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Anomalies in conservation laws in quantum mechanics

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It has been pointed out that a simple quantum-mechanical system, involving a charged particle moving in a uniform magnetic field, can exhibit what looks like an anomaly. This note analyzes the problem, and shows that in such cases the anomaly is (in some sense) already present at the classical level.

A couple of years ago, Manton[1] described a curious kind of anomaly which occurs in a simple quantum-mechanical system, namely, that of a charged particle moving on a flat two-dimensional torus in a constant background magnetic field. He pointed out that there is a translation (momentum) operator P which commutes with the Hamiltonian, but whose expectation value is not conserved. One way[2] of seeing why Ehrenfest's theorem is side-stepped in this example is to observe that the operator P does not preserve the domain on which the Hamiltonian H is Hermitian. The Schrödinger equation implies that

$$\frac{d}{dt}\langle P\rangle = i\langle [H,P]\rangle + i\langle (H^{\dagger}-H)P\rangle \ .$$

Even though $[H,P]=0$, and $H^{\dagger}=H$ when acting on a "physical state" ψ, the state $P\psi$ is not physical, and $\langle (H^{\dagger}-H)P\rangle$ is nonzero. So $d\langle P\rangle/dt$ does not vanish.

The purpose of this note is to examine this type of system in somewhat greater generality: namely, we look at a charged particle moving on some manifold M, in a background magnetic field that is not necessarily uniform. It turns out that if there is a *classical* constant of motion which is well defined, then the corresponding *quantum* quantity is also conserved. In other words, in the example[1] referred to above, the classical constant of motion is already ill defined, and so this is not really an example of an anomaly in the usual sense of word. But, of course, it remains relevant to the anomalous field theory (the Schwinger model) discussed in Ref. 1.

Consider a charged scalar particle moving on a manifold M, in the presence of a magnetic field. This means that we have a complex line bundle L over M, and the Hilbert space of wave functions is the space $\mathcal{H}$ of square-integrable sections of L. On L there is a connection D_j, which locally (with a choice of gauge) can be written[3] $D_j=\partial_j+iA_j$. The magnetic field is $F_{jk}=-i[D_j,D_k]$.

Let v^j be a vector field on M. The magnetic field is invariant under motion along v^j if its Lie derivative[4] $\mathcal{L}_v F_{jk}$ vanishes. Since we have

$$\mathcal{L}_v F_{jk}=\partial_j(v^l F_{lk})-\partial_k(v^l F_{lj}) \ , \tag{1}$$

this invariance condition can also be written

$$v^j F_{jk}=-\partial_k \Psi \tag{2}$$

for some scalar function Ψ (see Ref. 4, Eq. 1.7). However (and this is a crucial point), the vanishing of the expression (1) only guarantees that Ψ exists *locally* on M; as we shall see later, it may not exist globally.

Assuming for the time being that Ψ *does* exist globally, we know[4] that it appears in the constant of motion associated with the symmetry v^j. Let us first recall what this looks like at the classical level. We have to introduce a metric g_{jk} on M, with associated connection ∇_j, and we have to assume that v^j is a Killing vector:

$$\nabla_j v_k+\nabla_k v_j=0 \ , \tag{3}$$

i.e., v^j is a symmetry of g_{jk} as well as of the magnetic field. The classical equations of motion are

$$\ddot{x}^{\,j}=F^j{}_k\dot{x}^{\,k} \tag{4}$$

and the constant of motion is $\dot{x}^{\,j}v_j+\Psi$. Proof:

$$\begin{aligned}\frac{d}{dt}(\dot{x}^{\,j}v_j)&=\ddot{x}^{\,j}v_j+\dot{x}^{\,j}\dot{x}^{\,k}\nabla_k v_j\\ &=F_{jk}\dot{x}^{\,k}v^j+0 \quad \text{using (3) and (4)}\\ &=-\dot{x}^{\,k}\partial_k\Psi \ \text{by (2)}\\ &=-\frac{d}{dt}\Psi \ .\end{aligned}$$

We now want to extend this structure to the quantum-mechanical level. In order to do so, one has to "lift" the vector field v^j to a Hermitian operator V on the Hilbert space $\mathcal{H}$, compatible with the connection D_j. The lifting condition is that V should have the form

$$V = iv^j D_j + f \ , \tag{5}$$

where f is a real-valued function on M. The compatibility condition is

$$[D_j, V] = i(\nabla_j v^k)D_k \tag{6}$$

(acting on scalars). The significance of (6) is that it implies that V commutes with the Hamiltonian $H = -\frac{1}{2}g^{jk}D_j D_k$. Proof: acting on a scalar, the commutator is[5]

$$\begin{aligned}-2[H,V] &= D^j[D_j,V] + [D_j,V]D^j \\ &= iD^j(\nabla_j v^k)D_k + i(\nabla_j v^k)D_k D^j \\ &\quad - iR_{jk}v^j D^k, \text{ using (6)} \\ &= i(\Delta v^k)D_k + i(\nabla^j v^k)(D_j D_k + D_k D_j) - iR_{jk}v^j D^k \\ &= 0 \text{ using (3)} .\end{aligned}$$

In fact, the conditions (5) and (6) imply that

$$v^k F_{kj} = -\partial_j f \ ,$$

which is the same as (2), with $f = \Psi$.

To sum up, if v^j is a Killing vector, and the magnetic field F_{jk} satisfies (2) for some globally defined scalar Ψ, then the corresponding operator V commutes with the Hamiltonian, and its expectation value is constant in time. The point is that V maps physical states (i.e., elements of $\mathcal{H}$) to physical states, and so the kind of problem referred to at the beginning does not occur in this case. On the other hand, if Ψ is *not* globally well defined, then a problem *can* occur (see example 2 below).

Example 1. The Dirac monopole. Here the magnetic field on $M = \mathbb{R}^3 - \{0\}$, namely

$$F_{jk} = qr^{-3}\epsilon_{jkl}x^l \quad (q = \text{const}) \ ,$$

is spherically symmetric. So take v^j to be a rotation:

$$v^j = \epsilon^{jkl}x_k\omega_l \quad (\omega_l = \text{const}) \ .$$

Then (2) is satisfied, with

$$\Psi = qr^{-1}\omega_j x^j \ .$$

Since Ψ is well defined on M, no problems arise: angular momentum is conserved both classically and quantum mechanically.

Example 2. Constant magnetic field on a flat torus. Suppose each coordinate x^j is periodic: this defines a flat torus M (the number of dimensions is irrelevant, although the original example[1] was in two dimensions). Choose v^j to be constant, and F_{jk} to be constant (and nonzero). Then (2) holds, with

$$\Psi = -F_{jk}v^j x^k \ . \tag{7}$$

But this is not periodic, and therefore not well defined on M. So the classical "constant of motion" $\dot{x}^j v_j + \Psi$ is not single valued (or, alternatively, not continuous). And the operator V on $\mathcal{H}$ does not exist at all; or to put it another way, the operator $iv^j D_j + \Psi$ maps elements of $\mathcal{H}$ to things that are not in $\mathcal{H}$. The generalized momentum is not conserved either classically or quantum mechanically.

In a case such as example 2, the problem occurs because M is not simply connected. Conversely, if M *is* simply connected, then Ψ is always globally defined. Of course, it has been known for a long time that novel features can arise when configuration space is not simply connected; in particular, for example, one can get a θ-vacuum structure.

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[1]N. S. Manton, Ann. Phys. (N.Y.) **159**, 220 (1985).
[2]J. G. Esteve, Phys. Rev. D **34**, 674 (1986).
[3]The indices $j, k, l, \ldots$ are spatial indices, referring to local coordinates on M. Units are chosen so that $c = \hbar = e = m = 1$.
[4]R. Jackiw and N. S. Manton, Ann. Phys. (N.Y.) **127**, 257 (1980).
[5]The curvature conventions are $[\nabla_j, \nabla_k]\xi^l = R_{jkm}{}^l\xi^m$ and $R_{jk} = R_{jlk}{}^l$. If v^j is a Killing vector, then $\Delta v^j = R_k{}^j v^k$, where $\Delta = \nabla_j \nabla_j$.