Dyon solutions in the temporal gauge

M. P. Benjwal and D. C. Joshi
Department of Physics, Post Box No. 25, Garhwal University, Srinagar, Garhwal, 246 174, Uttar Pradesh, India
(Received 10 March 1986)

Using a new non-Abelian gauge field tensor, the Yang-Mills field equations are solved asymptotically to show that spontaneous breaking of SU(2) symmetry by the Higgs-triplet field leads to the dyon solutions in the temporal gauge as well.

It is well known that the gauge theories in which the electromagnetic group U(1) is taken to be a subgroup of a larger group such as SU(2) magnetic monopoles can be created as regular solutions of field equations.[1] As an extension of such theories, Julia and Zee[2] showed that a non-Abelian gauge theory with Higgs fields exhibits classical solutions that are both electrically and magnetically charged. Enlarging the gauge group, finite-energy monopole and dyon solutions have also been found in an SU(3) gauge theory.[3] In all such theories,[4–6] the monopoles appear as the static solutions of the field equations in the temporal gauge, while dyons correspond to the static solutions in the nontemporal gauge. No dyon solutions to our knowledge are known in the temporal gauge. We show in the present paper that the static dyon solutions may also be obtained in the temporal gauge, if a new non-Abelian field tensor[7] is used and the spontaneous breaking of the SU(2) symmetry by the triplet of Higgs fields is considered.

We consider first a pure triplet scalar Higgs-field system for which the Lagrangian density may be given by

$$\mathcal{L}_0=(\partial_\mu\phi^a)(\partial^\mu\phi^a)-V(\phi^a)\ , \tag{1}$$

where $V(\phi^a)$ describes the self-interaction of the scalar field and has the form

$$V(\phi^a)=\frac{\lambda}{4}(\phi^a\phi^a-f^2)^2 \tag{2}$$

in which λ and f are real constants with $\lambda \ll 1$ and ϕ^a denote the Higgs-triplet fields. Now let us introduce a gauge function

$$U=\exp[-i\Lambda^a(x)T^a]\ , \tag{3}$$

where $\Lambda^a(x)$ are real functions of space-time and T^a represent the group generators of gauge group SU(2) obeying

$$[T^a,T^b]=i\epsilon^{abc}T^c \tag{4}$$

in which ϵ^{abc} are the structure constants of the gauge group SU(2) with a, b, c running from 1 to 3. Under the gauge functions of SU(2) gauge transformations, the scalar fields ϕ transform as

$$\phi\rightarrow\phi'=U\phi\ . \tag{5}$$

Since the Lagrangian density (1) is not gauge invariant under the gauge transformations (5), we introduce the vector gauge fields A_μ^a and B_μ^a to form the gauge-covariant derivatives

$$(D_\mu\phi)^a=(\partial_\mu\delta^{ac}+e\epsilon^{abc}A_\mu^b)\phi^c \tag{6a}$$

and

$$(D'_\mu\phi)^a=(\partial_\mu\delta^{ac}+g\epsilon^{abc}B_\mu^b)\phi^c\ , \tag{6b}$$

where e and g are two coupling constants, which couple ϕ with the gauge fields A_μ^a and B_μ^a, respectively, in order to restore the gauge invariance of the Lagrangian density (1). With such an arrangement we observe that the covariant derivatives $(D_\mu\phi)^a$ and $(D'_\mu\phi)^a$ transform like Eq. (5): i.e.,

$$(D_\mu\phi)^a\rightarrow U(D_\mu\phi)^a \tag{7a}$$

and

$$(D'_\mu\phi)^a\rightarrow U(D'_\mu\phi)^a \tag{7b}$$

provided the gauge vector fields A_μ^a and B_μ^a transform as

$$A_\mu\rightarrow UA_\mu U^{-1}-\frac{1}{e}(\partial_\mu U)U^{-1} \tag{8a}$$

and

$$B_\mu\rightarrow UB_\mu U^{-1}-\frac{1}{g}(\partial_\mu U)U^{-1}\ . \tag{8b}$$

Now, in order to make the gauge fields true dynamical variables we need to add a term to the Lagrangian density (1) involving their derivatives, then the Lagrangian density becomes

$$\mathcal{L}=\tfrac{1}{2}[(D_\mu\phi)^a(D^\mu\phi)^a+(D'_\mu\phi)^a(D'^\mu\phi)^a]-V(\phi^a)-\tfrac{1}{4}F_{\mu\nu}^aF^{\mu\nu a} \tag{9}$$

in which the gauge field tensor $F_{\mu\nu}^a$ acquires the form

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+e^{abc}A_\mu^bA_\nu^c-\tfrac{1}{2}\delta_{\mu\nu\rho\sigma}(\partial^\rho B^{\sigma a}-\partial^\sigma B^{\rho a}+gf^{abc}B^{\rho b}B^{\sigma c})\ , \tag{10}$$

where $\delta_{\mu\nu\rho\sigma}$ is a completely antisymmetric field tensor with μ, ν, ρ, σ running from 0 to 3. The field tensor $F_{\mu\nu}^a$ transforms under the gauge functions (3) as

$$F_{\mu\nu}\rightarrow UF_{\mu\nu}U^{-1} \tag{11}$$

 ©1987 The American Physical Society

and the Lagrangian density (9) is gauge invariant under the transformations (7) and (11).

The Euler-Lagrange variation of the Lagrangian density (9) with respect to the potentials A_μ^a and B_μ^a give the following field equations, respectively:

$$(D_\nu F^{\mu\nu})^a = e\epsilon^{abc}\phi^b(D^\mu\phi)^c \quad (12a)$$

and

$$(D'_\nu \tilde{F}^{\mu\nu})^a = -g\epsilon^{abc}\phi^b(D'^\mu\phi)^c , \quad (12b)$$

where

$$\tilde{F}_{\mu\nu}^a = \tfrac{1}{2}\delta_{\mu\nu\rho\sigma}F^{\rho\sigma a} \quad (13)$$

is the dual of field tensor $F_{\mu\nu}^a$. The Euler-Lagrange variation of the Lagrangian density (9) with respect to ϕ gives the equation

$$(D_\nu D^\nu\phi)^a + (D'_\nu D'^\nu\phi)^a = \lambda\phi^a(\phi^a\phi^a - f^2) . \quad (14)$$

The energy of the system may be calculated from Lagrangian density (9) as

$$\mathcal{E} = \tfrac{1}{2}\int d^3\mathbf{x} \Big[F_{0i}^a F_{0i}^a + F_{jk}^a F_{jk}^a + (D_0\phi)^a(D_0\phi)^a + (D_i\phi)^a(D_i\phi)^a + (D'_0\phi)^a(D'_0\phi)^a + (D'_i\phi)^a(D'_i\phi)^a + \frac{\lambda}{2}(\phi^a\phi^a - f^2)^2 \Big] , \quad (15)$$

from which we may readily observe that the minimum of energy may be obtained by setting

$$F_{0i}^a = F_{jk}^a = 0 , \quad (16a)$$

$$|\phi|^2 = f^2 , \quad (16b)$$

and

$$(D_\mu\phi)^a = (D'_\mu\phi)^a = 0 ; \quad (16c)$$

i.e., the lowest-energy state is obtained at $|\boldsymbol{\phi}| = \pm f$. From Eq. (16b), we may also write $\boldsymbol{\phi} = \pm f\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector in the SU(2) gauge group space. This expression shows that three component ϕ has reduced to one component along the unit vector $\hat{\mathbf{n}}$ and thus loses its two components. We choose the unit vector $\hat{\mathbf{n}}$ along the third direction in the SU(2) group space, i.e., $\phi = (0,0,f)$, such that

$$\phi^a = f\delta^{a3} \quad (17)$$

and show that $\mathbf{A}_\mu \cdot \hat{\mathbf{n}}$, $\mathbf{B}_\mu \cdot \hat{\mathbf{n}}$, and $\mathbf{F}_{\mu\nu} \cdot \hat{\mathbf{n}}$ behave like electromagnetic potentials and fields while $\mathbf{A}_\mu \times \hat{\mathbf{n}}$ and $\mathbf{B}_\mu \times \hat{\mathbf{n}}$ become massive having masses ef and gf, respectively. For this purpose we consider the terms containing A_μ and ϕ as well as B_μ and ϕ in the Lagrangian density (8). Such terms are

$$-\frac{e^2}{2}[(A_\mu\phi)(A^\mu\phi) - (\phi\phi)(A_\mu A^\mu)] - e(\partial_\mu\phi \times \phi)A^\mu \quad (18a)$$

and

$$-\frac{g^2}{2}[(B_\mu\phi)(B^\mu\phi) - (\phi\phi)(B_\mu B^\mu)] - g(\partial_\mu\phi \times \phi)B^\mu . \quad (18b)$$

Using Eq. (17) these terms may be written as

$$-\frac{e^2f^2}{2}(A_\mu^3 A^{\mu 3} - A_\mu A^\mu) = \frac{{m_e}^2}{2}(A_\mu^1 A^{\mu 1} + A_\mu^2 A^{\mu 2}) \quad (19a)$$

and

$$-\frac{g^2f^2}{2}(B_\mu^3 B^{\mu 3} - B_\mu B^\mu) = \frac{{m_g}^2}{2}(B_\mu^1 B^{\mu 1} + B_\mu^2 B^{\mu 2}) . \quad (19b)$$

We see from Eq. (19) that the two components A_μ^1, A_μ^2 and B_μ^1, B_μ^2 acquire the masses $m_e = ef$ and $m_g = gf$ with the charges $\pm e$ and $\pm g$, respectively, while third components A_μ^3 and B_μ^3 remain massless. Thus, the quantized spectrum of spontaneously broken SU(2) theory contains two charged massive vector bosons of charges $\pm e$ and $\pm g$ with the masses ef and gf and a γ photon associated with the gauge fields A_μ^a and B_μ^a, respectively, and also separates out the electromagnetic fields from the three-vector potentials A_μ^a and B_μ^a of SU(2) gauge group.

It may also be observed that the scalar fields $\boldsymbol{\phi}$ must have at least one zero[1] and this $\boldsymbol{\phi} = 0$ may be taken as the origin of coordinates. The field configurations, which have finite energy and satisfy field equations (12), may now be obtained. This may be achieved by introducing the following boundary conditions:

$$|\boldsymbol{\phi}|^2 \underset{|\mathbf{x}|\to\infty}{\longrightarrow} f^2 , \quad (20)$$

$$F_{0i}^a, F_{jk}^a \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 0 , \quad (21)$$

and

$$(D_\mu\phi)^a, (D'_\mu\phi)^a \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 0 \quad (22)$$

for the finite energy (15). The conditions (20) allow us to write

$$\partial_\mu\boldsymbol{\phi} \times \boldsymbol{\phi} - e\phi^2\mathbf{A}_\mu + e\alpha_\mu\boldsymbol{\phi} = 0 \quad (23a)$$

and

$$\partial_\mu\boldsymbol{\phi} \times \boldsymbol{\phi} - g\phi^2\mathbf{B}_\mu + g\beta_\mu\boldsymbol{\phi} = 0 , \quad (23b)$$

where

$$\partial_\mu\boldsymbol{\phi} \times \boldsymbol{\phi} = \epsilon^{abc}\partial_\mu\phi^b\phi^c , \quad (24a)$$

$$\alpha_\mu = \frac{\boldsymbol{\phi} \cdot \mathbf{A}_\mu}{f} , \quad (24b)$$

and

$$\beta_\mu = \frac{\boldsymbol{\phi} \cdot \mathbf{B}_\mu}{f} . \quad (24c)$$

Thus from Eqs. (23) we have

$$\mathbf{A}_\mu = \frac{1}{ef^2}(\partial_\mu\boldsymbol{\phi} \times \boldsymbol{\phi}) + \frac{1}{f}\alpha_\mu\boldsymbol{\phi} \quad (25a)$$

and

$$\mathbf{B}_\mu = \frac{1}{gf^2}(\partial_\mu\boldsymbol{\phi} \times \boldsymbol{\phi}) + \frac{1}{f}\beta_\mu\boldsymbol{\phi} . \quad (25b)$$

Now using Eqs. (25) in Eq. (10), we may obtain the gauge field tensor $F_{\mu\nu}$ as

$$\mathbf{F}_{\mu\nu}=\hat{\boldsymbol{\phi}}\left[[\partial_\mu\alpha_\nu-\partial_\nu\alpha_\mu-\tfrac{1}{2}\delta_{\mu\nu\rho\sigma}(\partial^\rho\beta^\sigma-\partial^\sigma\beta^\rho)] -\phi\left[\frac{1}{ef^3}(\partial_\mu\boldsymbol{\phi}\times\partial_\nu\boldsymbol{\phi})-\frac{1}{2gf^3}\delta_{\mu\nu\rho\sigma}(\partial^\rho\boldsymbol{\phi}\times\partial^\sigma\boldsymbol{\phi})\right]\right], \tag{26}$$

where $\hat{\boldsymbol{\phi}}=\boldsymbol{\phi}/f$. It may be noted that the field strength (26) is parallel to the field $\boldsymbol{\phi}$ and that $\{(\boldsymbol{\phi}\cdot\mathbf{F}_{\mu\nu})/f\}$ is gauge invariant. The first part of field tensor (26), i.e., the terms contained in the small square brackets, resembles the Cabibbo-Ferrari[8] field tensor for the system of both the electric and magnetic charges in Abelian gauge theory and may be decoupled by setting $\alpha_\mu=0=\beta_\mu$. Now, working in the temporal gauge $A_0=0=B_0$ to get the static field $\boldsymbol{\phi}$, i.e., $\partial_0\boldsymbol{\phi}=0$, we obtain the electric and magnetic field strengths from Eq. (26) as

$$\mathbf{F}_{0i}=\frac{\boldsymbol{\phi}}{2gf^3}\epsilon_{ijk}(\partial_j\boldsymbol{\phi}\times\partial_k\boldsymbol{\phi}) \tag{27}$$

and

$$\mathbf{F}_{jk}=-\frac{\boldsymbol{\phi}}{2ef^3}\epsilon_{ijk}(\partial_j\boldsymbol{\phi}\times\partial_k\boldsymbol{\phi}) \; . \tag{28}$$

It may be noted from Eqs. (27) that electric field does not vanish. The fields ϕ in Eqs. (25), (27), and (28), satisfy the boundary conditions (20) at infinity and vanish at origin, may be used to set

$$\boldsymbol{\phi}\sim f\hat{\mathbf{x}} \; , \tag{29}$$

for which the asymptotic forms of the solutions (23) may be obtained as

$$A^{ia}\sim\epsilon^{iab}\frac{x^b}{e\,|\mathbf{x}|^2} \tag{30a}$$

and

$$B^{ia}\sim\epsilon^{iab}\frac{x^b}{g\,|\mathbf{x}|^2} \; , \tag{30b}$$

where $\epsilon^{iab}\Longrightarrow a\neq b$ and the corresponding values of electric [Eq. (27)] and magnetic [Eq. (28)] fields may be calculated as

$$\mathbf{E}\sim\frac{\mathbf{x}}{g\,|\mathbf{x}|^3} \tag{31}$$

and

$$\mathbf{H}\sim-\frac{\mathbf{x}}{e\,|\mathbf{x}|^3} \; , \tag{32}$$

respectively. The electric field given by Eq. (31) may be regarded as the field of electric charge of strength $1/g$ while Eq. (32) gives the magnetic field of a magnetic monopole of strength $1/e$. Thus our solutions given by Eqs. (31) and (32) describe a dyon without sources and behave properly at infinity and origin, as the origin of coordinates is at $\phi=0$. The fields A_i^a, B_i^a, and ϕ^a are regular everywhere for the finite-energy solutions to the field equations (12) and (14) with the boundary condition (29). A complete view of energy density (15) shows that it is strongly localized near the origin as the fields $\boldsymbol{\phi}$ vanish there, which may be considered the position of dyon.

Thus we may observe that, contrary to the nontemporal gauge conditions and usual gauge field tensor,[2] the dyon solutions are also possible in the temporal gauge with a new non-Abelian gauge field tensor. These solutions acquire significance in the sense that the possible existence of both electric and magnetic charges on a particle is predicted and that this possibility may be extended to consider the electric and magnetic sources[9] in non-Abelian gauge theories which may well be described in terms of the new non-Abelian field tensor [Eq. (10)] that finds its construction in the lines of the Cabibbo-Ferrari Abelian field tensor.[8]

[1] G.'t Hooft, Nucl. Phys. **B79**, 276 (1974).
[2] B. Julia and A. Zee, Phys. Rev. D **11**, 2227 (1975).
[3] E. Kyriakopoulos, Nuovo Cimento **52A**, 23 (1979).
[4] E. B. Bogomolny, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)].
[5] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).
[6] N. K. Pak and R. Percacci, Nucl. Phys. **B188**, 355 (1981).
[7] D. C. Joshi, M. P. Benjwal, and Sunil Kumar, Acta Phys. Pol. **B16**, 901 (1985).
[8] N. Cabibbo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962).
[9] R. A. Brandt and F. Neri, Nucl. Phys. **B145**, 221 (1978).