Gravitational acceleration of relativistic particles at finite temperature

M. Gasperini
Dipartimento di Fisica Teorica dell'Università, Corso M.D'Azeglio 46, 10125 Torino, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Torino, Italy
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Starting from the explicit expression of a generalized energy-momentum tensor, representing the effective source of gravity at finite temperature, it is shown that, in the case of a weak, static field and in the low-temperature limit, the thermal corrections to the gravitational acceleration of a relativistic test particle do not depend on its kinetic energy, and that the only effect is a constant, mass-dependent shift of the gravitational to inertial mass ratio, just as in the case, previously discussed, of nonrelativistic test bodies.

It has been recently demonstrated[1,2] that at finite temperature the gravitational and inertial masses are not the same. The motion of a test body is not geodesic, and the principle of equivalence is no longer satisfied. In the nonrelativistic limit and in the weak-field regime, considered in particular in Refs. 1 and 2, the acceleration induced by the Newton potential is different for particles of different mass, but this effect seems too small at accessible temperatures to become observable in a present Eötvös-type experiment.[1,2]

One may wonder however what happens in the case of a relativistic test particle, motivated also by the fact that anomalous velocity-dependent modifications of Newton's law are to be expected also as a consequence of many supersymmetric and multidimensional models of gravity[3] (the importance of measuring the gravitational properties of relativistic matter and antimatter has been stressed, in particular, in Ref. 4, and the results of a search for long-range interactions at highly relativistic velocities have been recently reported in Ref. 5).

To answer this question one may note that the difference between the inertial and gravitational masses, which from a thermodynamical point of view can be understood according to Ref. 6 as the low-momentum limit of the free energy and internal energy, respectively, is inversely proportional, at finite temperature, to the relativistic free energy of the particles itself.[1,2,6] It would seem, therefore, that the thermal corrections to the gravitational coupling take their maximum value in the static case, while they approach zero in the ultrarelativistic limit.

This conclusion should not be correct, however, and the aim of this paper is in fact to show with an explicit calculation that, on the contrary, the thermal corrections to the gravitational acceleration in the weak-field limit are not a function of the kinetic energy of the test body, and, at constant nonvanishing temperature T, a constant shift of the effective coupling is produced, depending only on T and on the rest mass of the particle, even in the ultrarelativistic case.

In order to include the thermal contributions, the acceleration of a relativistic particle in a weak static field can be obtained by approximating, in the weak-field limit, the equation of motion which generalizes the geodesic at finite temperature. Such an equation can be deduced, according to the procedure developed by Papapetrou,[7] by considering a narrow "world tube" containing the world line of the test particle (characterized by the energy-momentum tensor $T^{\mu\nu}$ nonvanishing only inside the world tube), by integrating the conservation equation for $T^{\mu\nu}$ over a three-dimensional hypersurface Σ and defining, in the limit in which the radius of the tube goes to zero:

$$\int_\Sigma d^3x'\sqrt{-g}\; T^{\mu\nu}(x') = \frac{p^\mu p^\nu}{E} , \tag{1}$$

where p^μ is the four-momentum of the particle and

$$E = p^4 = \int_\Sigma d^3x'\sqrt{-g}\; T^{44}(x') . \tag{2}$$

At finite temperature the inertia of a particle is increased by the interaction with the heat bath, and considering in particular a charged particle of mass m_0 in thermal equilibrium with a photon heat bath in the low-temperature limit, $T \ll m_0$, one obtains,[1,2] to first order in T^2,

$$E = ({m_0}^2 + |\mathbf{p}|^2 + \tfrac{2}{3}\alpha\pi T^2)^{1/2} \tag{3}$$

(m_0 is the renormalized $T=0$ mass and α the fine-structure constant).

Moreover, a detailed finite-temperature calculation[1,2] shows that, in the weak-field approximation, the effective source of gravity appearing in the linearized field equations, and expressed in the rest frame of the heat bath, is generalized as

$$\theta^{(0)\mu\nu} = T^{\mu\nu} - \frac{2}{3}\alpha\pi\frac{T^2}{E^2}\delta^\mu{}_4\delta^\nu{}_4 T^{44} . \tag{4}$$

It should be noted that, at finite temperature, the Minkowski vacuum is replaced by a thermal bath, and the introduction of a preferred frame (the one in which the blackbody radiation is isotropic, i.e., the frame at rest with the heat bath) breaks explicitly the Lorentz invariance of the finite-temperature vacuum[1,2] (for a recent detailed discussion of the spontaneous breakdown of Lorentz invariance at finite temperature see, in particular, Refs. 8 and 9). In this paper we shall assume that the lack of maximal symmetry of the vacuum leads to a finite-temperature theory of gravity in which the Lorentz group is not the dynamical symmetry group in the local tangent space, even if, in the world manifold, general covariance continues to hold: we are then in a quasi-Riemannian situation[10,11] in which, as we shall see, the effective matter current at finite temperature is covariantly conserved.

As we are concerned here with the thermal corrections to the equations of motion in the weak-field approximation, it is not necessary to start from the exact expression of the source term which includes the thermal contributions at all orders, but is enough to consider the simplest generalized source which reduces to Eq. (4) in the weak-field limit. We assume then, as an effective source at finite temperature,

$$\theta^{\mu\nu}=T^{\mu\nu}-\frac{2}{3}\alpha\pi\frac{T^2}{E^2}V^{\mu}{}_4V^{\nu}{}_4T^{44}\ , \tag{5}$$

where $V^{\mu}{}_i$ is the vierbein field and henceforth the index 4, whenever explicitly written in $E=p^4$, $V^{\mu}{}_4$, T^{44}, and so on, is always to be understood as a tangent space Lorentz index. This expression therefore transforms as a tensor under general coordinate transformations, but not under local Lorentz rotations [in the weak-field limit the source term is decoupled from gravity, so that one must replace $V^{\mu}{}_i$ and $\delta^{\mu}{}_i$ and Eq. (4) is then recovered].

Assuming that the direct contributions of the temperature to the geometry can be neglected (see, however, Ref. 12), the "left-hand side" of the Einstein equations is not modified and we have, at finite temperature, the field equations $G^{\mu\nu}=\theta^{\mu\nu}$, where $G^{\mu\nu}$ is the usual Einstein tensor. The contracted Bianchi identity $G^{\mu\nu}{}_{;\nu}=0$ implies then the following generalized continuity equation for $T_{\mu\nu}$:

$$T^{\mu\nu}{}_{;\nu}=\left[\frac{2}{3}\alpha\pi\frac{T^2}{E^2}V^{\mu}{}_4V^{\nu}{}_4T^{44}\right]_{;\nu} \tag{6}$$

(a semicolon denotes the usual Riemann covariant derivative) which, written explicitly in terms of the Christoffel symbols $\Gamma_{\mu\nu}{}^{\alpha}$, becomes

$$\partial_\nu(\sqrt{-g}\ T^{\mu\nu})+\Gamma_{\nu\alpha}{}^{\mu}\sqrt{-g}\ T^{\alpha\nu}=\partial_\nu\left[\sqrt{-g}\ \frac{2}{3}\alpha\pi\frac{T^2}{E^2}V^{\mu}{}_4V^{\nu}{}_4T^{44}\right]+\Gamma_{\nu\alpha}{}^{\mu}\sqrt{-g}\ \frac{2}{3}\alpha\pi\frac{T^2}{E^2}V^{\alpha}{}_4V^{\nu}{}_4T^{44}\ . \tag{7}$$

In order to obtain the equation governing the response of the test particle to a given external field, we shall integrate this equation over the spacelike hypersurface Σ intersecting the world line of the particle at $t=$const. By applying the Gauss theorem, expanding the gravitational field in power series around the coordinates x^{μ} of the center of mass according to the Papapetrou procedure,[7] and neglecting the coupling to internal momenta for a structureless pole particle,[7] we obtain, finally,

$$\begin{aligned}&\frac{d}{dx'^4}\int_\Sigma d^3x'\sqrt{-g}\ T^{\mu4}(x')+\Gamma_{\nu\alpha}{}^{\mu}(x)\int_\Sigma d^3x'\sqrt{-g}\ T^{\alpha\nu}(x')\\&\quad=\frac{d}{dx'^4}\left[\frac{2}{3}\alpha\pi\frac{T^2}{E^2}V^{\mu}{}_4(x)\int_\Sigma d^3x'\sqrt{-g}\ T^{44}(x')\right]+\Gamma_{\nu\alpha}{}^{\mu}(x)\frac{2}{3}\alpha\pi\frac{T^2}{E^2}V^{\alpha}{}_4(x)V^{\nu}{}_4(x)\int_\Sigma d^3x'\sqrt{-g}\ T^{44}(x')\ .\end{aligned} \tag{8}$$

Now we multiply by dx^4/ds and, in the limiting case in which the radius of the world tube approaches zero, i.e., $x'^{\mu}\to x^{\mu}$, we apply Eq. (1), putting $p^{\mu}=m\dot{x}^{\mu}$ and $E=m\dot{x}^4=m\dot{x}^{\nu}V_{\nu}{}^4$, where, according to Eq. (3), $m=(m_0{}^2+2\alpha\pi T^2/3)^{1/2}$ (an overdot denotes differentiation with respect to the proper time s along the particle world line). We have then

$$\ddot{x}^{\mu}+\Gamma_{\nu\alpha}{}^{\mu}\dot{x}^{\alpha}\dot{x}^{\nu}=\frac{d}{ds}\left[\frac{2}{3}\alpha\pi\frac{T^2}{mE}V^{\mu}{}_4\right]+\Gamma_{\nu\alpha}{}^{\mu}\frac{2}{3}\alpha\pi\frac{T^2}{m^2}V^{\alpha}{}_4V^{\nu}{}_4\ . \tag{9}$$

In the case of constant temperature $\dot{T}=0$ and then $\dot{m}=0$. Moreover $\dot{E}=m\ddot{x}^{\nu}V_{\nu}{}^4+m\dot{x}^{\nu}\dot{x}^{\beta}\partial_\beta V_\nu{}^4$. The acceleration of a test particle at finite temperature can be written then explicitly as a function of the velocity and of the first derivatives of the metric and of the vierbein fields as

$$\ddot{x}^{\mu}+\Gamma_{\nu\alpha}{}^{\mu}\dot{x}^{\alpha}\dot{x}^{\nu}=\frac{2}{3}\alpha\pi\frac{T^2}{mE}\dot{x}^{\nu}\partial_\nu V^{\mu}{}_4-\frac{2}{3}\alpha\pi\frac{T^2}{E^2}V^{\mu}{}_4(\ddot{x}^{\nu}V_\nu{}^4+\dot{x}^{\nu}\dot{x}^{\beta}\partial_\beta V_\nu{}^4)+\frac{2}{3}\alpha\pi\frac{T^2}{m^2}V^{\alpha}{}_4V^{\nu}{}_4\Gamma_{\nu\alpha}{}^{\mu}\ . \tag{10}$$

As noted before, this generalized equation of motion, obtained in the hypothesis that the gravitational sources are described by Eq. (5), is to be applied only in the weak-field limit. In particular we are interested in the response of a relativistic particle to a static field like that available in a terrestrial laboratory. Considering then the Schwarzschild metric in polar coordinates (r,ϑ,φ,t) we have $g_{\mu\nu}=\mathrm{diag}(-e^{\lambda},-r^2,-r^2\sin^2\vartheta,e^{\nu})$, where

$$e^{\nu}=e^{-\lambda}=1-\frac{2M}{r} \tag{11}$$

and $V^{\mu}{}_4=\delta^{\mu}{}_4e^{-\nu/2}$, $V_{\mu}{}^4=\delta_{\mu}{}^4e^{\nu/2}$. In the case of radial motion ($\dot{\vartheta}=0=\dot{\varphi}$) Eq. (10) gives $\ddot{\vartheta}=0=\ddot{\varphi}$, and

$$\left[1+\frac{2}{3}\alpha\pi\frac{T^2}{E^2}\right](\ddot{t}+\dot{\nu}\dot{t})=0\ , \tag{12}$$

$$\ddot{r}+\frac{\nu'}{2}\left[\dot{t}^2e^{\nu-\lambda}-\dot{r}^2-\frac{2}{3}\alpha\pi\frac{T^2}{m^2}e^{-\lambda}\right]=0\ , \tag{13}$$

where a prime denotes derivative with respect to the radial coordinate (remember that $E=m\dot{x}^{\nu}V_{\nu}{}^4=m\dot{t}e^{\nu/2}$ and $\nu'=-\lambda'$).

The integration of these equations can be easily performed, and gives

$$e^{\lambda}\dot{r}^2-e^{\nu}\dot{t}^2-\frac{2}{3}\alpha\pi\frac{T^2}{m^2}\nu=-1\ . \tag{14}$$

To obtain the radial acceleration in the weak-field limit, in which terms higher than linear in M/r are neglected, we can insert directly into Eq. (13) for $\dot{r}$ and $\dot{t}$ their values $\dot{r}_\infty$ and $\dot{t}_\infty$, estimated in the case of the vanishing gravitational field ($\nu=0=\lambda$), which satisfies $\dot{t}_\infty{}^2-\dot{r}_\infty{}^2=1$ [according to Eq. (14)]. We obtain then, to first order in M/r,

$$\ddot{r}=-\frac{M}{r^2}\left[1-\frac{2}{3}\alpha\pi\frac{T^2}{m^2}\right]\ . \tag{15}$$

Therefore, even in the case of a relativistic particle, the only thermal correction to the acceleration is a constant, mass-dependent shift of the ratio of gravitational to inertial mass, which to first order in T^2 is given by $m_g/m_i=1-(2\alpha\pi T^2/3m_0{}^2)$, just as in the static case.[1,2] This thermal effect induces a violation of the equivalence principle,[1,2] but in the weak-field and low-temperature limit it is not expected to become dominant at very high energies, unlike other mechanism inducing deviations from Newton's law.[3–5,13]

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