A parametrization of the covariant superstring

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We exhibit a parametrization of the covariant superstring which at one and the same time encodes the bosonic and fermionic components of the superstring in such a way as to satisfy all of the classical equations of motion.

I. INTRODUCTION

Recently,[1] we found a bilinear parametrization of the bosonic string in 3, 4, 6, and 10 dimensions based upon the following identity for two elements z and z' of a division algebra:

$$|z|\,|z'|=|zz'| \,, \tag{1.1}$$

where the parametrization identifies the two light-cone components of the vector $\partial X^\mu/\partial(\sigma+\tau)$ with $|z|^2$ and $|z'|^2$ and identifies the transverse components of the vector with the coefficients of the basis elements of zz'. [For octonions as the division algebra, (1.1) is known as the eight-squares theorem.[2,3]] This parametrization achieves two objectives: it automatically incorporates the null quadratic constraints of the theory and it permits a realization of the action of the Lorentz transformations on X^μ as a linear action upon the z and z' (specifically the components of z and z' transform as a spinor). However, this parametrization is apparently redundant, having twice as many variables as are necessary to describe the bosonic string. This would seem to indicate, as does the above spinor transformation, that the parametrization might also encode the fermionic coordinates of the superstring. In this paper we show that this is indeed possible. Furthermore, this parametrization gives classical solutions of all of the equations of motion of the Green-Schwarz[4] covariant superstring. By classical, we mean that all of the coordinates are treated as commuting variables. These solutions appear to be new.

In Sec. II we give the parametrization explicitly for the ten-dimensional case with octonions as the division algebra. (The cases of 3, 4, and 6 dimensions, associated with the real, complex, and quaternion division algebras are similar.) We then show that the parametrization solves the equations of motion. This proof involves the use of the identity familiar from super-Yang-Mills theories,[5]

$$\gamma_\mu\Psi_1\overline{\Psi}_2\gamma^\mu\Psi_3+\gamma_\mu\Psi_2\overline{\Psi}_3\gamma^\mu\Psi_1+\gamma_\mu\Psi_3\overline{\Psi}_1\gamma^\mu\Psi_2=0 \,, \tag{1.2}$$

which we establish by a new route which illuminates its dimension dependence. In fact, it becomes increasingly apparent that the special dimensionality associated with supersymmetric gauge theories is just a reflection of the properties of division algebras. In Sec. III we discuss the local supersymmetry and local bosonic invariances in the light of this new parametrization. In Sec. IV we explore the possibility of treating the fundamental spinor out of which the string coordinates are composed as a truly anticommuting object. We then find new and strange transformation properties for the bosonic variables (which are represented as fermionic bilinears).

II. THE PARAMETRIZATION OF THE SUPERSTRING

The Green-Schwarz covariant superstring action[4] takes the form

$$S=\frac{1}{\pi}\int d\sigma\, d\tau(L_1+L_2) \,, \tag{2.1}$$

where

$$L_1=-\tfrac{1}{2}\sqrt{-g}\,g^{\alpha\beta}\Pi^\mu_\alpha\Pi_{\beta\mu} \,, \tag{2.2}$$

$$L_2=-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar\theta^1\gamma_\mu\partial_\beta\theta^1-\bar\theta^2\gamma_\mu\partial_\beta\theta^2) -\epsilon^{\alpha\beta}\bar\theta^1\gamma^\mu\partial_\alpha\theta^1\bar\theta^2\gamma_\mu\partial_\beta\theta^2 \,, \tag{2.3}$$

$$\Pi^\mu_\alpha=\partial_\alpha X^\mu-i\bar\theta^A\gamma^\mu\partial_\alpha\theta^A \,. \tag{2.4}$$

$X^\mu(\sigma,\tau)$ are the ten bosonic coordinates of the world sheet and $\theta^A(\sigma,\tau)$, $A=1,2$, are the two 32-component spinors in the ten-dimensional spacetime. The two-dimensional metric $g^{\alpha\beta}$ has the signature $(-+)$ and the ten-dimensional metric used to contract the μ index has the signature $(-+++\cdots)$. The equations of motion which follow from this action after making the covariant (two-dimensional light-cone) gauge choice $g^{\alpha\beta}=\eta^{\alpha\beta}$ are

$$\Pi^\mu_-\Pi_{-\mu}=0,\quad \Pi^\mu_+\Pi_{+\mu}=0 \,, \tag{2.5}$$

$$\gamma_\mu\Pi^\mu_-\partial_+\theta^1=0,\quad \gamma_\mu\Pi^\mu_+\partial_-\theta^2=0 \,, \tag{2.6}$$

$$\partial_+\partial_-X^\mu-i\partial_-(\bar\theta^1\gamma^\mu\partial_+\theta^1)-i\partial_+(\bar\theta^2\gamma^\mu\partial_-\theta^2)=0 \,, \tag{2.7}$$

where the subscripts $\pm$ refer to $\sqrt{1/2}(\sigma\pm\tau)$. Note that these are the equations of motion whether the θ's are commuting or anticommuting variables. Following the ideas of the introduction and Ref. 1, we can write

$$\Pi^+_-=\chi^\dagger\chi \,,\quad \Pi^-_-=\psi^\dagger\psi \,,\quad \Pi^i_-=(\chi^\dagger\Lambda^i\psi) \,, \tag{2.8}$$

where χ and ψ are 8-component objects which together form a 32-component Majorana-Weyl spinor

$$\Psi^1 = \begin{bmatrix} \chi \\ \psi \\ 0 \\ 0 \end{bmatrix} .$$

There is a similar parametrization of the Π^μ_+ in terms of a Majorana-Weyl spinor Ψ^2. The matrix Λ^8 is given by $I\otimes I\otimes I$, and the real, antisymmetric matrices Λ^i $(i=1,\ldots,7)$ which satisfy $\{\Lambda^i,\Lambda^j\}=-2\delta^{ij}$ can be constructed as follows:[1,6]

$$\{\Lambda^i\} = \begin{cases} i\sigma_2\otimes I\otimes I \, , \\ i\sigma_3\otimes\sigma_2\otimes\sigma_3 \, , \\ i\sigma_1\otimes\sigma_2\otimes I \, , \\ i\sigma_3\otimes I\otimes\sigma_2 \, , \\ i\sigma_1\otimes\sigma_3\otimes\sigma_2 \, , \\ i\sigma_3\otimes\sigma_2\otimes\sigma_1 \, , \\ i\sigma_1\otimes\sigma_1\otimes\sigma_2 \, . \end{cases} \tag{2.9}$$

Equations (2.5) are automatically satisfied since [see (2.14) below]

$$\chi^\dagger\chi\psi^\dagger\psi = \sum_{i=1}^{8} (\chi^\dagger\Lambda^i\psi)(\chi^\dagger\Lambda^i\psi) \, . \tag{2.10}$$

If we define two octonions via

$$z=\chi^\alpha e_\alpha \quad\text{and}\quad z'=\psi^\alpha e_\alpha \, , \tag{2.11}$$

where the e_α are a basis for the octonions then (2.10), and therefore (2.5), are just versions of the eight-squares theorem (1.1).

This parametrization can also be written in the following form, for the Majorana-Weyl spinor Ψ^1, which makes its Lorentz transformation properties manifest:

$$\Pi^\mu_- = i\Psi^{1\dagger}\gamma^C\gamma^\mu\Psi^1 \, , \tag{2.12}$$

where the 32×32 matrices γ^μ are represented by

$$\begin{aligned} &\gamma^0 = i\sigma_2\otimes I\otimes\Lambda^8 \, , \\ &\gamma^i = \sigma_1\otimes i\sigma_2\otimes\Lambda^i \quad \text{for } i=1,\ldots,7 \, , \\ &\gamma^8 = \sigma_1\otimes\sigma_1\otimes\Lambda^8 \, , \\ &\gamma^9 = \sigma_1\otimes\sigma_3\otimes\Lambda^8 \, . \end{aligned} \tag{2.13}$$

In this representation the charge-conjugation matrix is $\gamma^C=i\gamma^0$ if the θ's are commuting variables and γ^0 if they are anticommuting variables. If Ψ^1 transforms like a (reducible) 32-component spinor of SO(1,9) then Π^μ_- transforms as a ten-dimensional vector. Notice that our γ matrices satisfy $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$. They differ by a factor of i from those of Green and Schwarz.

To see how the identities work, we need to develop a few relations satisfied by the Λ matrices. The fundamental relation which we shall use is

$$\sum_{i=1}^{7} \Lambda^i_{\alpha\beta}\Lambda^i_{\gamma\delta} = -2T_{\alpha\beta\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta} \, , \tag{2.14}$$

where $T_{\alpha\beta\gamma\delta}$ is the four-index tensor introduced in Refs. 7 and 8. It is not necessary for us to know anything about T other than that it is invariant under the action of SO(7) and it is antisymmetric in its four indices. Equation (2.14) is proved by noticing that the left-hand side is an SO(7)-invariant tensor, so it must be representable in terms of a linear combination of isotropic SO(7)-invariant tensors $(T_{\alpha\beta\gamma\delta}, \delta_{\alpha\gamma}\delta_{\beta\delta}, \delta_{\alpha\delta}\delta_{\beta\gamma}, \delta_{\alpha\beta}\delta_{\gamma\delta})$. Symmetry properties of the left-hand side dictate the allowable combinations of tensors on the right-hand side. The coefficients in the linear combination may be determined from a few specific choices for the indices. A version of this identity (2.14) is recorded in Ref. 8. Using (2.14) and saturating the indices $\alpha,\beta,\gamma,\delta$ with $\chi_\alpha,\psi_\beta,\chi_\gamma,\psi_\delta$, in consequence of the antisymmetry of T we obtain the eight-squares theorem.

The result (2.14) can also be used to prove (1.2). Using the identification (2.13) and the relations

$$\begin{aligned} &\sigma_{2ab}\sigma_{2cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \, , \\ &-I_{ab}I_{cd} + \sigma_{1ab}\sigma_{1cd} + \sigma_{3ab}\sigma_{3cd} = \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{cd} - 2\delta_{ab}\delta_{cd} \end{aligned} \tag{2.15}$$

we have

$$\begin{aligned} -(\gamma^{R,L}\gamma^\mu)_{a\alpha b\beta}(\gamma^{R,L}\gamma_\mu)_{c\gamma d\delta} &= (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - 2\delta_{ab}\delta_{cd})\delta_{\alpha\beta}\delta_{\gamma\delta} \\ &\quad + (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \\ &\quad\quad \times(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} - 2T_{\alpha\beta\gamma\delta}) \, , \end{aligned} \tag{2.16}$$

where $\gamma^{R,L}$ is the chiral projection $\frac{1}{2}(1\mp\gamma^{11})$ times γ^C. (In our representation $\gamma^{11}=-\sigma_3\otimes I\otimes I$.) Hence,

$$\begin{aligned} &(\gamma^{R,L}\gamma^\mu)_{a\alpha b\beta}(\gamma^{R,L}\gamma_\mu)_{c\gamma d\delta} + (\gamma^{R,L}\gamma^\mu)_{a\alpha c\gamma}(\gamma^{R,L}\gamma_\mu)_{d\delta b\beta} \\ &\quad + (\gamma^{R,L}\gamma^\mu)_{a\alpha d\delta}(\gamma^{R,L}\gamma_\mu)_{b\beta c\gamma} = 0 \, . \end{aligned} \tag{2.17}$$

The standard identity (1.2) is simply obtained by contracting (2.17) with three arbitrary Majorana-Weyl spinors of the same chirality $\Psi_{1b\beta}$, $\Psi_{2c\gamma}$, $\Psi_{3d\delta}$. (Remember that for these spinors $\overline{\Psi}=\Psi^\dagger\gamma^C$.) It is noteworthy that this identity is true whether the Ψ's commute or anticommute. We can also see the eight-squares theorem from this point of view by contraction of (2.17) with four identical Majorana-Weyl spinors.

Similar results hold in the case of 6 dimensions where $T_{\alpha\beta\gamma\delta}$ is replaced by $\epsilon_{\alpha\beta\gamma\delta}$ and in the case of four dimensions where no such tensor is necessary. There is no comparable identity in higher dimensions. This is most easily inferred from the observation that antisymmetric tensors such as $T_{\alpha\beta\gamma\delta}$ play the role of structure constants for the division algebras through the relationship

$$e_\alpha e_\beta = T_{\alpha\beta\gamma 8}e^\gamma + \delta_{\alpha 8}e_\beta + \delta_{\beta 8}e_\alpha - \delta_{\alpha\beta} \tag{2.18}$$

and there are no division algebras beyond the octonions.

The identity (2.17) also provides the basis for the establishment of the existence of nontrivial solutions to the spinor equations of motion (2.6). Suppose we choose the three spinors above to all be $\Psi^1_{b\beta}=(\chi_\beta,\psi_\beta)$. Then (2.14) is tantamount to

$$\Pi^\mu_- \gamma^C \gamma_\mu \Psi^1 = 0 \ . \tag{2.19}$$

Thus if we let $\partial_+\theta^1$ be proportional to Ψ^1, since γ^C is invertible, (2.6) is solved. In the standard analysis of the superstring only the trivial solution, θ^1 a function of just x_-, is considered. However, in our analysis, the proportionality factor $\partial_+ f$ can depend on both x_+ and x_-. The important distinction between this solution and the standard one is that here θ^1 and $\partial_- X^\mu$ are both constructed out of the same parameters Ψ^1. The other constraint for θ^2 in (2.6) is solved similarly by choosing $\partial_-\theta^2$ to be proportional to Ψ^2. For later convenience we will call the proportionality factor for θ^1, $\partial_+ f^1$ and for θ^2, $\partial_- f^2$.

It is amusing to note that from the point of view of the underlying division algebra, the existence of nontrivial solutions to (2.6) is associated with the simple result

$$z'(\bar{z}z) = (\bar{z}z)z' = \bar{z}(zz') \ . \tag{2.20}$$

The first equality is just commutativity of the scalar $\bar{z}z$. The second equality is alternativity, a weak form of associativity satisfied by octonions. In our previous identification we had $\Pi^i_- =$ the components of zz'. If we now identify the rows of $\gamma^\mu \partial_+ \theta^1$ with the components of z' and $\bar{z}$, then (2.20) is just another form of (2.19).

Straightforward substitution shows that the equation of motion (2.7) is automatically satisfied as long as Ψ^1 and therefore Π^μ_- depends only on x_- and similarly Ψ^2 and therefore Π^μ_+ depends only on x_+. Hence $\theta^1 = f^1\Psi^1$ and $\theta^2 = f^2\Psi^2$. We have also used the facts that X^μ is real and the Lagrangian is Hermitian.

This situation is somewhat reminiscent of the classical solutions of monopole theory in the Prasad-Sommerfield limit, where the Dirac spinors in the background field of the monopole are necessarily chiral and are parametrized by the same vectors as are employed in the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of the monopoles themselves.[9]

III. LOCAL INVARIANCES

Green and Schwarz[4] point out that the action (2.1) is invariant under the local bosonic transformations

$$\begin{gathered} \delta_\lambda \theta^1 = \partial_+ \theta^1 \lambda^1_- \ , \\ \delta_\lambda \theta^2 = \partial_- \theta^2 \lambda^2_+ \ , \quad \delta_\lambda X^\mu = i\bar{\theta}^A \gamma^\mu \delta_\lambda \theta^A \ , \end{gathered} \tag{3.1}$$

where the two-vector λ^1_α is anti-self-dual and the two-vector λ^2_α is self-dual, i.e., $\lambda^1_+ = \lambda^2_- = 0$. For our solutions, these transformations correspond to variations in the proportionality constants f^1 and f^2. On the space of our solutions $\partial_+\theta^1 = \partial_+ f^1 \Psi^1$. The only restriction on f^1 and f^2 arising from the equations of motion are that they be real, so the Lagrangian should remain invariant under independent real changes in f^1 and f^2. We can consider the variation $\delta_\lambda\theta^1$ to be entirely attributed to a change in f^1 of the form $f^1 \to f^1 + \delta f^1$, where $\delta f^1 = \partial_+ f^1 \lambda^1_-$ for arbitrary real λ^1_-, so that for our solutions $\delta\theta^1 = \partial_+\theta^1\lambda^1_-$ as desired. The other variations in (3.1) follow straightforwardly. [Romans[10] has recently shown that these bosonic transformations (3.1) can be ascribed to field redefinitions. This is in accord with our interpretation.]

Green and Schwarz[4] also point out that the action (2.1) is invariant under the local supersymmetry transformations

$$\begin{aligned} \delta_\kappa \theta^1 &= (2i\gamma\cdot\Pi_- - 8\partial_-\theta^1\bar{\theta}^1)\kappa^1_+ \ , \\ \delta_\kappa \theta^2 &= (2i\gamma\cdot\Pi_+ - 8\partial_+\theta^2\bar{\theta}^2)\kappa^2_- \ , \\ \delta_\kappa X^\mu &= -8\Pi^\mu_-\bar{\theta}^1\kappa^1_+ - 8\Pi^\mu_+\bar{\theta}^2\kappa^2_- + i\bar{\theta}^A\gamma^\mu\delta_\kappa\theta^A \ , \end{aligned} \tag{3.2}$$

where κ^1_α is anti-self-dual and a function of x_- alone, and κ^2_α is self-dual and a function of x_+ alone, i.e., $\kappa^1_- = \kappa^2_+ = 0$.

Using (2.14) or simply as a consequence of the $\mu\nu$ symmetry of $\Pi^\mu_-\Pi^\nu_-$ and (2.5), it is easy to see that

$$(\Pi^\mu_-\gamma_\mu)(\Pi^\nu_-\gamma_\nu) = 0 \ . \tag{3.3}$$

This shows that another class of solutions of the fermionic constraints (2.6) can be obtained by identifying $\partial_+\theta^1$ with $\Pi^\mu_-\gamma_\mu\kappa^1_+$, where here κ^1_+ is an arbitrary Majorana-Weyl spinor. It is clear that these solutions are different from the solutions discussed after (2.19) as these solutions are quadratic in χ and ψ whereas the first solutions were linear. These solutions are just the supersymmetry transformations acting on our original solutions. Green and Schwarz point out that a two-dimensional reparametrization has to be added to this transformation to maintain the light-cone version of the equations of motion [as in (3.2)].

Returning to our parametrization, we may enquire as to whether there exist transformations on Ψ^1 which leave the components of Π^μ_- invariant. Indeed there are in four and six dimensions. These are just "phase" transformations which act on the elements of the division algebra. The two elements z and z' must transform oppositely under these phase transformations:

$$z \to e^{\phi} z, \quad z' \to e^{\bar{\phi}} z' \ , \tag{3.4}$$

where ϕ is a constant, pure imaginary number in four dimensions and a constant, pure imaginary quaternion in six dimensions. Apart from an overall scale these transformations encode the redundancy in the bosonic parametrization allowing the usual choice of $z' = p_+$, the light-cone parametrization of Goddard, Goldstone, Rebbi, and Thorn.[11] Interestingly, in the case of ten dimensions, where one would analogously expect seven "phase" transformations, no such linear transformations exist as a consequence of nonassociativity of octonions.

IV. ANTICOMMUTATIVITY

In the analysis of Sec. II, it has been assumed that all the variables are commuting. From the point of view of quantization, we would prefer the basic spinor Ψ^1 to be Grassmannian. However, we would also like the Fourier components of Π^μ_- to satisfy a Heisenberg algebra. This does not work with our choice of (2.8) when we impose canonical anticommutation relations on Ψ^1. If from the outset we assume anticommutation relations for Ψ^1, then we can construct a set of bilinear quantities each of whose

Fourier components satisfy a Heisenberg algebra for the transverse degrees of freedom. It is clear that this must work as all we are demanding is a fermionic representation of the Kac-Moody algebra with finite subalgebra $[\mathrm{U}(1)]^{d-2}$. This is accomplished by substituting for the Λ matrices of Sec. II, a new set $\tilde{\Lambda}$ which are obtained from the Λ's by replacing σ_3 by I and $i\sigma_2$ by σ_1. The Λ^i's formed an anticommuting set, the new $\tilde{\Lambda}$'s form a commuting set. It is this change from anticommuting to commuting which allows the Fourier components to satisfy a Heisenberg algebra. Remarkably this replacement does not affect the analysis of Sec. II, i.e., Eqs. (1.2) and (2.5)–(2.7) are still satisfied where $\gamma^C\gamma^\mu$ is interpreted via (2.13) with the above substitutions. Furthermore, this construction works in any dimension $d=2^n+2$ since $i\sigma_2\otimes\tilde{\Lambda}$ are antisymmetric and one can always find a set of 2^n commuting antisymmetric matrices of dimension 2^{n+1}. The matrix elements with Ψ^1 then give the bosonic objects.

The strangest feature of this construction is that the light-cone components of Π^μ_- vanish, yet we still maintain Π^μ_- as a null vector. In fact, by juggling with signs it is possible to show that the first construction gives null vectors in spaces of signature $(1,d-1)$, $(3,d-3)$, etc., while the second, with anticommuting variables, realizes in a formal manner null vectors with signature $(0,d-2)$, $(2,d-4)$, etc., and this latter construction works for $d=2^n+2$.

Just as with the original representation, we should like to have linear transformations on Ψ^1 generate transformations on Π^i_- which preserve its null character. To see how this works, it is best to see how the Lorentz transformations are generated in the original representation. The action of the Lorentz group is generated by the following four types of matrices:

$$I\otimes[\Lambda_i,\Lambda_j],\quad \sigma_3\otimes\Lambda_i,\quad i\sigma_2\otimes\Lambda_i,\quad \sigma_1\otimes\Lambda_i \tag{4.1}$$

(where $i=1,\ldots,8$), which act on the column vector Ψ^1 in a representation where all the matrices are real. These matrices close under commutation on the Lorentz group in 10 dimensions. The number of matrices of the first type is 21 and of each of the other types is 8. This gives a total of 45, the correct dimension for SO(1,9). In the new representation, the matrices which generate the transformations which preserve the null character of Π^μ_- are obtained from the above transformation by the same substitutions, σ_3 by I and $i\sigma_2$ by σ_1. The first type of matrices are absent since the $\tilde{\Lambda}$'s commute. Of the second type, the substitution forces two of the groups of matrices to be identical. We are left with

$$\sigma_1\otimes\tilde{\Lambda}^i,\quad I\otimes\tilde{\Lambda}^i\,. \tag{4.2}$$

There are 2^{n+1} matrices of dimension 2^{n+1} which close under anticommutation. In fact, they form a Jordan algebra.

The transformations which stabilize the representations, in a similar manner to the phase transformations of Sec. I are in fact just the SO$(1,2+n)$ transformations acting on Ψ^1. The Lorentz transformations do not appear to be realized linearly on Ψ^1: transformations linear in Ψ^1 which move the components of Π^μ_- are the analogies of the phase transformations. Further elucidation of these mysteries, which are really a consequence of the fundamental nilpotent character of bilinears in a finite number of Grassmann variables will, we believe, require a deeper understanding of the role of ghost fields in maintaining covariant quantization.

V. FURTHER REMARKS

There is an intriguing prospect that the description of the superstring which we have constructed may be extended to incorporate the lattice framework of the heterotic string. In Ref. 3 Coxeter shows how to construct the root lattice of E_8 out of what he calls integral octonions, i.e.,

$$a=\sum_{i=1}^{8} a^i e_i\;, \tag{5.1}$$

where the coordinates a^i are either all integers or all half odd integers or 4 of them are half odd integers and the rest are integers, by regarding a as an eight-dimensional vector with scalar product $\frac{1}{2}(\bar{a}b+b\bar{a})$. This suggests the existence of a yet deeper interplay between the special properties of octonions and superstrings.

Recently, Derrick[12] has employed a parametrization equivalent to ours in four dimensions to describe particle motions. He also discusses the quantization of this system in terms of his parametrization. After this paper was typed, we became aware of a preprint by Bengtsson,[13] which covers similar ground.

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