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Observations on path-integral derivations of anomalies

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We make a number of observations on the path-integral derivations of anomalies by Fujikawa.

In recent years, Fujikawa has derived all known anomaly equations in the path-integral formalism in a series of remarkable papers.[1-5] Others have elaborated on various aspects of these derivations and offered alternatives or modified derivations of these results.[6-10] These results have also been extended in various ways.

We feel that in view of the remarkable success of Fujikawa's idea of interpreting anomalies as Jacobian factors, a deeper study of these results is necessary. In this context, we wish to offer a number of observations pertinent to these derivations which we feel have not been emphasized sufficiently or have been missed.

In order to be able to state our comments in the proper perspective and to introduce our notations we shall give a brief account of the derivation of the chiral anomaly by Fujikawa.[1] More details of the derivation can be found in Ref. 1. Our notations and conventions are those of Ref. 1.

Consider a Lagrangian in Euclidean space for a system of fermions and non-Abelian gauge fields:

$$\mathscr{L}=\frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})+\bar{\psi}(i\not{D}-m_0)\psi \ , \tag{1}$$

$$\not{D}\equiv\not{\partial}+\not{A} \ . \tag{2}$$

One encloses the system in a four-dimensional box so that $\not{D}$ has discrete eigenvalues and considers eigenfunctions of the Hermitian operator $\not{D}$:

$$\not{D}\phi_n=\lambda_n\phi_n \ , \quad \int \phi_n^\dagger\phi_m d^4x=\delta_{mn} \ . \tag{3}$$

One then expands ψ and $\bar{\psi}$ in terms these orthonormal eigenfunctions

$$\psi(x)=\sum_n a_n\phi_n(x) \ , \qquad \bar{\psi}(x)=\sum_n \phi_n^\dagger(x)\bar{b}_n \ , \tag{4}$$

where a_n and $\bar{b}_n$ are Grassmann variables. One then *defines* the measure to be

$$\mathscr{D}\psi\,\mathscr{D}\bar{\psi}\equiv\prod_{n=1}^{\infty}da_n\,db_n \ .$$

One then considers the behavior of the measure under infinitesimal local transformations:

$$\psi(x)\to\psi(x)+i\alpha(x)\gamma_5\psi(x) \ , \tag{5}$$

$$\bar{\psi}(x)\to\bar{\psi}(x)+i\alpha(x)\bar{\psi}(x)\gamma_5 \ , \tag{6}$$

used in the derivation of Ward-Takahashi (WT) identity involving $\partial^\mu J_\mu^5$, where

$$J_\mu^5\equiv\bar{\psi}\gamma_\mu\gamma_5\psi \ . \tag{7}$$

One finds that the Jacobian for the transformations of Eqs. (5) and (6) is

$$\begin{aligned} J&=1+\Delta \ , \\ \Delta&=-2i\int d^4x\,\alpha(x)\sum_n\phi_n^\dagger(x)\gamma_5\phi_n(x) \\ &\equiv-2i\int d^4x\,\alpha(x)A(x) \ . \end{aligned} \tag{8}$$

This leads to an extra term proportional to $\sum_n\phi_n^\dagger(x)\gamma_5\phi_n(x)$ in the equation for $\partial^\mu J_\mu^5$ and when evaluated leads to the anomaly term in the WT identity.

$\sum_n\phi_n^\dagger(x)\gamma_5\phi_n(x)$ is evaluated by first regularizing it by a cutoff, such as

$$\begin{aligned} A_M(x)&\equiv\sum_n\phi_n^\dagger(x)e^{-\lambda_n{}^2/M^2}\gamma_5\phi_n(x) \\ &=\sum_n\phi_n^\dagger(x)\gamma_5 e^{-\not{D}^2/M^2}\phi_n(x) \ . \end{aligned} \tag{9}$$

A_M is transformed into an integral over k by going to a plane-wave basis.[1] The result is

$$A_M(x)=M^4\int\frac{d^4k}{(2\pi)^4}e^{-k^2}\exp\left[\frac{D^2}{M^2}+\frac{i\sigma\cdot F}{2M^2}+\frac{2ik\cdot D}{M}\right] . \tag{10}$$

One then defines

$$A(x)=\lim_{M\to\infty}A_M(x) \tag{11}$$

yielding the result

$$A(x)=-\frac{1}{16\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{12}$$

which gives the anomaly term in the anomaly equation

$$\partial^\mu J_\mu^5(x)2m_0 i\bar{\psi}\gamma_5\psi-\frac{i}{8\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \ . \tag{13}$$

This, in brief, is the derivation of the chiral anomaly and a similar procedure when adopted for other anomalies has worked and given appropriate results.[2-5] Now, we shall make a number of observations.

(a) First we shall make a minor but basic observation useful in understanding later observations.

If one were to expand ψ and $\bar{\psi}$ in terms of eigenfunctions X_n of some Hermitian operator X, one would still get $A=\sum_n \chi_n^\dagger \gamma_5 \chi_n$ which is *formally* the same as $\sum_n \phi_n^\dagger \gamma_5 \phi_n$ (Ref. 1). When A is regularized as

$$A_M \equiv \sum_n \chi_n^\dagger(x)\gamma_5\chi_n(x)e^{-\lambda_n{}^2/M^2}=\sum_n \chi_n^\dagger \gamma_5 e^{-X^2/M^2}\chi_n \ ,$$

the result for $\lim_{M\to\infty} A_M$ crucially depends on the choice of X which is well known. Thus the Jacobian factor gives anomaly correctly only when regularized in a particular fashion in terms of eigenvalues of the operator $\not{D}$.

Further the regularization of the Jacobian can be directly obtained from a modified Lagrange density.[8] When the Lagrangian is so modified, Green's functions of the theory are also regularized in terms of the same cutoff M. All this suggests that the results of Fujikawa are rigorously valid in a scheme in which all Green's functions are regularized in terms of the eigenvalues of the energy operator $\not{D}$ and alike.

The modification of the Lagrange density in Ref. 8 is, however, necessarily of the nonpolynomial type. This essentially rests upon the properties required of the regularization function for which $f(0)=1$ and $f(\infty)=f'(\infty)=f''(\infty)=\cdots=0$ (Ref. 1). This precludes f from being a polynomial. Hence the usual theorems of renormalization which apply only to local polynomial Lagrangians do not apply to the modified Lagrange density in Ref. 8, which contains derivatives of fields to arbitrarily high order and is therefore essentially nonlocal.

Thus the formalism in which Fujikawa's results can be consistently and completely understood needs to be developed.

We also feel that since the anomaly is correctly obtained as Jacobian factors only when ψ and $\bar\psi$ are expanded in terms of eigenfunctions of $\not{D}$ (or functions of $\not{D}$) and analogous "energy operators"[1] this fact strongly indicates that there is something profound in regularizing all Green's functions in terms of eigenvalues of "energy operators."

(b) Next we make the following observation and elaborate on it. In the formalism in which Fujikawa's derivations of anomalies are valid, not only do the anomaly equations differ from their classical counterpart as they should, but *most equations of motion differ from their classical counterparts,* i.e., *are themselves "anomalous." Anomaly equations in Fujikawa's formalism are very special cases of the anomalous equations of motion themselves.*

To elaborate, an equation of motion say in ϕ^4 theory, dimensionally regularized, is an equation of the form

$$\left\langle \frac{\delta S}{\delta\phi(x)}F[\varphi(x)]\right\rangle + J(x)\langle F[\phi(x)]\rangle = 0 \ , \tag{14}$$

where $F[\phi(x)]$ is an arbitrary local functional of $\phi(x)$ and

$$\langle O(x)\rangle \equiv \frac{\int \mathscr{D}\phi \exp\left[iS[\phi]+i\int d^nx\, J(x)\phi(x)\right]O(x)}{\int \mathscr{D}\phi \exp\left[iS[\phi]+i\int d^nx\, J(x)\phi(x)\right]}$$

$$\equiv \frac{1}{Z[J]}O\left[-i\frac{\delta}{\delta J(x)}\right]Z[J] \ .$$

The above equation is usually obtained from $Z[J]$ by a change in the integration variable $\phi(x)\to\phi(x)+\epsilon F[\phi(x)]$ where ϵ is infinitesimal and equating the resulting change to zero noting that the Jacobian for the transformation is one because in dimensional regularization $\delta^n(0)$ and derivatives of $\delta^n(x)$ at $x=0$ vanish. See, for example, Ref. 11.

Thus the above equation in the dimensionally regularized functional formalism is not "anomalous" in the sense that they coincide with the classical result in form. This is not generally true in Fujikawa's formulation. This is so because the Jacobian for the transformation

$$\phi(x)\to\phi(x)+\epsilon F[\phi(x)]$$

is generally nontrivial and depends in a complicated fashion on the local functional $F[\phi(x)]$ and the equations of motion themselves are modified in a nontrivial way that depends on the functional $F[\phi(x)]$ in a characteristic manner, and the few anomaly equations are very special cases (linear combinations of) the above equations of motion.

As an example, if one makes the transformation of Eq. (5) *only* on the functional integral for the non-Abelian gauge theory (this is possible as ψ and $\bar\psi$ can be treated as independent Grassmann variables)

$$Z[J,\eta,\bar\eta] \equiv \int \mathscr{D}A\,\mathscr{D}\psi\,\mathscr{D}\bar\psi \exp\left[i\left[S_{\rm eff}+\int d^nx\,(J^\mu A_\mu+\bar\eta\psi+\bar\psi\eta)\right]\right]$$

one obtains

$$\left\langle -\frac{\delta S_{\rm eff}}{\delta\psi}\gamma_5\psi+\bar\eta\gamma_5\psi\right\rangle = \frac{1}{16\pi^2}{\rm Tr}(F_{\mu\nu}\tilde F^{\mu\nu}) \ , \tag{15}$$

whereas if one makes a transformation of Eq. (6) only, one obtains

$$\left\langle \bar\psi\gamma_5\frac{\delta S_{\rm eff}}{\delta\bar\psi}+\bar\psi\gamma_5\eta\right\rangle = \frac{1}{16\pi^2}{\rm Tr}(F_{\mu\nu}\tilde F^{\mu\nu}) \ . \tag{16}$$

The anomaly equation itself is only a linear combination of the above two equations and is obtained by noting

$$\partial^\mu J_\mu^5 \equiv i\left[\frac{\delta S_{\rm eff}}{\delta\psi}\gamma_5\psi-\bar\psi\gamma_5\frac{\delta S_{\rm eff}}{\delta\bar\psi}\right]+2im_0\bar\psi\gamma_5\psi$$

and

$$\langle\partial^\mu J_\mu^5\rangle = 2m_0 i\langle\bar\psi\gamma_5\psi\rangle + i\langle\bar\psi\gamma_5\eta+\bar\eta\gamma_5\psi\rangle - \frac{i}{8\pi^2}\langle{\rm Tr}(F_{\mu\nu}\tilde F^{\mu\nu})\rangle \ . \tag{17}$$

As a further example of this consider scalar electrodynamics:

$$\mathscr{L}=(D_\mu\phi)^*D^\mu\phi-m_0{}^2\phi^*\phi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ .$$

This theory does not have any standard anomalies associated with currents. Yet the equations of motion in Fujikawa's formalism are anomalous. For example, one obtains straightforwardly, following the procedure of Ref. 1,

$$\left\langle \frac{\delta S}{\delta\phi}\phi \right\rangle + \langle J\phi \rangle = \frac{1}{192\pi^2} F_{\mu\nu}F^{\mu\nu} ,$$

$$\left\langle \frac{\delta S}{\delta\phi^*}\phi^* \right\rangle + \langle J^*\phi^* \rangle = \frac{1}{192\pi^2} F_{\mu\nu}F^{\mu\nu} ,$$

in obvious notations. Similar but much more complicated equations are obtained in the case of general transformations. The vector current, however, does not have an anomaly because

$$\partial^\mu J_\mu \propto \frac{\delta S}{\delta\phi^*}\phi^* - \frac{\delta S}{\delta\phi}\phi$$

and the anomalous term cancels in the above linear combination.

We should mention that some of the anomalous equations of motion appear in Fujikawa's work itself.[2] Espriu[12] has also made an observation somewhat along similar lines in the special context of renormalization of gauge-invariant operators. We feel that this point deserves a great emphasis and deeper study. To reiterate, anomaly equations in Fujikawa's formulations are but a few special cases of anomalous equations of motion which are of fundamental importance to the theory.

(c) Derivations of anomalies along the lines of Refs. 1–5 have been often labeled as "nonperturbative" (see, e.g., Ref. 6), i.e., exact results independent of perturbation theory. This label has been generally applied to the chiral anomaly in various theories. We would like to emphasize that this is untrue. All the results obtained via this approach as of now are true only in the *one-loop approximation.* Nonrenormalization of the chiral anomaly allows one to maintain the misconception that these results are exact. We shall state a number of reasons why the results are of one-loop order.

(i) One does not know how to calculate any quantity in QCD exactly, i.e., without recourse to perturbation theory. Renormalization can be carried out only in the context of perturbation theory. As pointed out in (a) above, it is necessary to first develop a perturbation scheme based on a regularization involving eigenvalues of operator $\not{D}$ for calculating Green's functions. Only in the context of such a scheme can one possibly discuss higher-order corrections to anomalies. In other words, in the absence of such a scheme one cannot make any statement about the effects beyond one-loop order.

(ii) If the procedure of Refs. 1–5 were to yield exact nonperturbative results, one should be able to obtain exact trace anomaly[13] in the context of the path-integral method. As is well known this procedure only leads to the leading contribution to trace anomaly.[2] In fact the calculations of Jacobian factors have so far been done by treating the gauge field or gravitational field as external fields. Unless one knows how to handle these fields in higher orders in the context of a scheme described in (a) one cannot derive the results to all orders.

(iii) In the WT identity (i.e., anomaly equation) for $\partial^\mu J_\mu{}^5$ the anomalous term appears as a functional integral,

$$\int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} \left[\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x)e^{-\lambda_n{}^2/M^2} \right] \exp[i(S_{\text{eff}}+\text{source terms})] , \tag{18}$$

where $\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x)$ which is a functional of A_μ^α, the gauge fields. Thus

$$\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x)e^{-\lambda_n{}^2/M^2}$$

is *inside* the functional integral. We can evaluate

$$A_M(x) \equiv \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x)e^{-\lambda_n{}^2/M^2}$$

as an infinite series in $1/M^2$ for a finite M, viz.,

$$A_M(x) = -\frac{1}{16\pi^2}\text{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) + \frac{O(6)}{M^2} + \frac{O(8)}{M^4} + \cdots + \frac{O(2n)}{M^{2n-4}} + \cdots , \tag{19}$$

where $O(6)$, $O(8)$, ..., are gauge-invariant functionals of dimension 6,8, ..., respectively. $O(6)$, $O(8)$, ..., are themselves of $O(g_0{}^2)$ (or higher). Now in evaluating $\langle A_M(x) \rangle$ to $O(g_0{}^2)$, one needs to take tree Green's functions of $O(6)$, $O(8)$, ... which are always finite. Hence, *to this order*

$$\lim_{M^2\to\infty} \langle A_m(x) \rangle = \left\langle \lim_{M^2\to\infty} A_M(x) \right\rangle . \tag{20}$$

However, this is not true in higher orders. [The reasons for this are elaborated in the next observation (d) below.] This clearly indicates the *leading-order character* of the anomaly derivation.

(d) We now make a number of comments about Eq. (19). Firstly the series of Eq. (19) which represents the regularized Jacobian factor $A_M(X)$ will have to be dealt with if one is to prove Adler-Bardeen theorem or if one is to derive trace anomaly to all orders in the context of path-integral formalism. To see that the higher-order terms $O(2n)/M^{2n-4}$, $n=3,4,\ldots$, could contribute in higher orders of perturbation theory, imagine that a perturbation scheme as envisaged in (a) is used in which Green's functions are also calculated with a cutoff on the eigenvalues of operator $\not{D}$. Then the Green's functions of $O(2n)$ will generally contain divergences of order M^{2n-4} up to factors of logarithms of M^2. [There are no divergences of order M^{2n-2} as can be shown from gauge invariance of $O(2n)$.] This requires a detailed analysis presented elsewhere.[14] Now when the Green's functions of $O(2n)/M^{2n-4}$ are calculated for finite M^2 and then M^2 is let go to infinity, $\langle O(2n) \rangle / M^{2n-4}$ may contain finite pieces as well as pieces that diverge as $(\ln M^2)^p$. In any case, these terms will lead to nonvanishing and probably divergent contributions in higher orders.

We have analyzed this for chiral anomaly elsewhere (14) and from the *form* of the operators $O(2n)$ established that

$$\langle A_M(x)\rangle = -\frac{1}{16\pi^2}\mathrm{Tr}\langle F_{\mu\nu}\widetilde{F}^{\mu\nu}\rangle + g_0{}^4 f(g_0{}^2)\langle \partial^\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)\rangle \ , \qquad (21)$$

where $f(g_0{}^2)$ is an infinite series in $g_0{}^2$.

We have also analyzed the similar series in the context of trace anomaly and established that if the trace anomaly to all orders is to be given correctly by the Jacobian factor, the higher-order terms of the form $O(2n)/M^{2n-4}$, $n=3,4,\ldots$ must contribute and in fact yield a logarithmic divergent [$(\ln M^2)^p$-type] contribution.[15]

It has recently been pointed out[16] that one should distinguish generally between linear and nonlinear systems when dealing with higher-order effects. We should, however, point out that our observation made above in (c)(iii) and its elaboration in (d) regarding higher-order contributions to the Jacobian is equally valid for both the linear and the nonlinear systems.

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