im}\,A_0^{\mathrm{NL}}\left[1-\left[\frac{m_d-m_u}{m_s}\right]\left[\frac{1}{6}(\cos\theta-\sqrt{2}\sin\theta)^2+\tfrac{1}{3}\frac{m_\eta^2}{m_{\eta'}^2}(\cos\theta+\sin\theta/\sqrt{2})^2\right]\right]\\&=\mathrm{Im}\,A_0^{\mathrm{NL}}\left[1-0.39\left[\frac{m_d-m_u}{m_s}\right]\right],\\ \mathrm{Im}\,A_2&=\frac{\mathrm{Im}\,A_0^{\mathrm{NL}}}{3\sqrt{2}}\left[\frac{m_d-m_u}{m_s}\right]\left[(\cos\theta-\sqrt{2}\sin\theta)^2+2\frac{m_\eta^2}{m_{\eta'}^2}(\cos\theta+\sin\theta/\sqrt{2})^2\right]\\&=0.55\,\mathrm{Im}\,A_0^{\mathrm{NL}}\left[\frac{m_d-m_u}{m_s}\right],\end{aligned} \tag{A8}$$

so that

$$\begin{aligned}\epsilon'&=-\frac{\omega}{\sqrt{2}}\frac{\mathrm{Im}\,A_0^{\mathrm{NL}}}{\mathrm{Re}\,A_0}\left[1-0.39\left[\frac{m_d-m_u}{m_s}\right]-0.55\frac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2}\left[\frac{m_d-m_u}{m_s}\right]\right]\\&=-\frac{\omega}{\sqrt{2}}\frac{\mathrm{Im}\,A_0^{\mathrm{NL}}}{\mathrm{Re}\,A_0}(1-\Omega_{\mathrm{IB}})\end{aligned} \tag{A9}$$

with

$$\Omega_{\mathrm{IB}}\approx 0.55\frac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2}\left[\frac{m_d-m_u}{m_s}\right]\approx 0.33\to 0.44\,. \tag{A10}$$

In the case of $K\to 3\pi$, the mixing is different in the two classes of diagrams (a + b) and (c + d) because of the different relative weights of the η_8 and η_0 contributions. Working at the center of the Dalitz plot we have

$$\begin{aligned}\mathrm{Im}\,A(K^0\to\pi^+\pi^-\pi^0)&=\frac{\mathrm{Im}\,A_0^{\mathrm{NL}}}{6F_\pi}\left\{1+\left[\frac{m_d-m_u}{m_s}\right]\left[-\tfrac{5}{4}\left[\cos\theta-\frac{2\sqrt{2}}{5}\sin\theta\right](\cos\theta-\sqrt{2}\sin\theta)\right.\right.\\&\qquad+2(\cos\theta+\sqrt{2}\sin\theta)(\cos\theta-\sqrt{2}\sin\theta)\\&\qquad-\frac{m_\eta^2}{m_{\eta'}^2}\left[\cos\theta+\frac{5\sqrt{2}}{4}\sin\theta\right](\cos\theta+\sin\theta/\sqrt{2})\\&\qquad\left.\left.-\frac{m_\eta^2}{m_{\eta'}^2-m_K^2}\left[\cos\theta-\frac{\sin\theta}{\sqrt{2}}\right]\left[\cos\theta+\frac{\sin\theta}{\sqrt{2}}\right]\right]\right\}\\&=\frac{\mathrm{Im}\,A_0^{\mathrm{NL}}}{6F_\pi}\left[1-1.15\left[\frac{m_d-m_u}{m_s}\right]\right].\end{aligned} \tag{A11}$$

This generates ϵ'_{+-0} using Eq. (46),

$$\epsilon'_{+-0}=i\frac{\text{Im}\,A_0^{\text{NL}}}{\text{Re}A_0}\left\{\frac{1-1.15\left[\dfrac{m_d-m_u}{m_s}\right]}{1-\dfrac{\sqrt{2}\,\text{Re}\,A_2}{\text{Re}\,A_0}}-\left[1-0.39\left[\frac{m_d-m_u}{m_s}\right]\right]\right\}$$

$$=i\frac{\text{Im}\,A_0^{\text{NL}}}{\text{Re}\,A_0}\left[\frac{\sqrt{2}\,\text{Re}\,A_2}{\text{Re}\,A_0}-0.76\left[\frac{m_d-m_u}{m_s}\right]\right]$$

$$=i\sqrt{2}\frac{\text{Re}\,A_2}{\text{Re}\,A_0}\frac{\text{Im}\,A_0^{\text{NL}}}{\text{Re}\,A_0}\left[1-0.54\frac{\text{Re}\,A_0}{\text{Re}\,A_2}\left[\frac{m_d-m_u}{m_s}\right]\right]\equiv i\sqrt{2}\omega\xi(1-\bar{\Omega}_{\text{IB}}) \qquad \text{(A12)}$$

with

$$\bar{\Omega}_{\text{IB}}=0.54\frac{\text{Re}\,A_0}{\text{Re}\,A_2}\left[\frac{m_d-m_u}{m_s}\right]\approx\Omega_{\text{IB}}\ . \qquad \text{(A13)}$$

Without η-η' mixing this value would have been

$$\bar{\Omega}_{\text{IB}}=-0.5\Omega_{\text{IB}}\ . \qquad \text{(A14)}$$

APPENDIX B: THE FACTORIZATION MODEL

In many ways, the most unfamiliar aspect of our analysis is the inclusion of terms beyond the usual linear expansion of the matrix element. We now know from experiment that these terms are present in the amplitude, and theoretically we can understand why they should be present from the general framework of chiral Lagrangians.[13] However it is also useful to see how such terms arise in an explicit calculation. In this section we use the vacuum-saturation technique to study the origin of these higher-order terms. We will learn that the scale factor for the strength of the new terms, which enter at order $m_K{}^2/\Lambda^2$, is about 1 GeV, and that the higher-order terms may have a different *CP*-violating phase than that found in the lowest-order result. The vacuum-saturation method is *not* presented as a reliable approach in this regard, and the results are not to be trusted in detail. Rather our calculation is simply meant as an illustration of the types of effects which one can expect.

We focus on the $\Delta I=\frac{1}{2}$ part of the interaction and include only the operators O_I and O_5 in the Hamiltonian [see Eq. (2)], with

$$O_1=\bar{d}\gamma_\mu(1+\gamma_5)u\bar{u}\gamma^\mu(1+\gamma_5)s$$
$$-\bar{u}\gamma_\mu(1+\gamma_5)u\bar{d}\gamma^\mu(1+\gamma_5)s\ . \qquad \text{(B1)}$$

Because of some interest in charged-kaon decay, we present the analysis in the channel $K^+\to\pi^+\pi^+\pi^-$, but since the operators are purely $\Delta I=\frac{1}{2}$, the results can be easily transcribed to neutral-kaon decay.

The vacuum-saturation method involves current matrix elements such as

$$\langle 0\,|\,\bar{u}\gamma_\mu\gamma_5 d\,|\,\pi^-\rangle=-i\sqrt{2}F_\pi p_\mu\ ,$$
$$\langle\pi^+(p)\,|\,\bar{s}\gamma_\mu d\,|\,K^+(k)\rangle=(k+p)_\mu f_+[(k-p)^2]\ , \qquad \text{(B2)}$$
$$\langle\pi^+(p)\pi^-(q)\,|\,\bar{s}\gamma_\mu\gamma_5 u\,|\,K^+(k)\rangle=\frac{i\sqrt{2}}{F_\pi}\left[p_\mu+\frac{p\cdot p'}{m_K{}^2-m_\pi{}^2}p'_\mu\right]$$

with $p'_\mu=(k-p-q)_\mu$. We work in the SU(3) limit and use PCAC and crossing to determine other related matrix elements, such as

$$\langle\pi^+(p_1)\pi^-(q)\pi^+(p_2)\,|\,\bar{u}\gamma_\mu\gamma_5 d\,|\,0\rangle=\frac{i2\sqrt{2}}{F_\pi}\left[q_\mu-\frac{k\cdot q}{m_K{}^2-m_\pi{}^2}k_\mu\right] \qquad \text{(B3)}$$

with $k_\mu=(p_1+p_2+q)_\mu$. However, the form factors will in general be a function of q^2 such as

$$f_+(q^2)=1+\frac{q^2}{\Lambda^2}+\cdots\ . \qquad \text{(B4)}$$

In this case the q^2 variation of the form factors can be related to each other by PCAC. We find

$$\langle\pi^+\,|\,\bar{s}\gamma_\mu d\,|\,K^+\rangle=(k+p)_\mu\left[1+\frac{(k-p)^2}{\Lambda^2}\right]\ ,$$
$$\langle 0\,|\,\bar{u}\gamma_\mu\gamma_5 d\,|\,\pi^-\rangle=-i\sqrt{2}F_\pi(1+m_\pi{}^2/\Lambda^2)p_\mu\ ,$$
$$\langle 0\,|\,\bar{s}\gamma_\mu\gamma_5 u\,|\,K^+\rangle=-i\sqrt{2}F_\pi(1+m_K{}^2/\Lambda^2)p_\mu\ ,$$
$$\langle\pi^+(p)\pi^-(q)\,|\,\bar{s}\gamma_\mu\gamma_5 u\,|\,K^+(k)\rangle \qquad \text{(B5)}$$
$$=\frac{+i\sqrt{2}}{F_\pi}\left[p_\mu+\frac{p\cdot p'}{m_K{}^2-m_\pi{}^2}p'_\mu\right]\left[1+\frac{p'^2}{\Lambda^2}+\frac{(p+q)^2}{\Lambda_2{}^2}\right]\ .$$

In the last case Λ_2 is not determined by PCAC, and we will not include it in our example. Similar matrix elements can be defined for scalar and pseudoscalar densities

$$\langle\pi^+\,|\,\bar{s}d\,|\,K^+\rangle=A\left[1+\frac{(k-p)^2}{\Lambda^2}\right]\ ,$$
$$\langle 0\,|\,\bar{s}\gamma_5 d\,|\,K^+\rangle=-i\sqrt{2}F_\pi A(1+m_K{}^2/\Lambda^2)\ , \qquad \text{(B6)}$$
$$\langle\pi^+(p)\pi^-(q)\,|\,\bar{s}\gamma_5 u\,|\,K^+(k)\rangle=\frac{i\sqrt{2}A}{F_\pi}\frac{p\cdot p'}{m_K{}^2-m_\pi{}^2}\left[1+\frac{(k-p-q)^2}{\Lambda^2}\right]\ .$$

While in principle the scalar densities could have a different Λ than the currents, the quark equations of motion suggest that they are the same. Our experience with form factors suggests that Λ should be of order $m_\rho{}^2$ or $m_{K^*}{}^2$. Given these definitions it is straightforward to calculate the decay amplitudes in the vacuum-saturation method (we use $m_\pi{}^2=0$ for clarity):

$$\begin{aligned}\langle \pi_1^+\pi^-\pi_2^+ \mid O_1 \mid K^+\rangle &= -\tfrac{2}{3}[\langle \pi_1^+ \mid \bar{u}\gamma_\mu\gamma_5 d \mid 0\rangle\langle \pi^-\pi_2^+ \mid \bar{s}\gamma_\mu\gamma_5 u \mid K^+\rangle + (1\leftrightarrow 2) \\ &\quad -\langle \pi_1^+\pi^- \mid \bar{u}\gamma_\mu u \mid 0\rangle\langle \pi_2^+ \mid \bar{s}\gamma_\mu d \mid K^+\rangle - (1\leftrightarrow 2) + \langle \pi_1^+\pi^-\pi_2^+ \mid \bar{u}\gamma_\mu\gamma_5 d \mid 0\rangle\langle 0 \mid \bar{s}\gamma_\mu\gamma_5 u \mid K^+\rangle] \\ &= \tfrac{4}{9}[m_K{}^2 - \tfrac{3}{2}(s_- - s_0)] - \tfrac{4}{3}\frac{m_K{}^2}{\Lambda^2}(s_- - s_0) + \frac{2}{\Lambda^2}[(s_- - s_0)^2 - (s_1 - s_2)^2/3] \end{aligned} \tag{B7}$$

and

$$\begin{aligned}\langle \pi_1^+\pi^-\pi_2^+ \mid O_5 \mid K^+\rangle &= -\tfrac{32}{9}[\langle \pi_1^+ \mid \bar{u}\gamma_5 d \mid 0\rangle\langle \pi_2^+\pi^- \mid \bar{s}\gamma_5 u \mid K^+\rangle + (1\leftrightarrow 2) \\ &\quad -\langle \pi_1^+\pi^- \mid \bar{d}d \mid 0\rangle\langle \pi_2^+ \mid \bar{s}d \mid K^+\rangle - (1\leftrightarrow 2) + \langle \pi^+\pi^-\pi^+ \mid \bar{u}\gamma_5 d \mid 0\rangle\langle 0 \mid \bar{s}\gamma_5 u \mid K^+\rangle] \\ &= -\frac{128}{27}\frac{A^2}{\Lambda^2}[m_K{}^2 - \tfrac{3}{2}(s_- - s_0)] + \frac{32}{3}\frac{A^2}{\Lambda^4}[\tfrac{4}{3}m_K{}^2(s_- - s_0) - \tfrac{10}{9}m_K{}^4 + (s_- - s_0)^2/2 + (s_1 - s_2)^2/2] \ . \end{aligned} \tag{B8}$$

We see that the leading-order term is unique but at higher order the two operators have quite different behavior.

For the purposes of our example, we will normalize the coefficients of the operators such that the penguin operator O_5 contributes a fraction f to $K \to 2\pi$, O_1 contributes a fraction $(1-f)$, and we give the penguin contribution a phase ϕ:

$$\frac{G_F}{2\sqrt{2}}\cos\theta_1 \sin\theta_1 \langle \pi^+\pi^- \mid c_1 O_1 + c_5 O_5 \mid K^0\rangle = (1-f)A_0 + fA_0 e^{i\phi} \ . \tag{B9}$$

Note that in this picture our previously defined phase is $\xi = f\phi$. Again using vacuum saturation on $K \to 2\pi$, we find

$$\begin{aligned} c_1\langle \pi^+\pi^- \mid O_1 \mid K_0\rangle &= c_1 \frac{i4F_\pi}{3\sqrt{2}} m_K{}^2 \equiv (1-f)A_0 \ , \\ c_5\langle \pi^+\pi^- \mid O_5 \mid K^0\rangle &= c_5 \left[\frac{64}{9\sqrt{2}}\right] iF_\pi A^2 \left[\frac{2m_K{}^2}{\Lambda^2} + \frac{m_K{}^4}{\Lambda^4}\right] \equiv fA_0 e^{i\phi} \ . \end{aligned} \tag{B10}$$

Putting these together and factoring out the phase factor which is common with $K \to 2\pi$, we get

$$\begin{aligned}\langle \pi^+\pi^-\pi^0 \mid H_w \mid K^0\rangle = \frac{iA_0 e^{i\xi}}{6F_\pi}\Bigg[& 1 + \frac{3}{m_K{}^2}(s_3 - s_0) + \frac{6(s_3 - s_0)}{\Lambda^2} + \frac{39}{\Lambda^2 m_K{}^2}[(s_3 - s_0)^2 - (s_+ - s_-)^2/3] \\ & + \frac{fe^{i\xi(1-f)/f}}{\Lambda^2}\left[2m_K{}^2 - \tfrac{3}{2}(s_3 - s_0) + \frac{3}{2m_K{}^2}(s_+ - s_-)^2 - 9(s_3 - s_0)^2/m_K{}^2\right]\Bigg] \ . \end{aligned} \tag{B11}$$

Note that the higher-order terms do indeed induce a CP-violating phase within the (purely $\Delta I = \frac{1}{2}$) amplitude. Taking real and imaginary parts and using the definition of ϵ'_{+-0} [Eq. (46)] we find, at the center of the Dalitz plot,

$$\epsilon'_{+-0} \approx i\xi 2(1-f)\frac{m_K{}^2}{\Lambda^2} \ . \tag{B12}$$

If we estimate this quantity, using $\Lambda \approx 1$ GeV, ξ from Eq. (29) and $f \approx \frac{1}{2}$, we find

$$\epsilon'_{+-0} \approx i\,10\epsilon' \tag{B13}$$

consistent with our previous estimate of the effect of higher-order terms. This sample calculation has demonstrated then how, within a specific model, such higher effects occur and has reinforced our estimate of their magnitude.

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[10]The continuation of final-state phase shifts is problematic in lowest-order PCAC. We will throughout ignore these phase shifts, so that ϵ'_{+-0} is purely imaginary.

[11]As is well known, the charge asymmetry will generate $K_S \to \pi^+\pi^-\pi^0$ even in the absence of CP violation. This amplitude, however, is suppressed because it is $\Delta I = \frac{3}{2}$, and the resulting (CP-even) rate is small. The value of β is known via PCAC to be (see Ref. 12) $\beta = -\frac{27}{2}\omega$. In subsequent formulas, we will not explicitly include this term because it plays no role in our discussion.

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