PHYSICAL REVIEW D VOLUME 36, NUMBER 2 15 JULY 1987

Construction of multiloop superstring amplitudes in the light-cone gauge

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(Received 6 October 1986)

Inexact bosonic and fermionic modes are used to give a modular-invariant and holomorphic construction of multiloop amplitudes for heterotic and type-II superstring amplitudes in the light-cone gauge. Mapping methods are used to simplify this, and a short-string limit is deployed to replace external interaction vertices by all but two of the Koba-Nielsen sources. Superstring Feynman diagram rules are deduced to describe the resulting amplitudes. A further reduction at one loop is shown to lead to the result already obtained by dual resonance methods.

I. INTRODUCTION

In order to answer the question as to whether or not multiloop superstring amplitudes are finite it would seem useful to construct them in a reasonably complete fashion. The most satisfactory approach to such a construction would be expected to be by covariant methods. Yet a different, but more direct attack, may lead to explicit formulas which, while not initially covariant, may later be put into a covariant form. One such attack would be by the light-cone (LC) gauge construction of the second-quantized type-II superstring.[1] This may be reduced to a first-quantized version by methods developed for bosonic strings,[2] and so allow functional techniques, pioneered by Mandelstam,[3] to be used. These latter will lead to multiloop amplitudes expressed in terms of the various functions—first and third Abelian differentials—and the moduli of the closed Riemann surface Σ corresponding to the string world sheet. The particular moduli will arise in a very explicit form, being the interaction points and string widths in the loops. The interaction points will be those positions on Σ where the strings fuse. Such a picture describes the sole contribution to the g-loop amplitude for a closed superstring when Σ has genus g. Extension may then be made to open superstrings by inclusion of open or noncompact Riemann surfaces Σ; that will not be considered here.

The multiloop amplitude that results from the above LC gauge approach for type-II superstrings is very similar in form to that of the bosonic string except for extra factors associated with the interaction points. These factors were already present in the LC fermionic string,[4] but their multiloop implications have not apparently been investigated. If the multiloop amplitudes for superstrings are to be analyzed for their finiteness or to be used in physical applications the extra factors must be evaluated somewhat explicitly. This requires obtaining a detailed understanding of the positions of the interaction points, especially in their dependence on the positions of the external strings. This may be achieved by means of a suitable limiting process, the short-string limit,[3] in which all but two external strings have very small values of p^+ (in LC notation). In that limit the external interaction points, where the external strings fuse, may be shown to be close to the external strings when these are described by a suitable set of variables, the Koba-Nielsen variables. The short-string limit will be used here to determine a more specific form for the factors arising at the interaction points, especially the external ones.

Before the short-string limit can be used it is necessary to take complete account of both bosonic and fermionic inexact modes on Σ. The former enter in the global description of the first-quantized functional integral in order to preserve the noninteraction of the left and right modes,[5] or to preserve the holomorphicity of the amplitudes in the terminology used in covariant bosonic string analyses.[6] The fermionic inexact modes are apparently essential in order to give nonzero amplitudes with $g \geq 1$. The transformation under the modular group must be analyzed carefully in order to show that modular invariance is preserved. This is necessary since the second-quantized field theory should not apparently have any description in terms of a marking on the world sheet of the string.

Basic aspects of the construction of the multiloop amplitudes are considered in Sec. II. In the following section mapping methods are considered for the amplitudes. The short-string limit is then applied in Sec. IV to simplify the resulting expressions. The analysis of the tree kinematic factors is relegated to an appendix. The resulting amplitude is then analyzed from its graphical aspect. The relationship to known superstring tree and one-loop amplitudes is considered in Sec. V. Possible divergences of the multiloop amplitudes are considered briefly in Sec. VI, as well as further questions raised by the results.

This paper may be considered as a consolidation and extension of earlier papers on multiloop superstring amplitudes by the authors.[5,7]

II. FUNCTIONAL METHODS

In the LC gauge a bosonic string is described by the transverse embedding vectors $\mathbf{X}(\sigma)$ with $(d-2)$ components (if the embedding space is d dimensional). The formulation of string quantum theory guaranteeing unitarity is that of the second-quantized form, in terms of a quantum field $\phi(\mathbf{X})$. The construction of the action and of the generators of the symmetry groups in this formulation described, for example, for the superstring in Ref. 1 or the bosonic string in Ref. 2, does not involve any specification of a Riemann surface. The perturbation analysis of string scattering amplitudes does bring in such surfaces, by means of reducing the second-quantized amplitude to a first-quantized one.[2] At a given loop order the surface is a sphere with g handles, so of genus g. Integration over surfaces of genus g arises, since this corresponds to summing over different interaction times and string widths in the loops. However such summation must only be over conformally inequivalent surfaces, since the original theory did not distinguish between them. The first-quantized version can be written in a form which involves only the moduli of the surface up to global transformations (corresponding to the modular group). In order to factor out this residual invariance it is necessary to ensure that all amplitudes are modular invariant. Part of the analysis in this section will be to guarantee such invariance for LC gauge amplitudes defined functionally.

The basic step in the reduction of the second-quantized perturbation amplitudes to first quantized form is through the identification of the second-quantized string vertex with the first-quantized "sum over surfaces" by the formula[2]

$$\prod_{\sigma} \delta(\mathbf{X}_3(\sigma)-\mathbf{X}_1(\sigma))\delta(\mathbf{X}_3(\sigma)-\mathbf{X}_2(\sigma)) = \lim_{\tau\to 0} \int D\mathbf{X}\exp(-S_{\Sigma_\tau}) \prod_{i=1}^{3} \delta(\mathbf{X}(\sigma_i,\tau_i)-\mathbf{X}_i(\sigma_i)) \ . \qquad (2.1)$$

In (2.1) the surface Σ_τ is the two-string "fusion element" of Fig. 1, which denotes the fusion of two strings 1 and 2, with initial values $\mathbf{X}_1$ and $\mathbf{X}_2$, to string 3 with value $\mathbf{X}_3$ after a short time τ. The quantity S_{Σ_τ} is the usual LC bosonic string action for the element of area Σ_t. The reality of the phase in (2.1) denotes that continuation has already been made to Euclidean time; string positions on the world sheet will be denoted by $\rho=\tau+i\sigma$. The string widths are taken to be $\pi p_r^{+}=\pi\alpha_r$ for the rth string, and constancy of width in Fig. 1 corresponds to p^{+} conservation.

Decomposition of any LC strip diagram corresponding to a global Riemann surface Σ may be achieved into two-string fusion elements and rectangular strings. The above result for the fusion elements and a similar result for the propagator of a string on a rectangular strip[2] may then be used to rebuild the second-quantized perturbation amplitude as a first-quantized amplitude. Using the rules of Ref. 2 (and noting that an extra factor of α_r enters for any rectangular strip of width α_r due to the difference between the variable τ entering the first- and second-quantized amplitudes in Ref. 2) leads to a string amplitude

$$\left[\prod_{r=1}^{N} \alpha_r{}^{-1/2}\right] \sum{}' \int D\mathbf{X} \times \exp\left[-\frac{1}{4\pi}\int\int_{\Sigma} d\mathbf{X}\,{}^{*}d\mathbf{X} + \int\int \mathbf{X}\mathbf{J}\,d\sigma\,d\tau\right] . \qquad (2.2)$$

Note the remaining factor at the front of (2.2) arising as the remnant of the momentum-dependent vertex factors $\prod_{i=1}^{3}\alpha_i{}^{-1/2}$ at each fusion point as in Fig. 1, after due account has been taken of the above-mentioned factor α_r. The summation sign $\sum'$ in (2.2) denotes summing over internal loop momenta (widths), interaction times, and, if so required, over the genus g (after inclusion of a suitable coupling constant). Extension to open strings may be achieved by summing also over windows on Σ or cross caps (for nonorientable surfaces). While such an extension is possible it will not be attempted here. The source term involving τ is defined by integration over external strings coupled at $\tau=\pm\infty$. Only massless modes will be considered here, so τ only involves specification of momenta and polarizations of such states. When fermions are included external states will be included which specify the helicities. How this is to be done will be specified later. The bosonic action in (2.2) is written in a global fashion by means of a closed one-form $d\mathbf{X}$ and its dual ${}^{*}d\mathbf{X}$. Such a description appears necessary if the reconstruction of the Riemann surface from its decomposition into rectangular and fusion elements properly accounts for the possible bosonic modes on the surface. In particular inexact modes appear to be handled suitably in that manner, as will be seen shortly.

In the $SU(4)\times U(1)$ decomposition of $d=10$ closed superstrings[1] there are extra Grassmann-valued variables $\theta^A,\tilde{\theta}^A$ and their associated momenta $\lambda_A,\tilde{\lambda}_A$ (the tilde denoting opposite movement to the untilded and $1\le A\le 4$; type IIa corresponds to $\theta^A,\tilde{\theta}_A,\lambda^A,\tilde{\lambda}_A$). The fermionic action for these variables is proportional to

$$\int\int d\sigma\,d\tau(\lambda\partial_{\bar{\rho}}\theta+\tilde{\lambda}\partial_{\rho}\tilde{\theta}) \ , \qquad (2.3)$$

where the constant of proportionality is unimportant. The extra factors at each interaction point are factorizable into left- and right-moving terms V, $\tilde{V}$ with

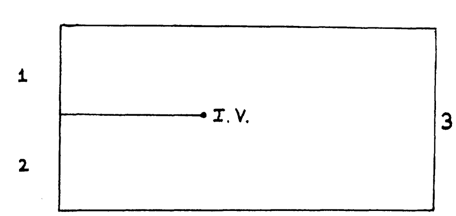

FIG. 1. Primitive interaction region for the fusion or splitting of two strings 1 and 2 into the third 3. The time τ of the interaction tends to zero.

$$
\begin{aligned}
&V=V(\sqrt{\epsilon}\partial\mathbf{X},\sqrt{\epsilon}\theta)\ ,\\
&V(Z,Y)=Z^{7+i8}-Y_i{}^2Z^i+Z^{7-i8}Y^4\ ,\\
&Y_i{}^2=(\rho_i)_{AB}\theta^A\theta^B\ ,\\
&\sqrt{\epsilon}=\lim_{\sigma\sim\pi\alpha}(\pi\alpha-\sigma)^{1/2}\ .
\end{aligned}
\tag{2.4}
$$

In (2.4) $\partial\mathbf{X}$ is the (1,0) part of $d\mathbf{X}$ and $\pi\alpha$ is the value of the string width at the interaction point. The $\sqrt{\epsilon}$ factor must clearly be handled with care; its presence and removal will be understood in the next section. Also $\tilde{V}=\tilde{V}(\sqrt{\epsilon}\bar{\partial}\mathbf{X},\sqrt{\epsilon}\tilde{\theta})$. The total amplitude for the type-II superstring is of the form, for given genus g,

$$
\left[\prod_{r=1}^{N}\alpha_r{}^{-1/2}\right]\int\prod_p d^2\tau_p\int\prod_{i=1}^{g}d^2\alpha_i\int D\mathbf{X}D\lambda D\theta D\tilde{\lambda}D\tilde{\theta}
\times\exp\left[-\frac{1}{4\pi}\int\int d\mathbf{X}^*d\mathbf{X}+\int\int(\lambda\partial_{\bar{\rho}}\theta+\tilde{\lambda}\partial_\rho\tilde{\theta})+\int\int\mathbf{J}\mathbf{X}+\int\int Q\theta+\int\int\tilde{Q}\tilde{\theta}\right.
\times\prod_p V(\sqrt{\epsilon}\partial\mathbf{X},\sqrt{\epsilon}\theta)\tilde{V}(\sqrt{\epsilon}\bar{\partial}\mathbf{X},\sqrt{\epsilon}\tilde{\theta})\ . \tag{2.5}
$$

In (2.5) integration over the interaction positions τ_p has been explicitly included as well as over the internal string loop momenta α_i; that these are complex for the closed-string case as compared to real in the open case has been thoroughly discussed in Ref. 8.

The heterotic string[9] amplitude may be deduced from (2.5) by removal of the terms involving θ and $\partial\mathbf{X}$, and addition of integration over a set 32 Neveu-Schwarz-Ramond (NSR) fermions ψ^I $(1\le I\le 32)$ with action $\int\int\psi^I\partial_\rho\psi^I d\sigma\,d\tau$. Summation over spin structures must also be performed, as discussed in Ref. 10. The resulting amplitude is thus

$$
\sum_\alpha\prod_{r=1}^{N}\alpha_r{}^{-1/2}\int\prod_p d^2\alpha_p\int\prod_{i=1}^{g}d^2\alpha_i\int D\mathbf{X}D\tilde{\lambda}D\tilde{\theta}D\phi_\alpha^I
\times\exp\left[-\frac{1}{4\pi}\int\int d\mathbf{X}^*d\mathbf{X}+\int\int\tilde{\lambda}\partial_\rho\tilde{\theta}+\int\int\psi^I\partial_\rho\psi^I+\int\int\mathbf{J}\mathbf{X}+\int\int\tilde{Q}\tilde{\theta}\right]\prod_\rho\tilde{V}(\sqrt{\epsilon}\bar{\partial}\mathbf{X},\sqrt{\epsilon}\tilde{\theta})\ . \tag{2.6}
$$

The summation over spin-structure leading to the O(16)×O(16) model may also be performed in (2.6), with a suitable signed factor giving opposite but equal contributions from the even and odd spin structures on Σ. It is not necessary to include vertex factors for the NRS fermions ψ^I since invariance in the compactified 16 dimensions has not been broken.

Modular invariance of the expressions (2.5) and (2.6) may be proved by a direct analysis of the effect of a modular transformation on the various factors. It will be shown later, associated with discussion of the short-string limit in Sec. IV, that the integration measure $\prod_{i=1}^{g}d^2\alpha_i$ is invariant when

$$
\begin{aligned}
\mathrm{Re}\alpha_i&=\mathrm{Im}[\rho(\gamma_iP)-\rho(P)]\ ,\\
\mathrm{Im}\alpha_i&=\mathrm{Im}[\rho(\delta_iP)-\rho(P)]\ .
\end{aligned}
\tag{2.7}
$$

In (2.7) $\{\gamma_i,\delta_j\}$ denote a canonical basis of the homology group $H_1(\Sigma,\mathbf{Z})$ corresponding to a fixed dissection on the surface. It can be seen that under the modular transformation[11]

$$
\begin{bmatrix}A&B\\C&D\end{bmatrix}\in\mathrm{Sp}(2g,\mathbf{Z}):\begin{bmatrix}\boldsymbol{\gamma}\\\boldsymbol{\delta}\end{bmatrix}\to\begin{bmatrix}A&B\\C&D\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}\\\boldsymbol{\delta}\end{bmatrix} \tag{2.8}
$$

(where $\boldsymbol{\gamma},\boldsymbol{\delta}$ are vectors with components γ_i,δ_j) then $(\mathrm{Re}\boldsymbol{\alpha},\mathrm{Im}\alpha)$ forms a $2g$-vector transforming under (2.8) by the same matrix. This is because the complex quantities α_i are obtained by finding the change of $\mathrm{Im}\rho$ around the γ_i or δ_i homology cycle. The transformation (2.8) relates the new homology basis to a linear combination of the old, so that $(\mathrm{Re}\boldsymbol{\alpha},\mathrm{Im}\alpha)$ must transform in a similar manner to the basis $(\boldsymbol{\gamma},\boldsymbol{\delta})$. The same result will be shown more explicitly in the next section by means of the short-string limit. The invariance of the measure follows immediately. It will also be shown in the next section, by using mapping methods, that the interaction positions ρ_p are invariant under (2.8). Modular invariance of the fermionic integration in the NSR sector for the heterotic amplitude (2.6) is guaranteed by the construction and appropriate summation over spin structures.[10,12]

It is necessary to consider the transformation properties of θ and λ (and $\tilde{\theta}$ and $\tilde{\lambda}$) on the world sheet under coordinate transformations. These properties are essential in order to determine the Green's function for θ and λ, and also the exact nature of the spaces of functions on which the operators $\partial_\rho,\partial_{\bar{\rho}}$ act in (2.3). These latter will determine the values of $\det\partial_\rho$, $\det\partial_{\bar{\rho}}$, which will arise on performing the functional integrations in (2.3), as we will see later.

Before LC gauge fixing the functions θ,λ were scalars with respect to arbitrary coordinate transformations on the world sheet.[13] In the process of gauge fixing a factor of $p^+=X^+$ has been absorbed into the action (2.3). The manner in which this may be achieved is ambiguous, in that a power $(p^+)^a$ may be absorbed into $\theta,(p^+)^{1-a}$ into λ, for any real number a. It is also relevant to note that although LC gauge fixing no longer allows arbitrary coordinate transformations on the world sheet the scale transformation $p \to \alpha p$, $\alpha \in \mathbf{R}$ is still allowed, and is a symmetry of the bosonic action in (2.2) and the fermionic one of (2.3) provided $\mathbf{X}$ are a set of scalars while

$$p \to \alpha p: \ \theta \to \alpha^{-a}\theta, \ \ \lambda \to \alpha^{a-1}\lambda, \ \ p^+ \to \alpha p^+ \ , \tag{2.8a}$$

where a is any real number.

At the first-quantized superstring level the weight a appears to be arbitrary. This does not seem to be so when the second-quantized action is considered, including the vertex factor (2.4) in the interaction term. To see this we note that the scaling transformation of the superstring second-quantized field Φ, may be read off, for example, from its equal-time commutator brackets [Eq. (3.7) of Ref. 1] to be

$$\Phi \to \left[\alpha^{-1} \prod_{\sigma} \alpha^{-4a}\right] \Phi \ ,$$

where $\prod_\sigma$ denotes the product along the length of string $\mathbf{X}$. Using this weight for ϕ it follows that the weight of the interaction term [(4.7) of Ref. 1 and including the interaction vertices (2.4)] is α^{-2}, as is necessary for a term in the Hamiltonian. Furthermore this implies that the vertex $|V\rangle$ of (4.11) of Ref. 1 must have scaling weight α^{-1}, on use of Eq. (4.32) of Ref. 1. The unique solution of the continuity equations for the $\mathbf{X}$, λ, and θ coordinates gives the vertex $|V\rangle$ transforming as α^{-1} provided $a=1$, as may be seen by direct inspection of Eqs. (4.19)–(4.24) of Ref. 1. This fixes the scaling weights of θ and λ to be -1 and 0, respectively, so that the space of θ's is that of vectors, the space of λ's is that of scalars. In particular ∂_ρ will act on (1,0) functions, $\partial_{\bar\rho}$ on (0,1) functions.

However as far as the Z plane is concerned, the SU(4)$\times$U(1) Grassmann variables λ and θ are chosen to be automorphic forms of weights (0,0) and (0,1), respectively, with respect to the covering group of Σ acting on the space of the Koba-Nielsen variables z. Thus if $\rho = F(z)$ is the automorphic mapping function the corresponding variables $\boldsymbol{\theta} = F^1\theta$ and $\boldsymbol{\lambda} = \lambda$ are forms of weights (1,0) and (0,0) under the group Γ of mappings, such that Σ is conformally equivalent to $\{z\}/\Gamma$. For λ is clearly a scalar under Γ, while if $\gamma \in \Gamma$ and since $\rho(z) = \rho(\gamma(z))$ by the automorphic character of F, then $\theta(\gamma(z)) = \gamma'(z)F'(z)\theta(z) = \gamma'(z)\boldsymbol{\theta}(z)$.

It may be helpful to remark here that although helicity (in the 7-8 directions) does not appear to be satisfied by the expression (2.5) due to the vertex factors V, $\tilde V$ (each with helicity -1) there is a compensating $+1$ unit of helicity from each of the δ function of θ and $\tilde\theta$ conservation at the interaction point. This latter can be shown directly in mode form,[1] or using contour-integration techniques in the field-theory case, using

$$\overline{\theta(z)\lambda(\omega)} = -\frac{1}{2\pi}\frac{1}{z-\omega} \ .$$

Thus overall J^{78} conservation occurs in the functional construction of (2.5), as it did in the original mode form (or second-quantized version) of Green and Schwarz.[1]

It is necessary to analyze the bosonic action in more detail before the functional integration over $\mathbf{X}$ can be performed. The closed one-form $d\mathbf{X}$ (dropping the vector label associated with the transverse modes) can be decomposed globally into an exact and an inexact part:

$$d\mathbf{X} = d\mathbf{X}_{\text{ex}} + \left[\sum_{i=1}^{g} Z_i V_i + \text{H.c.}\right] . \tag{2.9}$$

The inexact part in (2.9) has been expanded as a sum of harmonic one-forms, with complex basis being taken as the first Abelian differentials V_i on Σ $(1 \le i \le g)$. Thus besides the exact mode $\mathbf{X}_{\text{ex}}$, the usual embedding vector of the world sheet of the string, inexact modes arise described by $g(d-2)$ complex coordinates. The first Abelian differentials are taken to be normalized as usual: $\int_{\gamma_i} V_j = \delta_{ij}$, $\int_{\delta_i} V_j = \Pi_{ij}$ with Π the period matrix of Σ, with $\Pi^T = \Pi$, $\text{Im}\Pi > 0$. The bosonic part of the string action now becomes, with (2.9),

$$-\frac{1}{4\pi}\int\int d\mathbf{X} \wedge {}^*d\mathbf{X} = \frac{1}{4\pi}\int\int d\sigma\, d\tau\, \mathbf{X}_{\text{ex}} \Delta \mathbf{X}_{\text{ex}} - \frac{1}{\pi} Z^+ \, \text{Im}\Pi Z \ . \tag{2.10}$$

There is an ambiguity in the second contribution[5] on the right-hand side (RHS) of (2.10) since it cannot be detected on decomposition of Σ into rectangular and two-string fusion elements. The coefficient in front of this term will therefore be multiplied by a factor $(1+a)$. It will be shown in the next section that a may be chosen so that L and R modes do not interact. With this modification

$$\begin{aligned} &\overline{X^a_{\text{ex}}(z_1) X^b_{\text{ex}}(z_2)} = G(z_1,z_2)\delta^{ab} \ , \\ &\Delta G = -2\pi\delta^2 \ , \\ &\overline{Z_i^{a\dagger} Z_j^b} = \pi(1+a)^{-1} (\text{Im}\Pi)_{ij}{}^{-1}\delta^{ab} \ . \end{aligned} \tag{2.11}$$

The integration over $\mathbf{X}$ and $\boldsymbol{\lambda}$ may now be performed in (2.5) or (2.6). That over $\mathbf{X}$ leads, after translation of $\mathbf{X}$ by $G^*\mathbf{J}$, to the factor $(\det\Delta_0)^{-4}$, where Δ_0 is the Laplacian acting on scalars, together with contractions of factors $\mathbf{X}$ in the vertex factors V or $\tilde V$. These factors involve $\partial\mathbf{X}$ or $\bar\partial\mathbf{X}$, which is to be written, by (2.9), as

$$\partial\mathbf{X} = \partial\mathbf{X}_{\text{ex}} + Z_i v_i, \ \ \bar\partial\mathbf{X} = \bar\partial\mathbf{X}_{\text{ex}} + \bar Z_i v_i \ , \tag{2.12}$$

with $V_i = v_i dz$, $\partial = \partial/\partial z$, etc. The rules (2.11) are then to be used for the expressions (2.12) in V or $\tilde V$. In the process of integration over $\mathbf{X}$ a suitably modular invariant measure must be used for the separate variables (2.9). Under the modular transformation (2.8) it is necessary to transform V to preserve the normalization $\int_{\gamma_i} V_j = \delta_{ij}$, and Π, as

$$\begin{aligned} &\mathbf{V} \to (A + B\Pi)^{-1}\mathbf{V} \ , \\ &\Pi \to (C + D\Pi)(A + B\Pi)^{-1} \ . \end{aligned} \tag{2.13}$$

It is necessary to transform $\mathbf{Z}$ contragrediently to $\mathbf{V}$ to preserve (2.9)

$$\mathbf{Z}^T \to \mathbf{Z}^T(A + B\Pi) \ . \tag{2.14}$$

The invariant measure related to (2.14) is

$$\prod_{i=1}^{g} d^2Z_i(\det \mathrm{Im}\Pi)$$

since this is invariant under (2.14) and the transformation following from (2.13)

$$\mathrm{Im}\Pi \to (B\Pi + A)^{+-1}\,\mathrm{Im}\Pi(B\Pi+A)^{-1} .$$

Thus the modular-invariant measure for X is

$$D\mathbf{X}_{\mathrm{ex}} \prod_{i=1}^{g} d^2 Z_i(\det \mathrm{Im}\Pi)^{d-2} . \quad (2.15)$$

On integration over the Z_i's the final factor in (2.15) is eliminated by a similar but inverse factor arising from the exponential of the last term in (2.10).

The integration over λ and $\tilde{\lambda}$ leads to the factors $\prod_\sigma \delta^4(\partial_{\bar\rho}\theta)\delta^4(\delta_\rho\theta)$. The integration over the NRS fermions ψ^I in (2.6) gives[16] $(\det_\alpha \partial_\rho{}^{1/2})$, where α denotes the spin structure involved and the suffix $\frac{1}{2}$ on ∂_ρ denotes that ∂_ρ is acting on true spinors on the world sheet, i.e., on $(\frac{1}{2},0)$ forms. Finally the translations of $\mathbf{X}$ by $G^*\mathbf{J}$ produces the usual exponential factor $\exp(\frac{1}{2}\int\int \mathbf{J}G^*\mathbf{J})$. The translation $\partial G^*\mathbf{J}$ or $\bar\partial G^*\mathbf{J}$ must also enter into the $\partial\mathbf{X}$ or $\bar\partial\mathbf{X}$ term in V or $\tilde V$ at the interaction vertices.

III. MAPPING METHODS

In order to perform the remaining integration over θ and $\tilde\theta$ in (2.5) or over θ only in (2.6), as well as to be able to use the short-string limit explicitly it is appropriate to transform from the LC strip diagram, with variable ρ, to the Koba-Nielsen variable z. The use of this mapping was pioneered by Mandelstam[3] to construct the open-bosonic-string tree amplitude and extended to the one-loop level by Arfaei both for the open and closed bosonic string.[8,14] Mapping methods occur naturally in the dual-resonance-model (DRM) approach, and the open-bosonic-string amplitudes for arbitrary loops were directly derived in terms of a Schottky uniformization in the z plane.[15] More recently Mandelstam[16] has used the z plane to give a functional approach to construction of the multiloop bosonic string amplitudes. Green and Schwarz[17] have also derived tree and one-loop superstring amplitudes directly in terms of such variables by DRM techniques.

For a general closed Riemann surface there are powerful theorems[18] which allow a uniformisation of Σ by a domain D in the upper-half plane U. There will be a Fuchsian group Γ, a subgroup of $\mathrm{PSL}(2,\mathbf{R})$, which is isomorphic to $\Pi_1(\Sigma)$ and for which Σ is conformally equivalent to U/Γ. It is also possible to give a Schottky uniformisation of Σ in terms of a domain in the whole complex plane $\mathbf{C}$ outside a set of nonintersecting Jordan curves. This latter arose naturally in the DRM approach[15] through the work of Burnside. Other than noting the lack of absolute convergence of the Burnside Green's functions in certain regions of moduli space,[19] it is not necessary to choose a specific uniformisation. That is preferable since the amplitude was originally defined on Σ, so that only after achieving as much simplification as possible explicit forms of the various functions on Σ need to be used. Following Mandelstam[3] the mapping $\rho = F(z)$ introduced in the previous section is taken to be

$$F(z) = \sum_{r=1}^{N} \alpha_r \mathbf{G}(z,z_r) . \quad (3.1)$$

The Green's function $\mathbf{G}$ in (3.1), with real part defined by (2.11), may be constructed from the third Abelian differential with complex normalization. This is the function $\eta_{ab}(z)$ for which $d\eta_{ab}$ has poles at a and b with residues -1 and $+1$, respectively, and for which the normalization $\int_{\gamma_i} d\eta_{ab} = 0$; the notation of Lebowitz[20] will be used since it is most appropriate for discussion of the finiteness properties of the resulting multiloop amplitudes. It is also helpful to use that

$$\int_{\delta_i} d\eta_{ab} = 2\pi i[u_i(a) - u_i(b)] ,$$

where u_i are first Abelian integrals, $du_i = V_i$. The expression for $\mathbf{G}$ for which the real part is single valued on Σ (a necessary criterion if G is the Green's function for Δ associated with $\mathbf{X}$) is then[15]

$$\mathbf{G}_{ab} = \eta_{ab} + 2\pi i u_i(\mathrm{Im}\Pi)^{-1}{}_{ij}\,\mathrm{Im}[u_j(a) - u_j(b)] . \quad (3.2)$$

It is convenient to choose $b = z_N$, so $\mathbf{G}(z,z_r) = \mathbf{G}_{z_r z_N}(z)$. Then r is summed only up to $N-1$ in (3.1).

It is interesting to note how holomorphicity of the type-II amplitude is achieved[5] by the choice $a=1$ in (2.11). From (3.2) it is seen that

$$\partial_{z_1}\partial_{\bar z_2}\mathbf{G}(z_1,z_2) = -\frac{\pi}{2}u_i(z_1)(\mathrm{Im}\Pi)^{-1}{}_{ij}\bar u_j(z_2) . \quad (3.3)$$

For $a=1$, (2.11) reduces to

$$\overline{z_i \bar z_j} = (\pi/2)(\mathrm{Im}\Pi^{-1})_{ij} .$$

Then it follows from (2.12) that

$$\overline{\partial_{z_1}\mathbf{X}(z_1)\partial_{\bar z_2}\mathbf{X}(z_2)} = \partial_{z_1}\partial_{\bar z_2}\mathbf{G}(z_1,z_2) + z_i\bar z_j u_i(z_1)\bar u_j(z_2)$$
$$= 0 . \quad (3.4)$$

Equation (3.4) may be regarded as the LC version of holomorphicity in the internal variables discussed in the covariant string approach[6] for the vacuum amplitudes. The discussion in Ref. 6 is actually not directly related to the above analysis for the type-II superstring since in the latter case there is cancellation of the bosonic and fermionic z plane part of the determinants. The above analysis is thus only for the nondeterminantal part.

It is difficult to argue for $a \neq 0$ when starting from the second-quantized field theory. In that case, and for all $a \neq 1$, there will be explicit interactions between L and R movers, as given by the RHS of (3.3). In Sec. V it is shown that such terms vanish at one loop, in agreement with results obtained by DRM methods. The situation for higher loops is unclear, but they could still vanish. The natural choice for a would then be zero.

A similar question does not arise in the heterotic string, since the inexact modes do not contribute as they have no propagators, $\overline{z_i z_j} = 0$, so the modes can be integrated harmlessly away.

The interaction vertices ρ_p are defined by

$$\rho_p = F(z_p), \quad F'(z_p) = 0 . \quad (3.5)$$

For ρ near ρ_p it is possible to expand

$$\rho-\rho_p=\tfrac{1}{2}(z-z_p)^2F''(z_p)+O((z-z_p)^3)\ .$$

Near an interaction point the derivative ∂/∂_ρ therefore becomes

$$\partial/\partial\rho \sim F'(z)^{-1}\partial/\partial z \sim [2(\rho-\rho_p)F''(z_p)]^{-1/2}\partial/\partial z\ . \tag{3.6}$$

There is therefore a vanishing factor $\epsilon^{-1/2}$ in (3.6), where $\sqrt{\epsilon}$ is the same factor that was necessary in (2.4) to make the vertex factors V and $\tilde{V}$ well defined. The origin of these $\sqrt{\epsilon}$ factors is now clear; they arise due to the $\epsilon^{-1/2}$ divergence character of derivatives $\partial_\rho X, \partial_{\bar\rho} X$ at the interaction points. The effect of the transformation to the z plane on the interaction vertices is to introduce a factor $|F''(z_p)|^{-1}$ for the type-II superstring and a factor $F''(z_p)^{-1/2}$ for the heterotic string; ∂X then denotes $(\partial/\partial z)X$. There is still a $\sqrt{\epsilon}$ factor multiplying the Grassmann variables in (2.5) and (2.6). This is to be handled by solution of the δ-function condition $\partial_{\bar\rho}\theta=0$.

It is now appropriate to restrict the external sources solely to the massless string modes. This is satisfactory as far as low-energy problems are concerned, and since the higher modes are most likely unstable is very likely enough to describe the total theory. In any case the functional techniques allow for the calculation of higher mode amplitudes if they are stable in terms very similar to the massless modes.[3] The particularly interesting question of finiteness of amplitudes[21] will already be searchingly analyzed if only the massless modes are considered.

The external string states will now be described by momenta p_r, in the usual way, and by LC superfields $\phi_r(\theta_r,\zeta_r,u_r)$. The variables ζ_r are polarization vectors with $\zeta_r\cdot p_r=0$; they, as well as the p_r's will be chosen to be only six-component, with $\zeta^{7\pm i8}=p^{7\pm i8}=0$. Formulas obtained using such a choice are to be expressed in terms of the SO(6) invariants $\zeta_r\cdot p_s$, $\zeta_r\cdot\zeta_s$, $p_r\cdot p_s$, which may then be extended to the full SO(1,9) invariants. The u_r are helicity states which will not be considered in detail.

The solution to the constraint $\partial_{\bar\rho}\theta=0$ may now be written down. The boundary conditions are that $\theta\sim\theta_r$ as $\rho\sim\rho_r$. These are both satisfied by

$$\theta=\sum_{r=1}^{N-1}\alpha_r\partial_\rho \mathbf{G}(\rho,\rho_r)\theta_r+\sum_{i=1}^{g}\theta_i v_i \tag{3.7}$$

provided that

$$\sum_{r=1}^{N}\alpha_r\theta_r=0\ . \tag{3.8}$$

That (3.7) satisfies $\partial_{\bar\rho}\theta=0$ away from the sources is immediate from the construction of $\mathbf{G}$ so that $\partial_{\bar\rho}\partial_\rho\mathbf{G}(\rho,\rho_r)\neq 0$ if $\rho\neq\rho_r$ and the fact that the v_i are holomorphic on Σ. It should be pointed out that, as in (3.1), r is summed only up to $(N-1)$ when $b=z_N$ is chosen in (3.2). Then as

$$\rho\sim\rho_r\ (1\le r\le N-1),\quad F'(z)\sim -\alpha_r(z-z_r)\ ,$$

and

$$\partial_z\mathbf{G}(z,z_r)\sim -1/(z-z_r)\ ,$$

so $\theta\sim\theta_r$. A similar argument also applies to the higher modes as will be shown elsewhere.[22] For them the constants θ_r in (3.7) are replaced by the Fourier coefficients θ_{rn} of $\theta(\rho_r)$ and integration on the external string length is performed.

The first term on the RHS of (3.7) then becomes

$$\sum_{r=1}^{N-1}\alpha_r\int_0^\pi d\eta_r\partial_\rho\mathbf{G}(\rho,\rho_r)\cdots\sum_n\theta_{nr}e^{2in\eta_r}\ , \tag{3.7a}$$

where $\rho_r=\tau+i\alpha_r\cdot\eta_r$, $0\le\eta_r\le\pi$, and $\theta_{0r}=\theta_r$ of (3.7). It is to be noted that (3.7) and (3.7a) may be used to show that the periodic boundary conditions on the external strings are satisfied by (3.7) and (3.7a), since they reproduce the Fourier series for $\theta(\rho)$ as $\rho\sim\rho_r$, as can be shown by explicit calculation of the nth Fourier coefficient

$$\frac{1}{\pi}\int_0^\pi d\eta_r e^{-2in\eta_r}\theta(\eta_r)\ ,$$

where $\theta(\eta_r)$ is given by (3.7) and (3.7a). There is no further boundary condition on θ at an interaction vertex except that of periodicity around the whole string, in contradistinction to the case of the spinning string.[4] However the conformal weight of θ, discussed in the previous section, is important to remove the vanishing factor $\epsilon^{1/2}$ at the interaction vertex in (2.4), as will now be explained.

As $\rho\sim\rho_N$ then

$$\partial_z\mathbf{G}(z,z_r)\sim(z-z_N)^{-1}\ ,$$

$$F'(z)\sim\left[\sum_{r=1}^{N-1}\alpha_r\right]\Big/(z-z_N)\ ,$$

so that

$$\theta\sim\sum_{r=1}^{N-1}\alpha_r\theta_r\left[\sum_{r=1}^{N-1}\alpha_r\right]^{-1}\ .$$

This reduces to (3.8) using $\sum_{r=1}^{N}\alpha_r=0$. Furthermore this latter condition ensures the invariance of (3.8) under the global supersymmetry (SUSY) transformation $\delta\theta_r=\epsilon$. This global SUSY invariance is present in the original theory and may be used to determine θ_N, say, in terms of the remaining θ_r's; the determination is by means of (3.8). It is also useful to note that at an interaction point ρ_p the value of θ is expressible in terms of $\theta_{r1}=\theta_r-\theta_1$, with

$$\sqrt{\epsilon}\theta(\rho_p)=F''(z_p)^{-1/2}\left[\sum_{r=2}^{N-1}\alpha_r\partial_z\mathbf{G}(z_p,z_r)\theta_{r1}+\sum_{i=1}^{g}\theta_i v_i\right]\ . \tag{3.9}$$

This may be seen true since the coefficient of θ_1 in (3.9) reduces to $\alpha_1\partial_z\mathbf{G}(z_p,z_1)$ with use of (3.5). Expression (3.9) will be an important one in further determination of the amplitude in the next section.

To give a complete evaluation of the θ integration the measure must be analyzed in a similar manner to that of the $\mathbf{X}$ integration. The first step in this is to give an orthogonal decomposition with respect to the usual inner product on one-forms f,g given by $(f,g)=\int\int_\Sigma \bar f\wedge g$. Write

$$\theta=\theta_p+\theta_t\ ,\qquad \theta_t=-(4\pi)^{-1}\partial_\rho \mathbf{G}*\partial_{\bar\rho}\theta\ , \tag{3.10}$$

so that $\partial_{\bar\rho}\theta_p=0$, and, for two one-forms,

$$(\theta_1,\theta_2)=(\theta_{1p},\theta_{2p})+(\theta_{1t},\theta_{2t})\ .$$

Changing variables from $\partial_{\bar\rho}\theta$ to θ_p,θ_t in the δ function gives a Jacobian factor $(\det\partial_{\bar\rho})$, and leads to

$$D\theta\delta(\partial_{\bar\rho}\theta)=D\theta_p D\theta_t\delta(\theta_t)(\det\partial_{\bar\rho})^4\ . \tag{3.11}$$

Using (3.7) for θ_p leads to

$$(\theta_{1p},\theta_{2p})=\bar\theta_{1i}(\mathrm{Im}\Pi)_{ij}\theta_{2j}+\mathrm{const}\times\sum_{r=1}^{N-1}\alpha_r{}^2\theta_{1r}\theta_{2r}\ . \tag{3.12}$$

The constant factor in the second term on the RHS arises by regularizing the inner product by removing small circles of radii δ around the sources z_r. The resulting measure may be taken as

$$D\theta=\prod_{i=1}^{g}d^4\theta_i\prod_{i=1}^{g}d^4\tilde\theta_i(\det\mathrm{Im}\Pi)^{-4}\prod_{r=1}^{N}d^4\theta_r d^4\tilde\theta_r\delta^4\left[\sum_{r=1}^{N}\alpha_r\theta_r\right]\delta^4\left[\sum_{r=1}^{N}\alpha_r\tilde\theta_r\right]\ . \tag{3.13}$$

The argument for modular invariance of the first three factors on the RHS of (3.13) is identical to that for the inexact mode measure in (2.15) with $\boldsymbol{\theta}=(\theta_i)$ transforming as $\mathbf{Z}$ in (2.14). There appears to be a slight arbitrariness about the powers of α_r entering (3.13) associated with $d^4\theta_r d^4\tilde\theta_r$, but it only seems possible to obtain a sensible short-string limit in agreement with DRM results[17,23] if (3.13) is used.

It is to be noted that if the modular transformation is associated with $\rho\to\rho^1(\rho)$ then since F is defined on the surface, independently of a marking, $\partial F/\partial_{\rho^1}=(\partial F/\partial\rho)(\partial\rho/\partial\rho^1)$, and thence $\rho_p^1=\rho_p$. This was used earlier in discussion of the modular invariance of the amplitudes (2.5) and (2.6).

In summary the results of the functional integration over λ is to produce the constraint $\partial_{\bar\rho}\theta=0$, since λ acts as a Lagrange multiplier, while further integration over θ then reduces to integration over analytic θ's with boundary values θ_r at the external strings r and additional inexact holomorphic modes. Thus the integration over λ,θ is not Gaussian, so it does not lead to Green's functions joining all of the interacting vertex; only those functions joining an interacting vertex and an external string can arise. The detailed form these lines take will be analyzed later.

The net result of the above discussion is that the amplitude for the scattering of N external massless states, described by the superfields ϕ_r, takes the value from (2.5) for type-II superstrings

$$\begin{aligned}\int&\left[\prod_{r=1}^{N}\alpha_r{}^{-1/2}\right](\det\Delta_0)^{-4}(\det\partial_\rho)^4(\det\partial_{\bar\rho})^4\\&\times\prod_p d^2\rho_p\prod_{i=1}^{g}d^2\alpha_i\exp[p_r p_s G(z_r,z_s)]\prod_{r=1}^{N}d^4\theta_r d^4\tilde\theta_r\phi_r(\theta_r,\tilde\theta_r,\zeta_r,\tilde\zeta_r,u_r)\\&\times\prod_{i=1}^{g}d^4\theta_i d^4\tilde\theta_i(\det\mathrm{Im}\Pi)^{-4}\delta^4\left[\sum_{r=1}^{N}\alpha_r\theta_r\right]\delta^4\left[\sum_{r=1}^{N}\alpha_r\tilde\theta_r\right]\\&\times\prod_p V\left[F''^{-1/2}\left[\partial_z\overline{X_{\mathrm{ex}}}+\sum_{r=1}^{N-1}\partial_z\mathbf{G}(z_p,z_r)p_r\right],F''^{-1/2}\left[\sum_{r=2}^{N-1}\alpha_r\partial_z\mathbf{G}(z_p,z_r)\theta_{r1}+\sum_{i=1}^{g}\theta_i v_i\right]\right]\tilde V(\ ,\ )\ .\end{aligned} \tag{3.14}$$

Terms involving the bosonic inexact modes have been dropped, so that there are no contractions in (3.14) between $\partial_{z_i}X_{\mathrm{ex}}$ and $\partial_{\bar z_z}X_{\mathrm{ex}}$. A similar expression occurs for the heterotic string from (2.6), obtained by dropping all terms with a tilde in (3.14) and modifications of the differential operators as specified at the end of Sec. II. This leads to (for nongauge particles)[24]

$$\begin{aligned}\int&\left[\prod_{r=1}^{N}\alpha_r{}^{-1/2}\right](\det\partial_\rho)^{-12}\prod_p d^2\rho_p\prod_{i=1}^{g}d^2\alpha_i\exp[p_r p_s G(z_r,z_s)]\Theta(0\,|\,\Pi\otimes\Gamma)\\&\times\prod_{r=1}^{N}d^4\tilde\theta_r\phi_r(\tilde\theta_r,\tilde\zeta_r,\tilde u_r)\prod_{i=1}^{g}d^4\tilde\theta_i(\det\mathrm{Im}\Pi)^{-4}\delta^4\left[\sum_{r=1}^{N}\alpha_r\tilde\theta_r\right]\\&\times\prod_p V\left[\bar F''(Z_p)^{-1/2}\left[\partial_{\bar z}\overline{X_{\mathrm{ex}}}+\sum_{r=1}^{N-1}\partial_{\bar z}G(z_p,z_r)p_r\right],\bar F''^{-1/2}\left[\sum_{r=2}^{N-1}\alpha_r\partial_{\bar z}\mathbf{G}(z_p,z_r)\tilde\theta_{r1}+\sum_{i=1}^{g}\theta_i\bar v_i\right]\right]\ .\end{aligned} \tag{3.15}$$

In (3.15) ∂_ρ operates on scalars in the ρ plane and $\Theta(0\mid\Pi\otimes\Gamma)$ is the Θ function at zero for the matrix $\Pi\otimes\Gamma$, where Γ is the Cartan matrix of the even self-dual lattice Λ for $E_8\times E_8$ or spin $32/Z_2$. This expression is also explicitly modular invariant since $\prod_{i=1}^{g} d^4\tilde{\theta}_i$ transforms as $[\det(C\overline{\Pi}+D)]^{-4}$, the Θ function as $[\det(C\Pi+D)]^8$, and $(\det\partial_\rho)^{-12}$ as $[\det(C\Pi+D)]^{-12}$. The net factor of $|\det(C\Pi+D)|^{-8}$ is exactly compensated for by the transformation of $(\det \mathrm{Im}\Pi)^{-4}$. The total expression may be extended straightforwardly to the case of massless colored gauge vectors, with compactified momenta $\tilde{p}_r$, and gauge indices $[IJ]$, by including the factor

$$\exp\left[2\sum_{r<s}\tilde{p}_r\tilde{p}_s\eta(z_r,z_s)\right]\Theta(0\mid\Pi)^{-8}$$

and replacing $\Theta(0\mid\Pi\otimes\Gamma)\cdot\phi_r(\ )$ by $\Theta(v\mid\Pi\otimes\Gamma)\phi_r^{[IJ]}(\ )$, with

$$v_i^I=\sum_r\sum_{m=1}^{16}\Gamma_{lm}n_m^{(r)}u_i(z_r)$$

and

$$\tilde{p}_r^I=2^{-1/2}\sum_{m=1}^{16}n_m^{(r)}e_m^I\ ,$$

with $\mathbf{e}^I$ a set of basis vectors of Λ.

IV. SHORT-STRING LIMIT

The short-string limit is a technique initially introduced by Mandelstam[3,4] and since used by other workers[8,14,1] to simplify calculations of string and superstring tree and loop amplitudes. It is used to find the various functions M_i of the kinematic invariants (p_ip_j) in the expansion of an amplitude M

$$M=\sum K_iM_i\ .$$

The functions K_i are invariant functions of the external polarizations, helicity states, etc. Choice of a particular kinematic configuration may enable M to be obtained, as well as the functions K_i. Assuming that these are all independent then the resulting functions M_i can be calculated, and hence an explicitly covariant result be obtained for M. This technique will be used here with the soluble configuration being chosen as $\alpha_r\sim 0$ $(2\le r\le N-1)$, so that all but two of the strings are very narrow. The results we obtain by this technique will be ten-dimensionally Lorentz covariant. The results obtained seem to depend on the choice of nonshort strings chosen, but are known not to for the tree level with $N=4$ (Ref. 7). This remains true at all higher loops for $N=4$, as is shown by the explicit symmetry in (s,t,u) of the expression (5.23), so that the same covariant amplitude will be obtained whichever of the initial and final strings are taken short. A similar result should be valid for higher N; this will be discussed elsewhere.[11]

It is proposed to determine solutions of (3.5) with (3.1). For $b=z_N$ the resulting equation reduces to

$$\sum_{r=2}^{N-1}\alpha_r\mathbf{G}'(z_p,z_r)+\alpha_1\mathbf{G}'(z_p,z_1)=0\ . \tag{4.1}$$

It is convenient to denote $\mathbf{G}(z,z_1)$ by $L(z)$. Then there will be $(N-2)$ values of z_p arbitrarily close to an associated z_r as the α_r's ~ 0 $(2\le r\le N-1)$; the remaining $2g$ interaction points will not be close to the z_r's, and so will in general satisfy

$$L'(z_p)=0\ . \tag{4.2}$$

The values of those z_p's close to a z_r will be called external interaction points, to distinguish them from the solutions of (4.2), which will be denoted internal interaction points (living in the space of moduli of Σ and being unrelated to the external sources). The positions $\tilde{z}_r$ of the external interaction points can be obtained by a perturbation series in the small parameters α_r $(2\le r\le N-1)$. To achieve this suppose

$$\tilde{z}_s=z_s+a_s\alpha_s(1+b_{sr}\alpha_r)+O(\alpha_r^3)\ . \tag{4.3}$$

For $\tilde{z}_s$ given by (4.3) the following estimates may be used:

$$\begin{aligned}&\mathbf{G}'(\tilde{z}_s,z_r)=\mathbf{G}'(z_s,z_r)+O(\alpha_s)\quad(s\neq r)\ ,\\&\mathbf{G}(\tilde{z}_s,z_s)=-\ln(\tilde{z}_s-z_s)+\phi(\tilde{z}_s,z_s)\ ,\\&\mathbf{G}'(\tilde{z}_s,z_s)=-(a_s\alpha_s)^{-1}(1-b_{sr}\alpha_r)+\phi'(z_s,z_s)\\&\qquad+a_s\alpha_s\phi''(z_s,z_s)+O(\alpha_s^2)\ .\end{aligned} \tag{4.4}$$

The estimates in (4.4) may be inserted into (4.1) and terms of a given order in α_r equated. To $O(\alpha_r)$,

$$\begin{aligned}&a_s=[\alpha_1L'(z_s)]^{-1}\ ,\\&b_{sr}=-a_s\{(1-\delta_{rs})\mathbf{G}'(z_s,z_r)\\&\qquad+\delta_{rs}[\phi'(z_s,z_s)+a_s\alpha_1L''(z_s)]\}\ .\end{aligned} \tag{4.5}$$

The higher-order terms may be calculated in (4.3), if so desired. It will be seen later that the $O(\alpha_r^2)$ term in (4.3) does not enter the short-string limit for the closed superstring amplitudes under consideration.

It is now necessary to estimate the leading order in α_r in the various expressions in (3.14) or (3.15). This results in

$$\sum_{r=1}^{N-1}G'(\tilde{z}_s,z_r)p_r=\frac{1}{2}\left\{\sum_{r\neq s}\mathbf{G}'(z_s,z_r)p_r+[A_s-(a_s\alpha_s)^{-1}]p_s+O(\alpha_r)\right\} \tag{4.6}$$

with $A_s=\phi'(z_s,z_s)+(a_s\alpha_s)^{-1}b_{sr}\alpha_r$,

$$\sum_{r=2}^{N-1}\alpha_r\theta_{r1}\mathbf{G}'(\tilde{z}_s,z_r)=\sum_{r\neq s}\alpha_r\theta_{r1}\mathbf{G}'(z_s,z_r)+[A_s\alpha_s-(a_s)^{-1}]\theta_{s1}+O(\alpha_r^2)\ , \tag{4.7}$$

$$F''(\tilde{z}_s)=(a_s{}^2\alpha_s)^{-1}+O(1)\ . \tag{4.8}$$

The relevant vertex factors from the external interaction vertices (EIV's) are

$$\prod_{r=1}^{N-1}\Big[2\partial_{z_r}X^{7+i8}+\Big[2\partial_{z_r}X+A_rp_r+\sum p_s\mathbf{G}'(z_r,z_s)-(a_r\alpha_r)^{-1}p_r\Big]^{j_r}$$
$$\times a_r{}^2\alpha_r\Big[-a_r{}^{-1}\theta_{r1}+A_r\alpha_r\theta_{r1}+\sum_{s\neq r}\alpha_r\theta_{s1}\mathbf{G}'_{rs}+\theta_iv_i(z_r)\Big]^{2j_r}$$
$$+2\partial_{z_r}X^{7-i8}a_r{}^4\alpha_r\Big[-a_r{}^{-1}\theta_{r1}+A_r\alpha_r\theta_{r1}+\sum_{s\neq r}\alpha_s\theta_{s1}\mathbf{G}'_{rs}+\theta_iv_i(z_r)\Big]^4\Big] \tag{4.9a}$$

and from the internal interaction vertices (IIV's) are

$$\prod_{\alpha}\Big[2\partial_{\tilde z_\alpha}X^{7+i8}+\Big[2\partial_{\tilde z_\alpha}X+\sum p_r\mathbf{G}'(\tilde z_\alpha,z_r)\Big]^{j_\alpha}\Big[\theta_iv_i(\tilde z_\alpha)+\sum\alpha_r\mathbf{G}'(\tilde z_\alpha,z_r)\theta_{r1}\Big]^{2j_\alpha}$$
$$+2\partial_{\tilde z_\alpha}X^{7-i8}\Big[\theta_iv_i(\tilde z_\alpha)+\sum\alpha_r\mathbf{G}'(\tilde z_\alpha,z_r)\theta_{r1}\Big]\Big]^4 . \tag{4.9b}$$

The next step in the analysis is to consider the vertex factors V in (4.9). The product (4.9b) over the internal interaction vertices (IIV's) may be expanded in increasing powers of α_r. The lowest-order term, ~ 1, involves a product of two or four θ_i for each IIV, there being $2g$ of the latter. If the quadratic Grassmann factors at the IIV's are considered first, the overall product is therefore over $4g$ factors of the θ_i. Berezhin integration will thus set the θ_i to zero in all of the EIV factors. The integration over θ_i's from the IIV factors can now be performed explicitly and will lead to an overall factor

$$T_{\mathbf{j}}(\tilde{\mathbf{z}})=\int\prod_{i=1}^{g}d^4\theta_i\prod_{\alpha=1}^{2g}\Big[\sum\theta_{i_\alpha}v_{i\alpha}(\tilde z_\alpha)\Big]^{2j_\alpha} , \tag{4.10}$$

where $\mathbf{j}=(j_1,\ldots,j_{2g})$, $\tilde{\mathbf{z}}=(\tilde z_1,\ldots,\tilde z_{2g})$. The notation is $\theta^{2j}=\theta^{2\,AB}\rho^j_{AB}$, with normalization $\int d^4\theta\,\theta^{2i}\theta^{2j}=\delta^{ij}$. When $g=1$

$$T_{\mathbf{j}}(\tilde{\mathbf{z}})=[v(\tilde z_1)v(\tilde z_2)]^2\delta_{j_1j_2} \tag{4.11}$$

while for $g=2$

$$T_{\mathbf{j}}(\tilde{\mathbf{z}})=\sum_{1,2,3,4}\Big[v_1{}^2(\tilde z_1)v_1{}^2(\tilde z_2)v_2{}^2(\tilde z_3)v_2{}^2(\tilde z_4)\delta^{j_1j_2}\delta^{j_3j_4}-4v_1(\tilde z_1)v_2(\tilde z_1)v_1(\tilde z_2)v_2(\tilde z_2)v_1{}^2(\tilde z_3)v_2{}^2(\tilde z_4)\mathrm{Tr}(\rho^{i_2}\rho^{i_1}\rho^{i_3}\rho^{i_4})$$
$$+16\prod_{j=1}^{4}v_1(\tilde z_j)v_2(\tilde z_j)\epsilon^{ABCD}\epsilon^{EFGH}\rho^{i_1}_{AE}\rho^{i_2}_{BF}\rho^{i_3}_{CG}\rho^{i_4}_{DH}\Big] . \tag{4.12}$$

Each IIV will contribute a further factor $[F''(\tilde z_\alpha)]^{-3/2}$. This is to be multiplied by the term $O(\prod_{r=2}^{N-1}\alpha_r)$ arising from the other factors in (3.14).

The crucial factor under discussion is, for general N, and to within inessential factors

$$M_N^{(0)}=\int\prod_{r=1}^{N-1}d^4\theta_r\theta_r{}^{2i_r}\prod_{r=1}^{N}\zeta_r^{i_r}\Big[\sum_{s=1}^{N-1}\alpha_s\theta_s\Big]^{2i_N}\prod_{r=2}^{N-1}(a_r\alpha_r{}^{3/2})$$
$$\times\prod_{r=2}^{N-1}a_r{}^2\Big[2\partial_{z_r}\bar X+\sum_{s\neq r}p_s\mathbf{G}_{rs}-(a_r\alpha_r)^{-1}p_r\Big]^{j_r}\Big[-a_r{}^{-1}\theta_{r_i}+\sum_{s\neq r}\alpha_s\theta_{s1}\mathbf{G}_{rs}\Big]^{2j_r}$$
$$\times\prod_{\gamma=1}^{2g}[F''(\tilde z_\gamma)]^{-3/2}\Big[2\partial_{\tilde z_\gamma}\bar X+\sum_{r=1}^{N-1}p_r\mathbf{G}'(\tilde z_\gamma,z_r)]\Big]^{j_\gamma}T_{\mathbf{j}}(\tilde{\mathbf{z}}) . \tag{4.13}$$

Contractions may arise in (4.13) between X's at IIV's and EIV's; the number of these EIV-IIV Feynman lines will be denoted by r. There can then be contractions between s IIV's alone and between t EIV's alone. The resulting contribution to M_N of order $\prod_{r=2}^{N-1}\alpha_r{}^{3/2}$ will be denoted by $M_N^{(0)(r,s,t)}$, so that $M_N^{(0)}=\sum_{(r,s,t)}M_N^{(0)(r,s,t)}$. In particular the case $M_N^{(0)(0,s,t)}$ was evaluated as a function of the covariant scalar products of ζ_r's and p_r's in the first paper of Ref. 7 in some detail. The other cases with $r\neq 0$ can be evaluated in a similar manner. They are discussed in detail for $N=4$ in the Appendix.

It should be added that in considering solely the quadratic Grassmann factors in (4.9b) terms containing factors

$$\partial_{z_r}\overline{X^{7\pm i8}\partial_{z_s}X^{7\pm i8}}$$

have been dropped. This can be justified since each such term must have an associated quartic Grassmann factor $\alpha_s{}^2[\theta_{s1}+\theta_iv_i(z_s)]^4$. Since it is of $O(\alpha_r{}^2)$ the corresponding external wave function Grassmann factors must be $\theta_s{}^2\theta_1{}^4$, so that the above Grassmann factor must contribute $\theta_s{}^{2j}[\theta_iv_i(z_s)]^{2j}$. Since $2g$ IIV's already contribute $4g$ factors of θ_i, this term gives no contribution.

The same result follows for terms with factors

$$\partial_{z_r}\overline{X^{7\pm i8}\partial_{\tilde z_\alpha}X^{7\pm i8}} ,$$

which will arise from quartic Grassmann factors in (4.9b).

Such a term is zero if there is an unsaturated internal $\partial_{\bar{z}_\beta} X^{7+i8}$; another factor

$$\partial_{z_s} \overline{X^{7\pm i8} \partial_{\bar{z}_\beta} X^{7\pm i8}}$$

is necessary. But then the Grassmann factors are $[\theta_i v_i(\tilde{z}_\alpha)]^4[\theta_{s1}+\theta_i v_i(z_s)]^4 \theta_s{}^2 \theta_1{}^2$, and this is to be multiplied by $4(g-1)$ factors θ_1; the result on Berezhin integration must be zero. If there is no unsaturated internal $\partial_{\bar{z}_\beta} X^{7+i8}$ then there will be $4g+2$ factors θ_1; and again zero results.

There can be nonzero contribution from internally contracted factors

$$\partial_{\bar{z}_\alpha} X^{7+i8} \partial_{\bar{z}_\beta} X^{7-i8}$$

with associated Grassmann factor $[\theta_i v_i(z_\beta)]^4$. It is therefore necessary to replace the last two factors in (4.13) by

$$\sum_{\{\tilde{z}_s\},\{\text{pairs } \tilde{z}_{\alpha^1},\tilde{z}_{\beta^1})} \prod \left[2\partial_{\tilde{z}_\delta} \bar{X} + \sum_{r=1}^{N-1} \mathbf{G}'(\tilde{z}_\delta, z_r) p_r \right]^{j_\delta} \times \prod_{\text{pairs } (\alpha^1,\beta^1)} 8\partial_{\tilde{z}_{\alpha^1}}, \partial_{\tilde{z}_{\beta^1}} \mathbf{G}(\tilde{z}_{\alpha^1}, \tilde{z}_{\beta^1}) T'_{\mathrm{j}}(\tilde{\mathbf{z}}) \tag{4.14}$$

with

$$T'_{\mathrm{j}}(\mathbf{z}) = \int \prod_{i=1}^{g} d^4\theta_i \prod_{\{\tilde{z}_\delta\}} [\theta_i v_i(\tilde{z}_\delta)]^{2j_\delta} [\theta_i v_i(\tilde{z}_{\beta^1})]^4 , \tag{4.15}$$

where the set of points $\{\tilde{z}_\delta\} \cup \{\text{pairs } \tilde{z}_{\alpha^1}, \tilde{z}_{\beta^1}\}$ compromise all of the IIV's.

We may interpret (4.13) as arising from a set of bosonic lines from each IV or source. Between IV's or sources z_A, z_B the bosonic line factor is $4\partial_{z_A}\partial_{z_B}\mathbf{G}(z_A,z_B)$. Instead of such bosonic lines at any IV or source, z_A there is also sum of external bosonic line contributions $\sum_r \partial_{z_A}\mathbf{G}(z_A,z_r)p_r$, where $r \neq A$ if z_A is a source. The modification introduced by (4.14) is that a set of pairs of IIV's may only have the internal bosonic lines between them, without the external bosonic line contributions.

There are also terms of $O(\alpha_r), O(\alpha_r \alpha_s), \ldots, O(\prod_{r=2}^{N-1} \alpha_r)$ from the quadratic fermionic IIV factors, which are to be multiplied by the terms of complementary order in the remaining contribution from (4.9) to give terms of $O(\prod_{r=2}^{N-1} \alpha_r)$. These are discussed in detail in the case of $N=4$ in the Appendix.

A general analysis of these higher-order terms may be performed by introducing the concept of fermionic Feynman lines. The quadratic Grassmann factors at the IIV's will be considered first. The $O(\alpha_r), \ldots,$ terms in the IIV factors can only arise by replacing some of the factors: $[\theta_i v_i(\tilde{z}_{\alpha_r})]^{2j_{\alpha_r}}$ by either $[\theta_{r1}\theta_i v_i(\tilde{z}_{\alpha_r})]^{2j_{\alpha_r}} \mathbf{G}'(\tilde{z}_{\alpha_r}, z_r)$ or by $[\theta_{r1}\theta_{s1}]^{2j_{\alpha_{rs}}} \mathbf{G}'(\tilde{z}_{\alpha_{rs}}, z_r) \mathbf{G}'(\tilde{z}_{\alpha_{rs}}, z_s)$.

Each of the factors $G'(\tilde{z}_{\alpha_r}, z_r)$ arising in this way will be denoted a fermionic line. Either one or two fermionic lines can arise, respectively, from a given IIV from the above IIV quadratic factors which are the Grassmann variables. The factor generalizing $T_{\mathrm{j}}(\mathbf{z})$ of (4.15), associated with single fermionic lines from $z_{\alpha_{lm}}$ to z_l and z_m will be

$$T\{A_r\}\{j_{lm}\}\mathbf{j}(\mathbf{z}) = \int \prod_{i=1}^{g} d^4\theta_i \prod [\theta_{i_\gamma} v_{i_\gamma}(\tilde{z}_\gamma)]^{2j_\gamma} \prod [\theta_{i_\alpha} v_{i_\alpha}(z_r)]^{A_r} \prod [\theta_i v_i(z_l)]^{C_l} \theta_j v_j(z_m)^{D_m} \rho_{C_l D_m}^{j_{lm}} \tag{4.16}$$

corresponding to the extra fermionic lines

$$\prod \mathbf{G}'(\tilde{z}_{\alpha_r}, z_r) \prod \mathbf{G}'(\tilde{z}_{\alpha_{lm}}, z_l) \mathbf{G}'(\tilde{z}_{\alpha_{lm}}, z_m) .$$

The manner in which the external source factors combine with (4.16) are given in detail for the case $N=4$ in the Appendix. In the case of higher N up to four fermionic lines can be drawn from a given IIV to different EIV's, and it is expected that, as for the case of $N=4$ discussed in the Appendix, there is an exclusion principle preventing both bosonic and fermionic lines. This has been checked in specific cases, where similar factors to those in (A16) arise, which vanish if a fermionic and bosonic line go from an IIV to an EIV. The general Feynman rules that go into the construction of the L and R factors separately are the following.

(a) There is 1 bosonic Feynman line going from each IIV $\tilde{z}_\alpha$. This either goes to another IIV $\tilde{z}_\beta$, when the line has value $2\partial_{\tilde{z}_\alpha}\partial_{\tilde{z}\beta}\mathbf{G}(\tilde{z}_\alpha,\tilde{z}_\beta)$ or to an EIV z_r, when the line has value $\partial_{\tilde{z}_\alpha}\mathbf{G}(\tilde{z}_\alpha,z_r)p_r$ or $2\partial_{\tilde{z}_\alpha}\partial_{z_r}\mathbf{G}(\tilde{z}_\alpha,z_r)$.

(b) Fermionic Feynman lines may also occur from IIV's to EIV's, there being a maximum of four such lines from each IIV $\tilde{z}_\alpha$ and one to each EIV $z_r (2 \leq r \leq N-1)$; the contribution from such a line is $\partial_{\tilde{z}_\alpha}\mathbf{G}(\tilde{z}_\alpha,z_r)$.

(c) There is the "exclusion principle" that there is never both a bosonic and a fermionic line from an IIV to an EIV. This principle will turn out to be of crucial importance in the analysis of divergences.

(d) There is an overall tensor-spinor vertex function $T_{\mathrm{j}}(\mathbf{z};\tilde{\mathbf{z}})$ associated with the vectorial indices j_α and positions $\tilde{z}_\alpha$ of the IIV's and of the EIV's z_r which have fermionic lines going to them.

(e) There are similar contributions to (a)–(d) from right-moving vertex factors described earlier.

(f) An overall factor

$$(\det \mathrm{Im}\Pi)^{-4} \exp[p_r p_s G(z_r,z_s)] \prod_{\mathrm{IIV's}} |F''(\tilde{z}_\alpha)|^{-3} .$$

(g) Integration is performed over EIV's, IIV's, loop widths, and twists.

Further restrictions are also necessary to exclude diagrams apparently allowed by the above rules but not included in the short-string limit of the amplitudes (3.14) or (3.15). Before the final amplitude can be written down it is necessary to discuss the powers of the α_r's arising from the above considerations; only the type-II case will be discussed in detail. From each EIV there will be a net factor of $\alpha_1{}^{-2}\alpha_r{}^3$ and a further factor of $\alpha_1{}^{-3}$ from each IIV, since α_1 is the factor of proportionality times L in the mapping function F of (3.1). There is a further factor of $\alpha_1{}^8$ in the elimination of the integration over θ_N, and a factor of $\alpha_r{}^{-2}$ from each external state by its normalization.[1] The net factor of α's is therefore

$$\alpha_1{}^{-6g-2(N-2)+4}\left[\prod_{r=2}^{N-1}\alpha_r\right]$$

from the interaction vertices. Further factors of α arise from transformation of variables. It is natural to rescale the internal values of $\tilde{\rho}_\gamma$ and α_i by

$$\tilde{\rho}_\gamma=\alpha_1\hat{\rho}_\gamma,\quad \alpha_i=\alpha_1\cdot\hat{\alpha}_i$$

and the Jacobian for the transformation from ρ_r to z_r is

$$\alpha_1{}^{2(n-2)}\prod_{r=2}^{N-1}|L_r'|^2\ . \tag{4.17}$$

The net Jacobian is

$$\begin{aligned}J&=\partial(\rho_r,\tilde{\rho}_\gamma,\alpha_i)/\partial(z_r,\hat{\rho}_\gamma,\hat{\alpha}_i)\\&=\alpha_1{}^{6g+2(N-2)-2}\prod_{r=2}^{N-1}|L_r'|^2\ .\end{aligned}$$

The total combination of J and the net factor of α's at the IV's is therefore

$$\left[\prod_{r=1}^{N}\alpha_r\right]\prod_{r=2}^{N-1}|L_r'|^2\ . \tag{4.18}$$

The fact that the product of the factors α_r in (4.17) is now from 1 to N for r, instead of the range 2 to $N-1$, is important. Any other order of vanishing in the α_r's of the factors in (3.14) would not lead to the range of r in this first factor of (4.18). The ratio of determinants $\Delta=(\Delta_0/\det\partial_\rho\partial_{\bar\rho})^{-4}$ can only have contributions from the singular points of the metric on the string world sheet, so that

$$\Delta=\left[\prod_p|F''(z_p)|\right]^a\left[\prod_{r=1}^{N}\alpha_r\right]^b\ . \tag{4.19}$$

No choice of b could cancel a power of factors α_r unless the factor is $\prod_{r=1}^{N}\alpha_r$. There is thus the strong consistency check on the short-string limit (SSL), that it leads to the leading-order amplitudes with such factors of α_r that they can be canceled by Δ. That is so if

$$a=0,\quad b=-1\ . \tag{4.20}$$

The value of a in (4.19) may be evaluated from the conformal anomaly in two dimensions for manifolds with boundary;[25] the particularly relevant boundary is a set of small contours encircling the sources and the interaction points, following the analysis of Mandelstam.[16] The differential operator ∂ is denoted by ∇^z_{-1} acting on the space J^{-1}, in the notation of Alvarez,[25] so that (in that notation)

$$\Delta=(\det\nabla_z^1\Delta_0^z)/\det{}'(\nabla_z^0\nabla^z_{-1})\ .$$

Since $\det\nabla_0^z=(\det\nabla_z^0)^*$, $\det\nabla_z^1=(\det\nabla^z_{-1})^*$ then the conformal anomaly terms cancel exactly in the evaluation of $\ln\Delta$. However there are constant (σ-independent) terms which are to be chosen to ensure the Lorentz invariance of the resulting amplitude. From the values (4.20) it is necessary to choose the overall constant to agree with (4.20). The covariant amplitude for the type-II superstring that results can be written as

$$\begin{aligned}\text{amplitude}=&\int\prod_{r=2}^{N-1}d^2z_r\prod_{\alpha=2}^{2g}d^2\hat{\rho}_\alpha\prod_{i=1}^{g}d^2\hat{\alpha}_i(\det\text{Im}\Pi)^{-4}\exp[p_rp_sG(z_r,z_s)]\\&\times\left[\sum_{(f,r,s,t)}A_N^{(f,r,s,t)}(u,\zeta,p,\mathbf{G})\right]\left[\sum_{(g,u,v,w)}A_N^{(g,u,v,w)}(\tilde{u},\tilde{\zeta},p,\overline{\mathbf{G}}\right]\ ,\end{aligned} \tag{4.21}$$

where $A_N^{(f,r,s,t)}$ denotes the coefficient of $\prod_{r=2}^{N-1}a_r\alpha_r{}^{3/z}$ in $M_N^{(f,r,s,t)}$ and f, g denote the number of fermionic lines. It is to be noted that the factors $a_r{}^2$ in (4.13) are canceled by the second factor of J in (4.18) The explicit dependence of $A^{(f,r,s,t)}$ on the first Abelian differentials or the α_i's has also not been shown. The form of (4.21) shows clearly the holomorphic nature of the amplitude.

In order to calculate the heterotic amplitude (3.15) it is necessary to evaluate the conformal anomaly contribution to the factor $(\det\Delta_0)^{-6}$. This may be obtained by a similar manner to that associated with the type-II superstring discussed above. From the calculation of Mandelstam, the conformal anomaly factor in $(\det\Delta_0)^{-6}$ is[16]

$$(\det\Delta_0)^{-6}=\left[\prod_p|F''|\right]^{-1/2}\left[\prod|\alpha_r|\right]^{-1}\ . \tag{4.22}$$

In the SSL this reduces to

$$\begin{aligned}&\alpha_1^{-g-(N-2)/2-1}\left[\prod_{r=1}^{N}|\alpha_r|\right]^{-1/2}\\&\quad\times\left[\prod_{r=2}^{N-1}|L_r'|\right]^{-1/2}\left[\prod_{\alpha=1}^{2g}|L_\alpha''|\right]^{-1/2}\ .\end{aligned} \tag{4.23}$$

When combined with the factors arising from the vertex factors, the external state normalizations and the Jacobian J, a net factor in the α's of $\alpha_1{}^{2g+(N-2)/2+2}$, remains. It is therefore necessary to choose the constant factor

$$(\det\Delta_0)^{-6}_{\text{const}}=\alpha_1^{-2g-(N-2)/2+2} \tag{4.24a}$$

in order that the SSL be well defined and nonzero. The value of $(\det\Delta_0)^{-6}_{\text{const}}$ in (4.24) is to be compared to that resulting for the closed bosonic string by a similar approach, for which

$$(\det\Delta_0)^{-12}_{\text{const}}=\alpha_1^{-4g-(N-2)+4}\left[\prod_{r=1}^{N}|\alpha_r|\right]. \tag{4.24b}$$

It is possible to use (4.24b) as a definition of $(\det\Delta_0)^{-12}_{\text{const}}$, again to within an arbitrary constant factor. The ratio between the square of (4.24a) and (4.23b) is $(\prod_{r=1}^{N}|\alpha_r|)^{-1}$, which is equal to the constant factor for the type-II superstring given by (4.20). Such a relation between the heterotic, bosonic, and type-II constant factors follows from the formula for the determinants:

$$|\det{}_H|^2=|\det{}_B|^2|\det \text{II}|^2\,, \tag{4.25}$$

where

$$\begin{aligned}\det{}_H&=(\det\Delta_0)^{-4}(\det\nabla^{\rho}_{-1/2})^{16}(\det\nabla^{\rho}_{-1})^4\,,\\ \det{}_B&=(\det\Delta_0)^{-12}\,,\\ |\det{}_{\text{II}}|&=(\det\Delta_0)^{-4}|\det\nabla^{\rho}_{-1}|^8\,.\end{aligned} \tag{4.26}$$

Use of the Quillen formula[10]

$$|\det\nabla^{\rho}_{-1/2}|^2=(\det{}'\Delta_0)^{-1}\times(\text{known functions}) \tag{4.27}$$

immediately leads to (4.25), and hence to the relation between the integration constants discovered above. The total heterotic amplitude which results, including the remaining factors of (4.23) in (3.15), is then

$$\begin{aligned}\text{amplitude}=\int&\prod_{r=2}^{N-1}d^2z_r(\bar{L}_r'^{-3})(L_r'^{-1})\\ &\times\prod_{\alpha=2}^{2g}d^2\hat{\rho}_\alpha\prod_{i=1}^{g}d^2\hat{\alpha}_i\exp[p_rp_sG(z_r,z_s)](\det\text{Im}\Pi)^{-4}\\ &\times\sum_{\epsilon}\theta[[\epsilon](0|\Pi)]^{16}[P(\Sigma)]\prod_{\alpha=1}^{2g}[L''_\alpha]^{-1/2}\left[\sum_{(f,r,s,t)}A_N^{(f,r,s,t)}(u,\zeta,p,\mathbf{G})\right],\end{aligned} \tag{4.28}$$

where $P(\Sigma)$ is the partition function $(\det'\partial)_z^{-12}$ of the left-moving bosons evaluated in the z plane, with the nonholomorphic factor $(\det\text{Im}\Pi)^{-6}$ extracted. The wave functions for the external states have not been specified above, but may be taken as the set $\{\phi_j(x,u,\zeta,\theta),\ \phi_I(x,u,\zeta,\theta),\ \phi_{p_I}(x,u,\zeta,\theta)\}$. The superfield ϕ_j is the $d=10$ supergravity multiplet, while ϕ_I and ϕ_{p_I} are the "neutral" and charged super Yang-Mills sectors, where $p_I{}^2=2$ with p_I belonging to the even self-dual root lattice of $E_8\times E_8$ or spin $32/Z_2$.

The above expressions (4.28) may be described in terms of a set of "string" Feynman rules. The Feynman lines are of two types: (a) joining either EIV's or IIV's, but not both EIV's, with value $2\partial_A\partial_B G(A,B)$ (A here is the point or its uniformisation in the z plane); (b) from an EIV or IIV with value $\sum_{r=1}^{N-1}\partial_A G(A,z_r)p_r$. There is a factor $[F''(z_\alpha)]^{-3/2}$ associated with each IIV; there is also the joint factor $T(\{z\})$ for all of the IIV's. Then there can be r lines of type (a) joining EIV's to IIV's, s lines of type (a) joining IIV's to each other, t lines of type (b) to IIV's, and u lines of type (b) to EIV's. The associated factor generalizing $A^{(r,s,t)}$ of (4.4) and (4.22) will then be

$$\begin{aligned}A^{(r,s,t,u)}=&\prod_{A_r}\partial_{z_r}\partial_{\tilde{z}_{\alpha_r}}\mathbf{G}(z_r,\tilde{z}_{\alpha_r})\zeta_r^{\alpha_r}\prod_{A_s}\partial_{\tilde{z}_{\alpha_i}}\partial_{\tilde{z}_{\alpha_i'}}\mathbf{G}(\tilde{z}_{\alpha_i},\tilde{z}_{\alpha_i'})\delta_{j_{\alpha_i}j_{\alpha_i'}}\\ &\times\prod_{A_t}\left[\sum_{l=1}^{N-1}\partial_{\tilde{z}_\gamma}G(\tilde{z}_\gamma,z_l)p_l\right]^{j_\alpha}L_uT_j(\{\tilde{z}\})\prod_{\alpha=1}^{2s}[F''(\tilde{z}_\alpha)^{-3/2}]\,.\end{aligned} \tag{4.29}$$

In (4.29), A_r denotes the set of paired EIV's and IIV's $(z_r,\tilde{z}_{\alpha_r})$ for type (a) lines, and A_t the set of IIV's $\tilde{z}_\gamma$ for type (b) lines. The remaining factor L_u has been discussed in detail in the Appendix; it is a function only of the external source points z_u to which the (b) lines are attached and of their polarization vectors $\zeta_u^{i_u}$. These latter occur either in scalar products with each other or with the external momenta of other EIV's. The method used in this paper answers the question as to whether or not there can be a global choice of light-cone gauge condition $X^+=x^++p^+\tau$ on a surface of higher genus. The method starts from a quantum field theory of strings in the LC gauge which is well defined. The reduction to a first-quantized version of string theory then proceeds to calculate a matrix element for a particular process in two steps. The first of these is a local decomposition on the associated set of Riemann surfaces and specifies the integration measure over the conformally inequivalent classes by obtaining it from the second-quantized string field theory. In the second step the phase factor entering with each surface is defined in a global manner on the surface by means of the additional inexact modes. The contribution from these latter are completely specified by requiring holomorphicity (or no L-R interactions) and the measure associated with them is determined by modular invariance. The resulting expression is a global realization of the first-quantized string theory in the LC gauge.

V. THE FINAL REDUCTION FOR THE ONE-LOOP AMPLITUDE

We are going to show in this section that the multiloop amplitude (4.21) reduces in the one-loop case to the Green-Schwarz expression.[23] The reduction program we are going to follow has been pioneered by Mandelstam[16] for the explicit evaluation of multiloop amplitudes, for the bosonic string. The expression (4.21) involves Feynman lines which explicitly contain nonanalytic terms in the τ variable, that is $1/\text{Im}\tau$ factors, which have to be eliminated before the final reduction can be performed. Having done that elimination we use analyticity in the τ variable to show that the only dependence of the integrand on that variable is through the $F_s(\tau)$ factor of Green and Schwarz. Finally by performing an explicit evaluation in a particular configuration, we obtain the reduced form of the amplitude. The expression for the one-loop amplitude for the scattering of four external massless multiplets is (4.21) for $N=4$, which can be written as

$$\int d\tilde\rho_\pm d\tilde\rho_2 d\tilde\rho_3 d\alpha_L d\beta_L (\text{Im}\tau)^{-4}\exp[p_r p_s G(z_r,z_s)] \times |\partial_{\tilde z_+}{}^2F(\tilde z_+)\partial_{\tilde z_-}{}^2F(\tilde z_-)|^{-3} \times |\partial_{z_2}G(z_2,z_1)\partial_{z_3}G(z_3,z_1)|^{-6}\mathcal{L}_F \,, \tag{5.1}$$

where $\tilde\rho_\pm=\rho(\tilde z_+)-\rho(\tilde z_-)$ and $\mathcal{L}_F$ is the contribution of the fermionic and bosonic lines which we give explicitly in the Appendix. A typical term is, for example,

$$|\partial_{\tilde z_1}u(\tilde z_+)\partial_{\tilde z_-}u(\tilde z_-)|^4 \times |\partial_{z_2}G(z_2,z_1)\partial_{z_3}G(z_3,z_1)\partial_{z_3}G(z_3,z_2) \times\partial_{\tilde z_+}\partial_{\tilde z_-}G(\tilde z_+,\tilde z_-)|^2 \,. \tag{5.2}$$

It is convenient to change variables $\tilde\rho_r\to u_r=u(z_r)$, $r=2,3$, and $(\tilde\rho_\pm,\alpha_L,\beta_L)\to(u_1,\tau)$, $u_1=u(z_1)$. We have

$$J_1=\left|\frac{\partial(\tilde\rho_2,\tilde\rho_3)}{\partial(u_2,u_3)}\right| =\left|\frac{\partial_{z_2}G(z_2,z_1)\partial_{z_3}G(z_3,z_1)}{\partial_2 u(z_2)\partial_3 u(z_3)}\right|^2$$

and we also denote

$$J_2=\left|\frac{\partial(\tilde\rho_\pm,\alpha_L,\beta_L)}{\partial(u_1,\tau)}\right| \,. \tag{5.3}$$

We may now rewrite (5.1) as

$$\int d^2\tau\, d^2u_1 d^2u_2 d^2u_3 (\text{Im}\tau)^{-5}\exp[p_r p_s G(z_r,z_s)]f_1 f_2 \,, \tag{5.4}$$

where

$$f_1=\text{Im}\tau J_2\,|\partial_{u_+}{}^2F(\tilde z_+)\partial_{u_-}{}^2F(\tilde z_-)|^{-1} \tag{5.5}$$

and

$$f_2=J_1\mathcal{L}_F\,|\partial_{\tilde z_+}{}^2F(\tilde z_+)\partial_{\tilde z_-}F(\tilde z_-)|^{-2} \times|\partial_{z_2}G(z_2,z_1)\partial_{z_3}G(z_3,z_1)|^{-6} \times|\partial_{\tilde z_+}u(\tilde z_+)\partial_{\tilde z_-}u(\tilde z_-)|^{-2} \,. \tag{5.6}$$

All the $\partial/\partial z$ derivatives in f_2 may be replaced by $\partial/\partial u$ derivatives by using all the $\partial_z u$ factors. A typical term is then of the form (left)×(right), where the (left) term has the structure

$$\frac{\partial_{u_2}G(z_2,z_1)\partial_{u_3}G(z_3,z_2)\partial_{u_+}\partial_{u_-}G(\tilde z_+,\tilde z_-)}{\partial_{u_+}{}^2F\partial_{u_-}{}^2F} \tag{5.7}$$

and a similar expression for the (right) one but in terms of u^* derivatives. The expression (5.7) is then manifestly invariant under projective transformations on the z plane under which

$$\tau\to\tau\,, \quad u(z)\to u(z)+h+m\tau \quad (n \text{ and } m \text{ integers})\,. \tag{5.8}$$

It is also invariant under modular transformations

$$\tau\to\frac{a\tau+b}{c\tau+d},\quad u\to\frac{u}{c\tau+d}\,. \tag{5.9}$$

In fact, f_1 and f_2 are both invariants under those transformations.

Following Mandelstam[16] one may show that $f_1=1$. To show that f_2 is the kinematic factor K_{1234} of Green and Schwarz[23] it is necessary to remove all traces of $\text{Im}\tau$ in f_2. This may be done by introducing variables α_r, β_r by

$$u_r=\alpha_r\tau+\beta_r \quad (r=2,3) \tag{5.10}$$

with the Jacobian

$$\partial(\text{Re}u_r,\text{Im}u_r)/\partial(\alpha_r,\beta_r)=(\text{Im}\tau)\,.$$

In all terms in f_2 except the factors $\partial_{u_+}\partial_{u_-}G(\tilde z_+,\tilde z_-),\partial_{u_\pm}\partial_{u_r}G(\tilde z_\pm,z_r)$ $(r=2,3)$ it is possible to replace the nonanalytic expressions $(\text{Im}u_r/\text{Im}\tau)$ by α_r. In the former factors the sole nonanalyticity arises from factor $(\text{Im}\tau)^{-1}$. Thus f_2 can be expressed as

$$f_2=\sum_{t=0}^{4}(\text{Im}\tau)^{-r}f_2^{(t)}(\tau,\alpha_r,\beta_r) \tag{5.11}$$

with $1\le r\le 3$ and $f_2^{(t)}$ analytic in τ in the upper half plane and at least infinitely differentiable in the α,β variables.

The first step in proving independence of f_2 on τ is to show the vanishing of the functions $f_2^{(r)}$, $r>0$. It will be shown shortly that f_2 is bounded as $\text{Im}\tau\to\infty$ for given α_r, β_r provided $0<\alpha_r<1$ $(r=1,2,3)$, We note that this condition on the α_r's corresponds to the fundamental domain $0<\text{Im}v_r<\text{Im}\tau$ in v_r. From the modular invariance of f_2,

$$f_2(\tau,\alpha_r,\beta_r)=f_2\left[\frac{a\tau+b}{c\tau+d},a\alpha_r+b\beta_r,c\alpha_r+d\beta_r\right]\,, \quad ad-bc=1\,,\quad a,b,c,d\in\mathbf{Z}\,, \tag{5.12a}$$

it follows by taking the inversion $a=d=0$, $c=1=-b$, $\lim_{\mathrm{Im}\tau\to 0} f_2(\tau,\alpha_r,\beta_r)$ is finite for a given α_r, β_r, and $\mathrm{Re}\tau$. f_2 is also finite for any τ in the upper half plane and fixed α_r, β_r since there is no degeneration of the sources or interaction points. The analyticity of $f_2^{(r)}$ in τ then indicates that the RHS of (5.11) can only be finite as $\mathrm{Im}\tau\to 0$ provided $f_2^{(r)}\equiv 0$ for $r>0$; otherwise the inverse powers of $\mathrm{Im}\tau$ on the RHS of (5.11) could not be canceled by any terms involving $\mathrm{Re}\tau$ in the τ dependence of $f_2^{(r)}$. Such cancellation could be achieved at isolated singularities of $f_2^{(r)}$ for $s<r$; since this would still only allow $f_2^{(r)}$ to be nonzero at a set of isolated points then smoothness of $f_2^{(r)}$ guarantees its vanishing. Thus $f_2=f_2(\tau,\alpha_r,\beta_r)$ depends analytically on τ in the upper half plane for fixed α_r,β_r.

The next step proves independence of f_2 on τ by using the periodicity in each α_r,β_r independent of that for the other α_s,β_s, as can be seen by direct inspection of f_2. From this periodicity

$$f_2(\tau,\alpha_r,\beta_r)=\sum_{\mathbf{n},\mathbf{m}} A_{\mathbf{nm}}(\alpha)e^{2\pi i(\mathbf{n}\cdot\alpha+\mathbf{m}\cdot\beta)}, \tag{5.12b}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\beta=(\beta_1,\beta_2,\beta_3)$, and $\mathbf{n},\mathbf{m}$ are two three-vectors with components $\in\mathbf{Z}$. The discussion will now be given explicitly for one of these components, and can trivially be extended to the three-vector case.

The functions $A_{nm}(\tau)$ will therefore satisfy the modular property

$$A_{nm}\left[\frac{a\tau+b}{c\tau+d}\right]=A_{mb+nd\;ma+nc}(\tau)\;. \tag{5.13}$$

The subgroup of modular transformations for which $mb+nd=n$, $ma+nc=m$ is given by

$$a=1+nml,\quad b=n^2l,\quad c=-m^2l\;,\qquad d=-nml+1,\quad l\in\mathbf{Z}\;. \tag{5.14}$$

Then if $m=0$, (5.13) and (5.14) lead to

$$A_{n0}(\tau+n^2l)=A_{n0}(\tau)\;. \tag{5.15}$$

By inversion

$$A_{n0}\left[-\frac{1}{\tau}\right]=A_{0n}(\tau)\;. \tag{5.16}$$

For a given $\tau_0\in$upper half plane, then (5.15) and (5.16) result in

$$A_{n0}(\tau 0)=A_{0n}[-(\tau_0+n^2l)^{-1}]\;. \tag{5.17}$$

Taking $l\to\infty$ then $A_{n0}(\tau_0)$ becomes independent of τ_0; so therefore does $A_{0n}(\tau_0)$ with $A_{0n}=A_{n0}$. The argument of Siegel[11] may then be used. The transformations

$$T^{\pm}:\ (n,m)\to(n\pm m,m)\;,$$
$$T'^{\pm}:\ (n,m)\to(n,m\pm n)$$

generate $\mathrm{SL}(2,Z)/Z_2$. Defining $\|(n,m)\|=|n|+|m|$ and $nm\neq 0$ then

$$\min[\|T^{\pm}(n,m)\|,\|T'^{\pm}(n,m)\|]<\|(n,m)\|\;.$$

By repeated application of the transformations generated by $T^{\pm}$, $T'^{\pm}$ it is always possible to reduce n or m to zero. Hence $A_{nm}(\tau)$ is independent of τ provided one of n or $m\neq 0$. A similar argument can be deduced for $A_{00}(T)$ since it is modular invariant. Thus

$$A_{00}(\tau_0)=A_{00}(\tau_0,l)=A_{00}(-1/(\tau_0+l)) \tag{5.18}$$

and letting $l\to\infty$ in (5.18) leads to the independence of A_{00} on τ.

The third step is to prove the independence of $f_2(\alpha_r,\beta_r)$ on α_r,β_r. This uses the relation (5.13), where A_{nm} is now independent of τ. Equation (5.13) implies that

$$A_{n1}=A_{01}=a_1\quad\text{(say)}\;.$$

From the independence of the functions $e^{2\pi i(n\alpha+m\beta)}$ the contribution to f_2 in (5.12) with β dependence $e^{2\pi i\beta}$ may be discussed independently of the other function $e^{2\pi in\beta}$. This term's contribution to f_2 is

$$a_1e^{2\pi i\beta}\sum_n e^{2\pi in\alpha}=a_1e^{2\pi i\beta}\delta_\alpha\;. \tag{5.19}$$

The continuity of f_2 in α implies that (5.19) must vanish, so $a_1=0$. A similar proof can be given for the coefficient of any function $e^{2\pi im\beta}$, with

$$A_{rm,m}=A_{0m}=a_m\quad\text{(say)}$$

with resulting contribution to f_2 being $a_me^{2\pi im\beta}\delta_{m\alpha}$. Similarly a_m must be zero by continuity of this function in α. It was earlier shown that any A_{nm} can be reduced to an A_{0m}. Thus only the term A_{00} is allowed in (5.12b). Hence f_2 is also independent of α_r,β_r.

The penultimate step is to obtain the value of f_2 in any suitable configuration of values for τ,α_r,β_r. This will be taken as $\mathrm{Im}\tau\to\infty$, where at the end of the calculation we take $\alpha_r\to\alpha$, all r. In the limit $\mathrm{Im}\tau\to\infty$ it is possible to approximate $\mathbf{G}(z,z_r)$ by the tree value

$$-\ln(z-z_s)+\ln(z-z_N)+\alpha_r u\quad\text{if }|\alpha-\alpha_r|<1\;. \tag{5.20}$$

Then the interaction points are at $\tilde{z}_{\pm}=z_1(1+\epsilon_{\pm})$, with

$$\tilde{z}_+=z_1\alpha_1(1+\alpha_1)^{-1},\quad \tilde{z}_-=z_N(1+\alpha_1{}^{-1})\;. \tag{5.21}$$

Provided $|\alpha_1-\alpha_r|<1$, $|\alpha_1-\alpha_N|<1$ the approximation leads to a finite value for all of the terms in f_2 when $\mathcal{L}_F$ is calculated from the various diagrams of Fig. 2. This justifies the earlier claim that f_2 is finite for fixed α_r,β_r as $\mathrm{Im}\tau\to\infty$.

The final step is to use the special configuration $\alpha_r\to 0$, $r=1,2,3$ with $\alpha_r\ll\alpha_N\ll 1$. Then all the expressions in f_2 given by the diagrams of Fig. 2 may be calculated explicitly, using the values of $\mathbf{G}$ and $\tilde{z}_{\pm}$ given in (5.20) and (5.21) and that

$$F''(\tilde{z}_+)F''(\tilde{z}_0)\sim\alpha_N{}^2,\quad \partial_{\tilde{u}_+}G(\tilde{z}_+,z_r)\sim\alpha_N\;,\qquad \partial_{\tilde{u}_-}G(\tilde{z}_-,z_r)\sim\alpha_N,\quad \partial_{\tilde{u}_+}\partial_{\tilde{u}_-}G(\tilde{z}_+,\tilde{z}_-)\sim 0 \tag{5.22}$$

(where all derivations from now on are now with respect to the u variables). Direct analysis may be performed of the various contributions to the diagrams of Fig. 2 (which are given analytically in the Appendix). All of them are zero in this limit except for (c). The result in this case has the same coefficients of the various kinematic invariants as

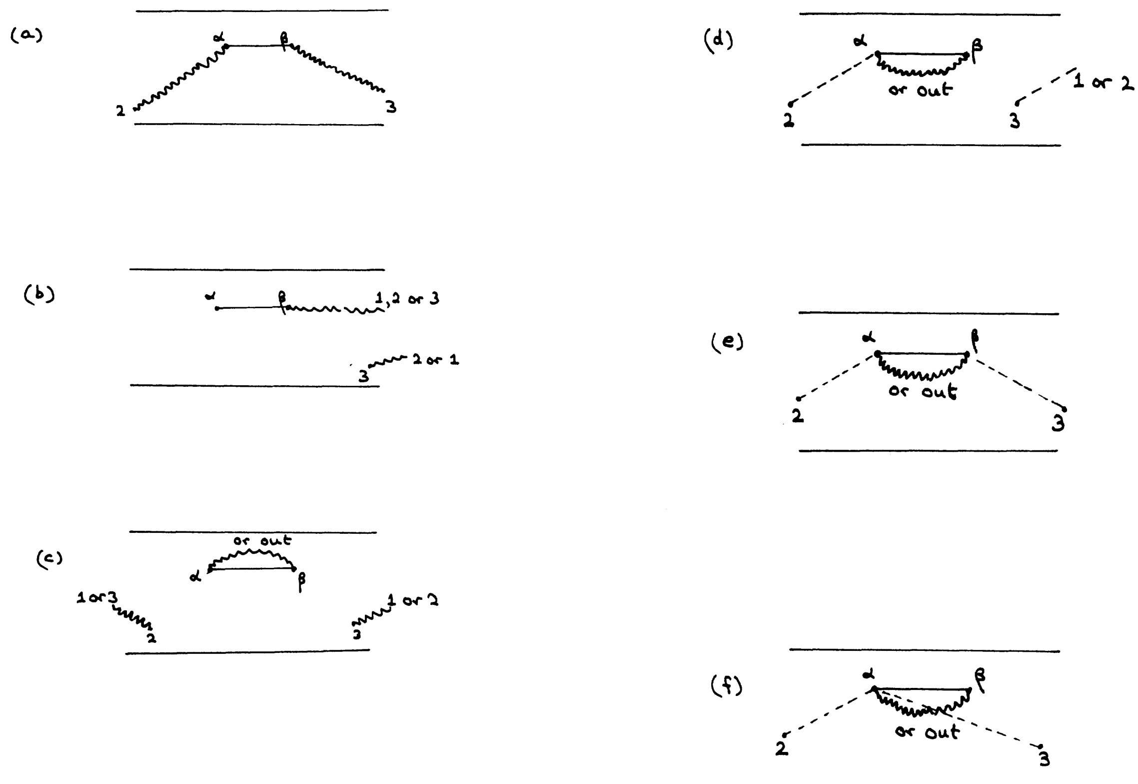

FIG. 2. Diagrammatic representation of the various superstring Feynman diagrams for four external strings ($N=4$) at one loop ($g=1$). Crosses denote interaction points. A wiggly line joining two IV's z_A, z_B is a "bosonic" line and denotes $\partial_{z_A}\partial_{z_B}G(z_A,z_B)$, while such a line with one one end at an IV to an external source z_r denotes $\partial_{z_A}G(z_A,z_r)p_r$. The dashed lines are "fermionic" lines from an IV z_A to a source z_r and denote $\partial_{z_A}G(z_A,z_r)$; the symbol "or out" denotes either the wiggly line joining the two IV's (as shown) or two single wiggly lines, one to each of the IIV's $\tilde{z}_\alpha, \tilde{z}_\beta$, each going out to the external sources though not so as to transgress the "exclusion" rules discussed in the text. (a) $M^{(2,s)}$ of Eq. (A6) (with $s=0$). (b) $M^{(1,s)}$ of Eq. (A8) (with $s=0$). (c) $M^{(0,s)}$ of Eq. (A11) (with $s=0$ or 1). (d) Representation of Eq. (A16). (e) Representation of Eq. (A17a). (f) Representation of Eq. (A17b).

the tree-level contributions[7] to within the vertex factors arising from the loop interaction points [see Eq. (A11) of the Appendix].

It is simplest to consider the coefficient of, say, the invariant $(\zeta_1,\zeta_2)(\zeta_3,\zeta_4)$, where the ζ_i's are the polarization vectors of the external massless states. The associated coefficient is then

$$Y_{1234}=(L_2'L_3'+2L_2'G_{32}'-2L_3'G_{23}')tF_{23} \ , \qquad (5.23)$$

where F_{23} arises from the loop vertex factors as given in the Appendix. Integration by parts, using the exponential factor $\exp[-\frac{1}{2}(sG_{12}+tG_{23}+uG_{13})]$ leads to a reexpression of Y_{1234} of (5.23) in the form

$$-2uL_2'L_3'F_{23}+2L_2'(\partial_{z_2}-\partial_{z_3})F_{23} \ . \qquad (5.24)$$

The last term in (5.24) contributes $O(\alpha_N)$ to f_2, so it is dropped. The value of the contribution of the first term to f_2 reduces, after inclusion of a further factor of t (arising from F_{23}) to

$$utF''(\tilde{z}_+)F''(\tilde{z}_-)L_2'L_3'\partial_{\tilde{u}_+}G(\tilde{z}_+,z_2)\partial_{\tilde{u}_-}G(\tilde{z}_-,z_3) +(2\leftrightarrow 3) \ . \qquad (5.25)$$

The value of (5.25) in the limiting configuration above is unity. Thus using (5.4) and that $f_1=1$, $f_2=ut$, $(\zeta_1,\zeta_2)(\zeta_3,\zeta_4)$ results in the known DRM value. A similar reduction of the one-loop heterotic amplitude to the DRM form given in Refs. 9 and 26 can be obtained from the LC constrained functional version of Ref. 24; the relation of this to the NRS version presented here will be considered elsewhere.

VI. DISCUSSION

In this final section it is appropriate to summarize the results obtained in the paper before commenting on their possible implications, extensions, and applications. The paper commences with deduction of the basic structure of the functional-integral expression of any multiloop superstring amplitude from the second-quantized field theory in the light-cone gauge. The functional integrations are then performed to produce a somewhat primitive version of the amplitude. This is simplified by considering only massless states. This primitive form is then made more compact by use of the short-string limit, and leads to a set of "raw" amplitudes. These are described in terms of the

Feynman-diagram language of field theory. A set of Feynman rules are deduced which allow these superstring diagrams to be written down. At the tree level the raw amplitude has elsewhere[7] been shown to agree with results already obtained by dual-resonance-model techniques. At one loop the superstring Feynman diagram contributions are already far more numerous than those obtained by DRM methods. In the previous section reduction of the raw amplitudes was shown to agree with the DRM value for four external strings.

It might be questioned as to how these equalities at tree and one loop between DRM amplitude and those obtained above by functional techniques can really be valid due to the presence of G'' factors arising from the bosonic Feynmann lines at the world sheet following the superstring Feynman rules of Sec. IV. No such factors appear in the DRM amplitudes. This has already been discussed at the tree level in the third paper of Ref. 7, where integration by parts was used to replace the G'' terms by those involving G' alone. Such partial integration can be done at the arbitrary loop level, and was a crucial step in the proof of finiteness of the type-II massless amplitudes.[27] Thus the apparent discrepancy between the functional and DRM versions of superstring amplitudes can be removed by partial integration, and no true discrepancy occurs.

The results summarized above are clearly only the first step in a number of ones to be taken before multiloop amplitudes are to be regarded as expressed in their final, most concise form. Thus the next step is to extend the results of the previous section to the case $g \geq 2$.

The technique of the last section appears difficult to extend directly. The initial step taken there was remove the nonanalytic terms involving $\text{Im}\tau$. It proved possible to do that by introducing the variables α_r, β_r as $\nu_r = \alpha_r \tau + \beta_r$. Such a transformation removed all of the terms involving $(\text{Im}\tau)^{-1}$ in the Green's functions in the superstring Feynman diagram factors. At the same time the regions of integration in the new external variables did not depend on $\text{Im}\tau$. An extension of this to $g > 1$ would need to remove factors of $(\text{Im}\Pi)_{ij}{}^{-1}$ in Green's functions as well as the dependence of the fundamental region on the elements of $\text{Im}\Pi$. This dependence is expected to be of the form that the boundary of a fundamental domain is given by $u_i(z) = \Pi_{ij} n_j + m_i$ $(1 \leq i,\ j \leq g)$, where $\mathbf{m}$ and $\mathbf{n}$ are g-component vectors with components being integers. However it appears difficult to introduce an independent set of variables $u_i(z_r)$ for each external variable r and $1 \leq i \leq g$; the dependent variables may produce further nonanalyticity in Tiechmüller space. Thus it is not possible to see if the conjecture of Belavin and Knizhnik[6] that superstring amplitudes may be written in a holomorphic form in Tiechmüller space is valid. It does appear likely to be able to show that it is valid for $N = 2$, however, and even that these self-energy diagrams are zero for all $g \geq 1$. Further analysis may indicate how such difficulties may be overcome for higher N and the important conjecture proved or disproven; alternatively it may turn out more appropriate to start from the covariant string approach in terms of which holomorphicity of the bosonic amplitudes was proven.[6]

Another step that needs to be taken is to extend the construction of amplitudes to massive states. This may be done directly in a similar manner to that for the tree-level analysis for the bosonic string.[28,8] For massive states (3.9) is extended by adding to the RHS the expression

$$-[F''(z_p)]^{1/2} \sum_{n \neq 0} \sum_r p_r{}^+ H_n(z_p, z_r) \theta_n^{(r)} , \tag{6.1a}$$

where

$$H_n(z, z_r) = \oint dz' [F_r(z')]^n \partial_{z'} \partial_z G_{z_N}(z, z') , \tag{6.1b}$$

$F_r(z) = [\exp F(z)]^{1/p_r^+}$, and $\theta_n^{(r)}$ is the nth Fourier mode of θ on the rth external string. Similar expression can be given for the bosonic contributions, so generalizing the tree-level analyses of Refs. 8 and 28. A complete expression for the amplitude can then be constructed extending (3.14) in a straightforward fashion. This will allow direct analysis of self-energy and other effects to be given for the massive modes instead of deducing such features by unitarity from the amplitudes solely for massless external states. It is interesting to note that since $H_n(z, z') \sim (z - z')^{-n-1}$ the expression (6.1) corresponds to the function θ having an essential singularity at $z = z_r$. A similar result also holds for the bosonic contribution G^*J. This is to be expected since the totality of variables on a string can only be described by the set of residues a function has with poles of arbitrary high order at the puncture z_r. This seems to present an interesting problem for a totally stringy approach to physics, and one which calls for appropriate discussion in purely Riemann surface constructions of string theory. The details of this and finiteness aspects for amplitudes for massive modes will be given elsewhere.[22]

It was remarked at the end of the previous section that the heterotic string amplitudes of Eq. (3.15) require further analysis in order to be reduced, at one loop, to the known DRM result. In order to fully justify the alternative construction[24] which does lead directly to the known one-loop result, it would be useful to develop a second-quantized field theory for the constrainted LC approach. That would also be valuable to help in developing a covariant string field theory of the heterotic string.

Another step yet to be made is that of obtaining expressions for open superstring amplitudes. That may be done using the techniques developed here, and leads to very similar formulas for the amplitudes as those in (3.14); that will be reported elsewhere.[29]

One important application of the results described here is the analysis of the divergences of the amplitudes. A report on such an analysis has been given in brief[21] and more complete[27] form elsewhere, using the raw amplitudes. Both closed type-II and heterotic massless amplitudes were shown to be finite at all loops. Proof of the above results may be considerably simplified if further reduction were possible of the multiloop amplitudes, along the lines discussed above. Such analysis seems valuable since the massless amplitudes are finite and undoubtedly deserve further consideration. Divergence analysis has still to be performed for massive amplitudes, but can now proceed for the raw amplitudes obtained in Ref. 22.

The final step is to apply the above results to the physical world. In order to do so string perturbation theory may be unsuitable, and nonperturbative techniques would be preferable. Such features are outside the realm of the present analysis; development of a covariant field theory of superstrings would appear to be a useful preliminary step before such a program can be realized.

ACKNOWLEDGMENTS

The authors would like to thank W. J. Harvey and S. Mandelstam for helpful discussions.

APPENDIX: THE SHORT-STRING LIMIT FOR $N=4$

In this appendix the details of the calculation of the short-string limit of the multiloop amplitude are given for the holomorphic part, for $N=4$, and for solely bosonic external states. Simplification is obtained by the choices.

$$\begin{aligned}&\zeta_r^{7\pm i8}=p_r^{7\pm i8}=0 \quad (r=1\text{–}4),\\ &\alpha_2,\alpha_3\sim 0, \quad (\zeta_r\cdot p_r)=0 \ .\end{aligned} \tag{A1}$$

The external superfields are

$$\phi_r=\zeta_r^{i_r}\rho^{i_r}_{A_rB_r}\theta_r^{A_r}\theta_r^{B_r} \ .$$

The amplitude (2.5) of the text becomes in this case proportional to (without integration over the moduli or inessential factors)

$$\begin{aligned}M_4=\int \prod_{i=1}^{g} d^4\theta_i \prod_{r=1}^{3} d^4\theta_r\theta_r^{2A_rB_r}\\
&\times\prod_{r=1}^{4}\zeta_r^{i_r}\rho^{i_r}_{A_rB_r}(\alpha_1\theta_1+\alpha_2\theta_2+\alpha_3\theta_3)^{2A_4B_4}a_2a_3(\alpha_2\alpha_3)^{1/2}\\
&\times V_2\{2\partial_{z_2}\overline{X}+A_2p_2+p_3\mathbf{G}'_{23}+p_1\mathbf{G}'_{21}\\
&\qquad -(a_2\alpha_2)^{-1}p_2,a_2\sqrt{\alpha_2}[-a_2{}^{-1}\theta_{21}+A_2\alpha_2\theta_{21}+\alpha_3\theta_{31}\mathbf{G}'_{23}+\theta_iv_i(z_2)]\}\\
&\times V_3\{2\partial_{z_3}\overline{X}+A_3p_3+p_z\mathbf{G}'_{32}+p_1\mathbf{G}'_{31}\\
&\qquad -(a_3\alpha_3)^{-1}p_3,a_3\sqrt{\alpha_3}[-a_3{}^{-1}\theta_{31}+A_3\alpha_3\theta_{31}+\alpha_2\theta_{21}\mathbf{G}'_{32}+\theta_iv_i(z_3)]\}\\
&\times\prod_{\gamma=1}^{2g}[L''(\tilde{z}_\gamma)]^{-1/2}V\left[2\partial_{\tilde{z}_\gamma}\overline{X}+\sum_{r=1}^{N-1}\mathbf{G}'(\tilde{z}_\gamma,z_r)p_r,L''(\tilde{z}_\gamma)^{-1/2}\right.\\
&\qquad\left.\times\left[\sum_{r=2}^{N-1}\alpha_r\theta_{r1}\mathbf{G}'(z_\gamma,z_r)+\theta_iv_i(\tilde{z}_\gamma)\right]\right] \ .\end{aligned} \tag{A2}$$

In (A2), $\theta^{2AB}=\theta^A\theta^B$, and the δ function $\delta(\sum_{r=1}^{4}\alpha_r\theta_r)$ has been used to eliminate θ_4. The leading term in α_2,α_3 in (A2) is $O(\alpha_2{}^{3/2}\alpha_3{}^{3/2})$, since the terms $O(\alpha_2{}^{1/2}\alpha_3{}^{1/2})$, $O(\alpha_2{}^{3/2}\alpha_3{}^{1/2})$, $O(\alpha_2{}^{1/2}\alpha_3{}^{3/2})$ may all be shown by inspection to contain factors $(\zeta_r\cdot p_r)$ which vanish. It is therefore necessary to obtain terms ~ 1, $O(\alpha_2)$, $O(\alpha_3)$, and $O(\alpha_2\alpha_3)$ from the vertex factors V in (A.2). In this evaluation it is useful to note that

$$\overline{X^{7+i8}(z_1)X^{7+i8}(z_2)}=\overline{X^{7-i8}(z_1)X^{7-i8}(z_2)}=0 \ ,$$

so that each contraction between two X's is quartic in the Grassmann variables. Moreover the contributions from the vertex factors must be combined with suitable orders in α_2 and α_3 from $(\alpha_1\theta_1+\alpha_2\theta_2+\alpha_2\theta_3)^{2A_4B_4}$ to produce terms of $O(\alpha_2\alpha_3)$.

As argued in the text in Sec. IV, we may modify the contributions from the purely quadratic Grassmann factors in the IIV factors at ~ 1 in a straightforward manner to obtain the quartic Grassmann factor contributions as we will see later.

The former of these will be considered first, and the change in $T_j(\bar{z})$ given by (4.15) will be specified later to include the quartic terms. The next step is to consider the term $M_4^{(0)}$ in (A2) arising from the ~ 1 term in the IIV factors. It is convenient to use the tensor $T_j(\bar{z})$ defined in Eq. (4.10). The expression for $M_4^{(0)}$ in (A2) now takes the more compact form

$$M_4^{(0)}=\int \prod_{r=1}^{3} d^4\theta_r \theta_r{}^{2i_r} \prod_{r=1}^{4} \zeta_r^{i_r}(\alpha_1\theta_1+\alpha_2\theta_2+\alpha_3\theta_3)^{2i_4}[2\partial_{z_2}\overline{X}+p_3\mathbf{G}'_{23}+p_1\mathbf{G}'_{21}-(a_2\alpha_2)^{-1}p_2]^{j_2}a_2\alpha_2$$

$$\times(-a_2{}^{-1}\theta_{21}+\alpha_3\theta_{31}\mathbf{G}'_{23})^{2j_2}[2\partial_{z_3}\overline{X}+p_2\mathbf{G}'_{32}+p_1\mathbf{G}'_{31}-(a_3\alpha_3)^{-1}p_3]^{j_3}a_3\alpha_3$$

$$\times(-a_3{}^{-1}\theta_{31}+\alpha_2\theta_{21}\mathbf{G}'_{32})^{2j_3}\prod_{\gamma=1}^{2g}[L''(\tilde{z}_\gamma)]^{-3/2}\left[2\partial_{\tilde{z}_\gamma}\overline{X}+\sum_{r=1}^{N-1}\mathbf{G}'(\tilde{z}_\gamma,z_r)p_r\right]^{j_\gamma}T_{\mathrm{j}}(\tilde{\mathbf{z}}) . \quad \text{(A3)}$$

The opposite sign in (A3) for the $a_1{}^{-1}$ terms compared to that of Ref. 7 arises due to the opposite sign for the residues in the third Abelian differentials; the end result will be independent of such a choice. The term of $O(\alpha_2\alpha_3)$ in (A3) may be evaluated by expanding the vertex factors at z_2 and z_3 as terms of ~ 1, $O(\alpha_2)$, $O(\alpha_3)$, and $O(\alpha_2\alpha_3)$ and multiplying by the appropriate terms in the expansion of $(\alpha_1\theta_1+\alpha_2\theta_2+\alpha_3\theta_3)^2$ in powers of α_r's. The various contributions may also be classified by the number r of contractions between the X's at the EIV's and the IIV's as well as by s, the number of such contractions between the IIV's; $0\le s\le g-1-(r/2)$. Then the $O(\alpha_2\alpha_3)$ contribution to $M_4^{(0)}$ of (A3) may be expressed as

$$M_4^{(0)}=\sum_{r=0}^{2}\sum_{s=0}^{g-1}M_4^{(r,s)} . \quad \text{(A4)}$$

For $r=2$ the relevant factor from z_2 and z_3 is

$$4\alpha_2\alpha_3\overline{\partial_{z_2}X^{j_2}\partial_{z_3}X^{j_3}}\theta_{21}{}^{2j_2}\theta_{31}{}^{2j_3} . \quad \text{(A5)}$$

The relevant term from $(\alpha_1\theta_1+\alpha_2\theta_2+\alpha_3\theta_3)^2$ is solely $\alpha_1{}^2\theta_1{}^2$, which combines with the term $\theta_1{}^{2i}$ in (A3) to set θ_1 to zero in all other factors. Upon performing all the Grassmann integrations over $\theta_1,\theta_2,\theta_3$ on the relevant terms in (A3)

$$M_4^{(2,s)}=\alpha_2\alpha_3\sum 4\partial_{z_2}\partial_{\tilde{z}_\alpha}G(z_2,\tilde{z}_\alpha)\partial_{z_3}\partial_{\tilde{z}_\beta}G(z_3,\tilde{z}_\beta)\prod_{i=1}^{s}4\partial_{\tilde{z}_{\alpha_i}}\partial_{\tilde{z}_{\alpha'_i}}G(\tilde{z}_{\alpha_i},\tilde{z}_{\alpha'_i})\delta_{j_{\alpha_i}j_{\alpha'_i}}$$

$$\times\prod\left[\sum_{r=1}^{N}\mathbf{G}'(\tilde{z}_\gamma,z_r)p_r\right]^{j_\gamma}T_{\mathrm{j}}(\tilde{\mathbf{z}})(\zeta_1\zeta_4)\zeta_2^{j_\alpha}\zeta_3^{j_\beta}\prod_{\delta=1}^{2g}[L''(\tilde{z}_\delta)]^{-3/z} . \quad \text{(A6)}$$

The unmarked summation in (A6) is over all distinct pairs $(\alpha,\beta),(\alpha_1,\alpha'_1),\ldots,(\alpha_s,\alpha'_s)$ from $(1,2g)$ and the unmarked product is over all γ unequal to such values.

In a similar manner the terms in M_4 for $r=1$ are $O(\alpha_2)$ and $O(\alpha_2\alpha_3)$, being

$$2\partial_{z_2}X^{j_z}[\alpha_2\alpha_3(p_2\mathbf{G}'_{32}+p_1\mathbf{G}'_{31})^{j_3}-\alpha_2L'_3p_3^{j_3}]\theta_{21}{}^{2j_2}\theta_{31}{}^{2j_3} . \quad \text{(A7)}$$

When combined with the relevant factors in the remaining part of (A3) and integration over θ_1, θ_2, and θ_3 is performed there results (powers of α_1 being dropped)

$$M_4^{(1,s)}=\alpha_2\alpha_3\sum 2\partial_{z_2}\partial_{\tilde{z}_\alpha}G(z_2,\tilde{z}_\alpha)\prod_{i=1}^{s}4\partial_{\tilde{z}_{\alpha'_i}}\partial_{\tilde{z}_{\alpha_i}}G(\tilde{z}_{\alpha_i},\tilde{z}_{\alpha'_i})\delta_{j_{\alpha_1}j_{\alpha'_1}}$$

$$\times\prod\left[\prod_{r=1}^{N-1}\mathbf{G}'(\tilde{z}_\gamma,z_r)p_r\right]^{j_\gamma}\{(\zeta_1\cdot\zeta_4)[\zeta_3\cdot(p_2\mathbf{G}'_{32}+p_1\mathbf{G}'_{31})]-2L'_3(\zeta_1\zeta_4\zeta_3p_3)\}$$

$$\times\zeta_2^{j_\alpha}T_{\mathrm{j}}(\tilde{\mathbf{z}})\prod_{\delta=1}^{2g}[L''(\tilde{z}_\delta)]^{-3/2} , \quad \text{(A8)}$$

where $\sum$ and $\prod$ have the same meaning as in (A6) except for deletion of β, and $(p_1,p_2,p_3,p_4)=\mathrm{tr}(\rho^a\rho^b\rho^c\rho^d)p_1^ap_2^bp_3^cp_4^d$. A similar term arises by interchanging 2 and 3 in $M_4^{(1,2)}$ of (A8).

The term $M_4^{(0,s)}$ has already been considered briefly in the first paper in Ref. 7, but will be given more fully here. Only the factors at the EIV's need be considered here, since the IIV's give the loop factor

$$L_g=\prod_{\gamma=1}^{2g}[L''(\tilde{z}_\gamma)]^{-3/2}\left[2\partial_{\tilde{z}_\gamma}X+\sum_{r=1}^{N-1}\mathbf{G}'(\tilde{z}_\gamma,z_r)p_r\right]^{j_\gamma}T_{\mathrm{j}}(\tilde{\mathbf{z}}) . \quad \text{(A9)}$$

In (A9) the contractions on the X's are only to be taken between each other, as the value $r=0$ indicates. The EIV's factor V_2V_3 has the following contributions:

$$\begin{aligned}
&\sim 1:\ \ a_2 a_3 p_2^i p_3^j L_2'^2 L_3'^2 \theta_{21}{}^{2i} \theta_{31}{}^{2j}\ ;\\
&O(\alpha_2):\ \ \alpha_2[a_2{}^2 a_3 L_2'^2 L_3'^2 (p_3\mathbf{G}'_{23}+p_1\mathbf{G}'_{21})^i p_3^j \theta_{21}{}^{2i}\theta_{31}{}^{2j} - 2a_2 a_3 L_2'^2 L_3' p_2^i \mathbf{G}'_{32} p_3^j \theta_{21}{}^{2i}(\theta_{31}\theta_{21}^j)]\ ; \qquad \text{(A10)}\\
&O(\alpha_2\alpha_3):\ \ \alpha_2\alpha_3\{-4G''_{23}\theta_{21}{}^{2l}\theta_{31}{}^{2l} + 4a_2 a_3 p_2^i p_3^j L_2' L_3' (\theta_{31}\theta_{21})^i (\theta_{31}\theta_{21})^j G'_{23} G'_{32}\\
&\qquad + 2a_2 a_3{}^2 L_2'^2 L_3' \mathbf{G}'_{32} p_2^i p_2^j \theta_{21}{}^{2i} (\theta_{31}\theta_{21})^j + (p_1\mathbf{G}'_{21}+p_3\mathbf{G}'_{23})^i (p_1\mathbf{G}'_{31}+p_2\mathbf{G}'_{32})^j \theta_{21}{}^{2i}\theta_{31}{}^{2j}\}\ .
\end{aligned}$$

Equation (A10) is now to be combined with the various terms of ~ 1, $O(\alpha_2)$, $O(\alpha_3)$, and $O(\alpha_2\alpha_3)$ in $(\alpha_1\theta_1+\alpha_2\theta_2+\alpha_3\theta_3)^{2i4}$ and integration performed over θ_1, θ_2, and θ_3. In the process, traces of products of SU(4) matrices $\rho^i{}_{AB}$ with polarization vectors ζ_r or moments p_r result. Thus

$$M_4^{(0,s)} = K_4 L_g \qquad \text{(A11)}$$

and K_4, the kinematic factor, may be obtained from (A2) and (A10) to be

$$\begin{aligned}
K_4 = \alpha_2\alpha_3 a_2 a_3 \Bigg[-\left[\prod_{r=1}^{4} \zeta_r^{i_r}\right] & 8p_2^i p_3^j \left(t^{ii_1 j i_3 i_4 i_2} L_2' L_3' + \mathbf{G}'_{32} L_2' t^{j i_2 i i_1 i_4 i_3} + \mathbf{G}'_{23} L_3' t^{i i_3 j i_1 i_4 i_2}\right) - 64\mathbf{G}''_{23}(\zeta_1\zeta_4)(\zeta_2\zeta_3)\\
& + 16\mathbf{G}'_{23}\mathbf{G}'_{32}(\zeta_1\zeta_4) p_2^i p_3^i \zeta_2^{i_2}\zeta_3^i 3 t^{i i_2 j i_3} + 16 L_3' \zeta_2^{i_2}(p_3\mathbf{G}'_{23}+p_1\mathbf{G}'_{23}) p_3^{j_3} \zeta_1^{i_1}\zeta_3^{i_3}\zeta_4^{i_4} t^{i_1 i_4 i_3 j}\\
& + 16 L_2' \zeta_3^{i_3}(p_2\mathbf{G}'_{32}+p_1\mathbf{G}'_{31}) p_2^j \zeta_1^i \zeta_2^{i_2}\zeta_4^{i_4} t^{i_1 i_4 i_2 j} - 64(\zeta_1\zeta_4)[(p_1\mathbf{G}'_{21}+p_3\mathbf{G}'_{23})\zeta_2][(p_1\mathbf{G}'_{31}+p_2\mathbf{G}'_{32})\zeta_3] \Bigg]\ . \qquad \text{(A12)}
\end{aligned}$$

In (A12), $t^{abcd}=\mathrm{tr}(\rho^a\rho^b\rho^c\rho^d)$, $t^{abcdef}=\mathrm{tr}(\rho^a\rho^b\rho^c\rho^d\rho^e\rho^f)$. The various terms in (A12) are to be evaluated in terms of scalar products of the SO(6) vectors ζ_r, p_r and extended to SO(1,9) vectors in the obvious manner. At one loop and higher there are also contributions from terms of higher order in the IIV factors. There are three such terms, one of $O(\alpha_2)$, one of $O(\alpha_3)$, and the third of $O(\alpha_2\alpha_3)$. The first of these arises by replacing one IIV quadratic fermionic term $[\theta_i v_i(\tilde{z}_\alpha)]^{2j_\alpha}$ by $[\theta_{21}\theta_i v_i(\tilde{z}_\alpha)]^{2j_\alpha} G'(z_\alpha,\alpha_2)$, the second by interchanging the labels 2 and 3. There will be an associated modification of the factor $T_{\mathrm{j}}(\tilde{\mathbf{z}})$ of (4.10). The remaining θ_i must be picked up from one of the two EIV factors, so will involve $\theta_i v_i(z_r)$ ($r=2$ or 3). There will thus be a generalization of $T_{\mathrm{j}}(\mathbf{z})$ to

$$T_{AB\mathrm{j}}(z_A,z_B;\tilde{\mathbf{z}}_{-\alpha}) = \int \prod_{i=1}^{g} d^4\theta_i \prod_{\beta\neq\alpha} [\theta_{i\beta} v_{i\beta}(\tilde{z}_B)]^{2j_\beta} [\theta_i v_i(z_A)]^A [\theta_j v_j(z_B)]^B\ , \qquad \text{(A13)}$$

where $z_A=z_2$ or z_3, $z_B=\tilde{z}_\alpha$; $\mathbf{j}=\{j_\alpha\}$, $\tilde{\mathbf{z}}_{-\alpha}=\{\tilde{z}_\beta\}$.

The $O(\alpha_2\alpha_3)$ factor is of two forms. The first has two changes of the above form, so replaces the factors $[\theta_i v_i(\tilde{z}_\alpha)]^{2j_\alpha}[\theta_j v_j(\tilde{z}_\beta)]^{2j_\beta}$ by $[\theta_{21}\theta_i v_i(\tilde{z}_\alpha)]^{2j_\alpha}[\theta_{31}\theta_j v_j(\tilde{z}_\beta)]^{2j_\beta}\mathbf{G}'(\tilde{z}_\alpha,z_2)\mathbf{G}'(\tilde{z}_\beta,z_3)$. The other term involves replacing a single IIV factor by $[\theta_{21}\theta_{31}]^{2j_\alpha}\mathbf{G}'(\tilde{z}_\alpha,z_2)\mathbf{G}'(\tilde{z}_\alpha,z_3)$. The associated factors extending $T_{\mathrm{j}}(\mathbf{z})$ of (4.10) are now

$$T_{ABCD\mathrm{j}}(z_A z_B z_C z_D;\tilde{\mathbf{z}}_{-\alpha-\beta}) = \int \prod_{\gamma\neq\alpha,\beta} [\theta_{i\gamma} v_{i\gamma}(\tilde{z}_\gamma)]^{2j_\gamma} \prod_{A,B,C,D} [\theta_{i\alpha} v_{i\alpha}(z_A)]^A \prod_{i=1}^{g} d^4\theta_i\ , \qquad \text{(A14)}$$

where $z_A, z_B = z_r, z_3$ and $z_C, z_D = \tilde{z}_\alpha, \tilde{z}_\beta$ or

$$T_{ABjc\mathrm{j}}(z_A z_B z_C;\tilde{\mathbf{z}}_{-\alpha}) = \int \prod_{i=1}^{g} d^4 g_i \prod_{\gamma\neq\alpha} [\theta_{i\gamma} v_{i_\gamma}(\tilde{z}_\gamma)]^{2j_\gamma} \prod_{A} [\theta_{i_\alpha} v_{i_\alpha}(z_A)]^A [\theta_{i\delta} v_{i\delta}(z_c)]^{2j_c}\ , \qquad \text{(A15)}$$

where $z_A, z_B, z_C = z_2, z_3, \tilde{z}_\alpha$, respectively. In general these new terms may be interpreted as corresponding to additional fermionic lines $\mathbf{G}'(z,z_r)$ from an IIV to an EIV. There are at most two such lines. The expressions (A13)–(A15) are special cases of the tensor spinor of (4.16).

These new terms also give contributions with different numbers r of contractions between the EIV's and IIV's. Without trying to disentangle these contractions the value of the short-string limit similar to (3.5) is, for the $O(\alpha_2)$ contributions from the IIV's, terms proportional to

$$\begin{aligned}
&[(\zeta_1\zeta_4\zeta_3 p_3) p_2^{j_2}\zeta_2^{i_2}\mathrm{tr}[\rho^{j_2}\rho^{i_2}\rho^{j_\alpha} T_{\mathrm{j}}(z_2,\tilde{z}_\alpha;\tilde{\mathbf{z}}_{-\alpha})] + \mathrm{tr}(\not{\zeta}_1\not{\zeta}_4\not{\zeta}_2\not{p}_2 T_{\mathrm{j}}\rho^{j_\beta}\not{\zeta}_3\not{p}_3)]\\
&\qquad\qquad \times\mathbf{G}'(z_3,z_1)\mathbf{G}'(\tilde{z}_\alpha,z_2)\prod_{\beta=1}^{2g}\left[2\partial_{\tilde{z}_\beta}\bar{X} + \sum \mathbf{G}'(\tilde{z}_\beta,z_r)p_r\right]^{j_\beta} \qquad \text{(A16a)}
\end{aligned}$$

[where $(p_1 p_2 p_3 p_4)=\mathrm{tr}(\rho^{a_1}\rho^{a_2}\rho^{a_3}\rho^{a_4})p_1^{a_1}p_2^{a_2}p_3^{a_3}p_4^{a_4}$], to

$$\mathrm{tr}(\rho^{j_3}\rho^{i_3}\rho^{j_\alpha}T_{\mathrm{j}})p_3^{j_3}\zeta_3^{i_3}(\partial_{z_2}\bar{X}+\cdots)^{j_2}\zeta_2^{j_2}(\zeta_1\cdot\zeta_4)\mathbf{G}'(\tilde{z}_\alpha,z_3)\mathbf{G}'(z_2,z_1)\prod_{\beta=1}^{2g}(2\partial_{\tilde{z}_\beta}\bar{X}+\cdots)^{j_\beta}\ , \qquad \text{(A16b)}$$

to

$$\mathrm{tr}(\rho^{j_2}\rho^{i_2}\rho^{j_3}\rho^{j_\alpha}\rho^{i_3}T_{\mathrm{j}})p_3^{j_3}\zeta_3^{i_3}p_2^{j_z}\zeta_2^{i_2}(\zeta_1\zeta_4)\sum_{\beta=1}^{2g}(2\partial_{\tilde{z}_\beta}\bar{X}+\cdots)^{j_\beta}\mathbf{G}'(z_3,z_2)\mathbf{G}'(\tilde{z}_\alpha,z_3) \tag{A16c}$$

and to interchanges of 2 and 3 in (A16b) and (A16c). Similar contributions from the $O(\alpha_2\alpha_3)$ terms (A14) and (A15) in the EIV factors are proportional to

$$(\rho^{j_3}\rho^{i_3}\rho^{j_\alpha})_{A_2A_3}(\rho^{j_2}\rho^{i_2}\rho^{j_\beta})_{A_1A_4}T_{A_1A_2A_3A_4}(z_2,z_3,\tilde{z}_\alpha,\tilde{z}_\beta;\tilde{\mathbf{z}})\zeta_2^{i_2}\zeta_3^{i_3}p_2^{j_2}p_3^{j_3}(\zeta_1\zeta_4)\mathbf{G}'(\tilde{z}_\alpha,z_3)G'(\tilde{z}_\beta,\tilde{z}_2)\prod_{\gamma=\alpha,\beta}^{2g}(\partial_{\tilde{z}_\gamma}\bar{X}+\cdots)^{j_\gamma} \tag{A17a}$$

and to

$$\mathrm{tr}[\rho^{j_3}\rho^{i_3}\rho^{i_\alpha}\rho_{i_2}\rho_{j_2}T_{\mathrm{j}}(z_2,z_3,\tilde{z}_\alpha;\tilde{\mathbf{z}})]\zeta_2^{i_2}\zeta_3^{i_3}p_2^{j_2}p_3^{j_3}(\zeta_1\zeta_4)\mathbf{G}'(\tilde{z}_\alpha,z_2)\mathbf{G}'(\tilde{z}_\alpha,z_3)\prod_{\gamma=\alpha}^{2g}(\partial_{\tilde{z}_\gamma}\bar{X}+\cdots)^{j_\gamma}\ . \tag{A17b}$$

As can be seen from the above there are at most two fermionic lines from each IIV and at most one from each EIV. These properties follow by direct inspection of the terms (A6), (A8), (A9), (A13)–(A16). There is the further nontrivial property of these expressions that there is no more than one line from any IIV to an EIV. This is automatic for the terms $M_4^{(r,s)}$ of (A6), (A8), and (A9), since these terms involve no fermionic terms in the first place. If a term with a bosonic line $G'(\tilde{z}_\alpha,z_r)p_r$ and a fermionic line $G'(\tilde{z}_\alpha,z_r)$ are constructed from (A13)–(A16) it can be seen from the detailed structure of (A16) that a factor of $p_r{}^2\not{\zeta}_r+(p_r\zeta_r)\not{p}_r$; this is zero since p_r is massless and orthogonal to ζ_r.

Similar expressions arise when the $X^{7\pm i8}$ coordinates are taken into account. Since the maximum order is $O(\alpha_2\alpha_3)$ in the contribution to the vertex factors from the IIV's there can still be at most two fermionic lines from each IIV, and again these will only be to different external sources. There will still be only one fermionic line per given external source. The contribution of $O(\alpha_2\alpha_3)$ from the IIV factors can nearly all be shown to vanish since only the lowest-order terms can then be allowed from the EIV factors; these terms will always give rise to a factor $(p_2\cdot\zeta_2)$ or $(p_3\cdot\zeta_3)$ after Berezhin integration over the θ_r's and the θ_i's, except for similar contributions to (A16) and (A17), with obvious modifications to T_{j} by reduction of the number of vector indices due to the replacement of quadratic terms in the θ_i by quartic or constant ones. The contributions of $O(\alpha_2)$ have a similar form to those of the change (A13) of (A16), (A8) and (A11), again with modification to T_{j}. The $O(1)$ contribution from the IIV factors only arises from contractions

$$\partial_{z_A}\overline{X^{7+i8}\partial_{z_B}X}^{7-i8}$$

when z_A and z_B are IIV's. An expression similar to (A11) is obtained with modification of L_g in (A9) by replacing an even subset of the vertex factors

$$(\partial_{\tilde{z}_\gamma}\bar{X}+\sum_r G'(\tilde{z}_\gamma z_r)p_r)^{j_\gamma}$$

by

$$\partial_{\tilde{z}_{\gamma+}}\overline{X^{7-i8}\partial_{\tilde{z}_{\gamma-}}X}^{7-i8}\ .$$

$T_{\mathrm{j}}(z)$ in (4.10) is modified by replacing the quadratic Grassmann factors by quartic factors at the points $\tilde{z}_{\gamma-}$ and by 1 at the points $\tilde{z}_{\gamma+}$. All of these extra contributions from the quartic Grassmann terms are therefore seen to correspond to extending the quadratic terms obtained earlier by replacing the range of the vector variable j to 7,8, with associated Dirac matrices $\rho^{7\pm i8}=1$. Diagrammatic representation of the various contributions (A6), (A8), (A11), (A16), and (A17) is given in Fig. 2.

[1]M. B. Green and J. H. Schwarz, Nucl. Phys. **B243**, 475 (1984).

[2]M. Kaku and K. Kikkawa, Phys. Rev. D **10**, 1110 (1974); **10**, 1823 (1974).

[3]S. Mandelstam, Phys. Rep. **13C**, 260 (1974).

[4]S. Mandelstam, Nucl. Phys. **B69**, 77 (1974).

[5]A. Restuccia and J. G. Taylor, Phys. Lett. (to be published).

[6]A. A. Belavin and V. G. Knizhnik, Phys. Lett. **168B**, 201 (1986); J. B. Jost and T. Jolicoeur, *ibid.* **174B**, 273 (1986); R. Catenacci, M. Cornalba, M. Martellini, and C. Rein, *ibid* **172B**, 328 (1986); C. Gomez, *ibid.* **175B**, 32 (1986).

[7]A. Restuccia and J. G. Taylor, Phys. Lett. **174B**, 56 (1986); **177B**, 39 (1986); King's College London report, 1986 (unpublished).

[8]H. Arfaei, Nucl. Phys. **B112**, 256 (1976).

[9]D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. **54**, 502 (1985); Nucl. Phys. **B256**, 253 (1985).

[10]L. Alvarez-Gaume, G. Moore, and C. Vafa, Commun. Math. Phys. **106**, 40 (1986).

[11]C. L. Siegel, *Topics in Complex Function Theory* (Wiley, New York, 1971).

[12]L. Dixon and J. Harvey, Princeton report (unpublished); L. Alvarez-Gaume, P. Ginsparg, G. Moore, and C. Vafa, Phys. Lett. **171B**, 155 (1986).

[13]M. B. Green and J. H. Schwarz, Phys. Lett. **136B**, 367 (1984); J. H. Schwarz, in *Supersymmetry and Its Applications,* edited by G. Gibbons, S. Hawking, and P. Townsend (Cambridge University Press, Cambridge, 1986); A. Dar, Report No. SLAC-PUB 3961, 1986 (unpublished).

[14]H. Arfaei, Nucl. Phys. **B85**, 535 (1975).

[15]C. Lovelace, Phys. Lett. **32B**, 7003 (1970); V. Alessandrini, Nuovo Cimento **2A**, 321 (1971).

[16]S. Mandelstam, in *Proceedings of the Workshop on Unified*

String Theory, Santa Barbara, 1986, edited by M. Green and D. Gross (World Scientific, Singapore, 1986).

[17]M. B. Green and J. H. Schwarz, Phys. Lett. **151B**, 21 (1985).

[18]See, for example, M. H. Farkas and I. Kra, *Riemann Surfaces* (Springer, Berlin, 1980).

[19]T. Azaka, Nagoya Math. J. **24**, 43 (1964).

[20]A. Lebowitz, in *Advances in the Theory of Riemann Surfaces Annals of Mathematical Studies,* edited by L. V. Ahlfors *et al.* (Princeton University Press, Princeton, New Jersey, 1971).

[21]J. G. Taylor, in 60th birthday celebration volume for E. S. Fradkin (unpublished); A. Restuccia and J. G. Taylor, Phys. Lett. **187B**, 267 (1987); **187B**, 273 (1987).

[22]A. Restuccia and J. G. Taylor (unpublished).

[23]J. H. Schwarz, Phys. Rep. **89**, 223 (1982).

[24]P. Bressloff, A. Restuccia, and J. G. Taylor, KCL report, 1986 (unpublished).

[25]O. Alvarez, Nucl. Phys. **B216**, 125 (1983); in *Proceedings of the Workshop on Unified String Theories, Santa Barbara, 1985* (Ref. 16); Berkeley Report No. UCB-PTH-85/50 (unpublished).

[26]S. Tahikozawa, Phys. Lett. **166B**, 135 (1986).

[27]A. Restuccia and J. G. Taylor, Commun. Math. Phys. (to be published).

[28]S. Mandelstam, Nucl. Phys. **B64**, 205 (1973).

[29]A. Restuccia and J. G. Taylor (unpublished).