Solitons in the supersymmetric CP^{N-1} model

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We propose an exact, factorized, S matrix for the two-dimensional supersymmetric CP^{N-1} model, which is compatible with the $1/N$ expansion. We find that antiparticles are bound states of $N-1$ particles.

The CP^{N-1} model and its supersymmetric version (SUSY CP^{N-1} model) has been the subject of several studies,[1] which recently focused on finite-temperature effects.[2] As is well known, the classically integrable CP^{N-1} model exhibits an anomaly in its quantum version,[3] preventing its integrability. The determination of its zero-temperature spectrum is thus an open problem.

Yet in the SUSY CP^{N-1} model there occurs a cancellation of anomalies[3] allowing its quantum integrability. In this paper we verify the expectation that the exact spectrum and two-body S matrix can be computed. Our calculations, relying on the Yang-Baxter factorization conditions and $1/N$ expansions, should provide a firmer ground for the study of finite-temperature effects in this model.

The $1/N$ Feynman rules of this model were obtained in Ref. 1 where we refer the reader. To establish some notation we will write the defining functional integral:

$$Z=\int |dZ||dZ^*||d\psi||d\bar{\psi}||d\omega||dA_\mu||d\sigma||d\pi||d\xi||d\bar{\xi}|$$
$$\times\exp\left[\frac{i}{2\alpha}\int d^2x\sum_{j=1}^{N}\left\{(\partial_\mu-iA_\mu)Z_j^*(\partial^\mu+iA^\mu)Z_j+\omega(Z_j^*Z_j-1)+\bar{\psi}_j i(\partial_\mu-iA_\mu)\gamma^\mu\psi_j\right.\right.$$
$$\left.\left.-\frac{1}{\sqrt{2}}\bar{\psi}_j(\sigma+i\pi\gamma_5)\psi_j+\bar{\xi}\psi_jZ_j^*+\bar{\psi}_jZ_j\xi\right\}\right] .$$

Using these rules one observes that all particle-antiparticle reflection amplitudes vanish in order $1/N$ and the absence of particle production to order $1/N^2$. Thus in constructing the exact S matrix one will, as in the chiral Gross-Neveu model,[4] start with vanishing reflection amplitudes from the very beginning. Transmission amplitudes are now defined introducing the symbols $b_\alpha(\theta_i)$ $[\bar{b}_\alpha(\theta_i)]$ and $f_\alpha(\theta_i)$ $[\bar{f}_\alpha(\theta_i)]$ to denote bosons [antibosons] and fermions [antifermions], respectively, where the variables θ_i are related to energy-momentum by $P_0^i=m\cosh\theta^i$, $P_1^i=m\sinh\theta_i$, and $P_1P_2=m\cosh\theta$ $(\theta=\theta_1-\theta_2)$:

$$^{\rm out}\langle b_\beta(\theta_1)b_\delta(\theta_2)\,|\,b_\alpha(\theta_1)b_\gamma(\theta_2)\rangle^{\rm in}=v_1(\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+v_2(\theta)\delta_{\alpha\delta}\delta_{\gamma\beta} ,$$
$$^{\rm out}\langle f_\beta(\theta_1)f_\delta(\theta_2)\,|\,f_\alpha(\theta_1)f_\gamma(\theta_2)\rangle^{\rm in}=u_1(\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+u_2(\theta)\delta_{\alpha\delta}\delta_{\gamma\beta} ,$$
$$^{\rm out}\langle f_\beta(\theta_1)b_\delta(\theta_2)\,|\,f_\alpha(\theta_1)b_\gamma(\theta_2)\rangle^{\rm in}=c_1(\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+c_2(\theta)\delta_{\gamma\delta}\delta_{\gamma\beta} ,$$
$$^{\rm out}\langle b_\beta(\theta_1)f_\delta(\theta_2)\,|\,f_\alpha(\theta_1)b_\gamma(\theta_2)\rangle^{\rm in}=d_1(\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+d_2(\theta)\delta_{\gamma\delta}\delta_{\gamma\beta} ,$$
$$^{\rm out}\langle b_\beta(\theta_1)\bar{b}_\delta(\theta_2)\,|\,b_\alpha(\theta_1)\bar{b}_\gamma(\theta_2)\rangle^{\rm in}=v_1(i\pi-\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+v_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta} ,$$
$$^{\rm out}\langle f_\beta(\theta_1)\bar{f}_\delta(\theta_2)\,|\,f_\alpha(\theta_1)\bar{f}_\gamma(\theta_2)\rangle^{\rm in}=u_1(i\pi-\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+u_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta} ,$$
$$^{\rm out}\langle b_\beta(\theta_1)\bar{b}_\delta(\theta_2)\,|\,f_\alpha(\theta_1)\bar{f}_\gamma(\theta_2)\rangle^{\rm in}=d_1(i\pi-\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+d_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta} ,$$
$$^{\rm out}\langle b_\beta(\theta_1)\bar{f}_\delta(\theta_2)\,|\,b_\alpha(\theta_1)\bar{f}_\gamma(\theta_2)\rangle^{\rm in}=c_1(i\pi-\theta)\delta_{\alpha\beta}\delta_{\gamma\delta}+c_2(i\pi-\theta)\delta_{\alpha\gamma}\delta_{\beta\delta} .$$

It is straightforward to get the following amplitudes with the $1/N$ Feynman rules:

$$v_1(\theta)=1-\frac{i\pi}{N}\coth,\quad v_2=-\frac{2i\pi}{N\theta} ,$$

$$u_2=\frac{2i\pi}{N\theta},\quad c_2=-\frac{2i\pi}{N\theta} ,$$

$$d_1=-\frac{\pi i}{N\sinh\frac{\theta}{2}},\quad d_2=0+O\left[\frac{1}{N^2}\right] .$$

The computation of the transmission amplitudes $c_1(\theta)$ and $u_1(\theta)$ present a problem yielding the indeterminate form 0/0. This arises from the singular behavior of the mixed A_μ-π propagator at zero momentum transfer. We circumvent this problem by inserting an infinitesimal momentum transfer δ at one vertex, making use of the identity

$$\bar{u}(P_2)\epsilon^{\mu\nu}(P_2-P_1)_\nu\gamma_5 u(P_1)=\frac{(P_1-P_2)^2}{2m}\bar{u}(P_2)\gamma^\mu u(P_1)$$

and finally letting δ go to zero. With this prescription we get

$$c_1(\theta)=1+O\left[\frac{1}{N^2}\right],$$
$$u_1(\theta)=1+\frac{i\pi\coth\theta/2}{N}.$$

The calculation of exact scattering amplitudes from the bootstrap approach was done[5] and reviewed[6] many times so we will limit ourselves to stating the result. Paying attention to signs due to statistics and fixing two free parameters by comparison with the $1/N$ perturbative expressions one gets the exact amplitudes ($\lambda=2/N$)

$$c_1(i\pi\Phi)=\prod_{l=0}^{\infty}\frac{\Gamma\left[\frac{\Phi}{2}+\frac{\lambda}{2}+l\right]\Gamma\left[1-\frac{\Phi}{2}+l\right]}{\Gamma\left[1-\frac{\Phi}{2}+\frac{\lambda}{2}+l\right]\Gamma\left[\frac{\Phi}{2}+l\right]}\times\prod_{l=0}^{\infty}\frac{\Gamma\left[\frac{\Phi}{2}-\frac{\lambda}{2}+l\right]\Gamma\left[1-\frac{\Phi}{2}+l\right]}{\Gamma\left[\frac{1}{2}-\frac{\Phi}{2}-\frac{\lambda}{2}+l\right]\Gamma\left[\frac{\Phi}{2}+l\right]}, \quad (1a)$$

$$v_1(i\pi\Phi)=\frac{\sin\frac{\pi}{2}(\Phi-\lambda)}{\sin\frac{\pi\Phi}{2}}c_1(i\pi\Phi), \quad (1b)$$

$$u_1(i\pi\Phi)=\frac{\sin\frac{\pi}{2}(\Phi+\lambda)}{\sin\frac{\pi\Phi}{2}}c_1(i\pi\Phi), \quad (1c)$$

$$d_1(i\pi\Phi)=-\frac{\sin\frac{\pi\lambda}{2}}{\sin\frac{\pi\Phi}{2}}c_1(i\pi\Phi), \quad (1d)$$

$$v_2(\theta)=-\lambda\frac{i\pi}{\theta}v_1(\theta),\quad u_2(\theta)=\frac{\lambda i\pi}{\theta}u_1(\theta),$$
$$c_2(\theta)=-\frac{\lambda i\pi}{\theta}c_1(\theta),\quad d_2(\theta)=-\frac{\lambda i\pi}{\theta}d_1(\theta). \quad (1e)$$

Let us now discuss some interesting properties of the solution (1a)–(1e). The pole contained in $c_1(\theta)$ generates the spectrum

$$m_n=m\frac{\sin\frac{n\pi\lambda}{2}}{\sin\frac{\pi\lambda}{2}},\quad n=1,2,\ldots,N-1.$$

In particular we have $m_{N-1}=m$. The fact that the bound state of $N-1$ particles has the same mass as the original particles is one ingredient in the proof that the symmetry of our solution (1a)–(1e) is SU(N). In fact there are two bound states of $(N-1)$ particles with mass m in the $\bar{N}$ channel and one verifies that they have identical scattering amplitudes to the original antiparticles $\bar{b}$ and $\bar{f}$. This entails the following identification of antiparticles with bound states of $(N-1)$ particles: the antiboson is identified with the bound state of $(N-1)$ fermions and the antifermion is the bound state of $(N-2)$ fermions with one boson. Symbolically we may write

$$\bar{b}=f_1f_2\cdots f_{N-1},$$
$$\bar{f}=f_1\cdots f_{N-2}b_{N-1}+f_1\cdots f_{N-3}b_{N-2}f_{N-1}+\cdots+b_1f_2\cdots f_{N-1}.$$

This implies that our fundamental particles obey generalized statistics, which should be treated as in Ref. 7.

For $N=2$, Eqs. (1a)–(1e) prevent the existence of the elementary O(3) triplet in the supersymmetric nonlinear σ model. This is also signalized by the vanishing of the (would-be) fermion-boson reflection amplitude in the O(3) case.[5] Hence we expect the O(3) spectrum to consist only of SU(2) kinks with scattering amplitudes given by Eqs. (1a)–(1e).

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