BRST-invariant transitions between closed and open strings

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We consider the problem of how closed strings couple to open strings. We obtain a Becchi-Rouet-Stora-Tyutin- (BRST-) invariant "first-quantized" operator Υ which codes off-shell transitions between a single closed-string state and a single open-string state. Υ is obtained in both the bosonic and the fermionic language for the ghosts. We confirm that Υ implies the correct scattering amplitudes for one closed string in its tachyonic and massless states and any number of tachyon open strings. Our BRST-invariant transition operator is then used in a novel analysis of the Cremmer-Scherk-Higgs mechanism, in which the open-string photon and the closed-string antisymmetric tensor get a mass. We close with some remarks on the possible usefulness of our methods for string field theory.

I. INTRODUCTION

The problem of formulating a field theory for closed strings has so far resisted a compelling solution.[1,2] On the other hand, the field theory of open strings, as formulated by Witten,[3] seems to be satisfactory. It provides us with a complete theory of closed strings in interaction with open strings. This is because, as has been known since the early days of dual resonance models, closed strings appear consistently as particle singularities in loop diagrams of the open-string field theory. Thus each of the string models which contain an open-string sector can presumably be formulated as a string field theory of the Witten type. This is not a completely satisfactory theory of closed strings, of course, since there are several interacting string models of closed strings alone. Among these, one can view the nonheterotic closed-string models as limiting cases of the ones containing open strings: simply formulate the larger theory as a function of N, the internal-symmetry group, and then take $N \rightarrow 0$.

In these theories we may begin with considering only open-string states, but closed-string states occur automatically as intermediate states. It is possible to extract all closed-string scattering amplitudes by factoring, on closed-string poles, larger diagrams in which these closed strings decay into open strings. In fact, the transition amplitude was found long ago by Cremmer and Scherk[4] and Clavelli and Shapiro,[5] who factored the closed-string singularity in the one-loop nonplanar amplitude for open strings. They found the operator $\check{\Upsilon}$ which gives the transition between arbitrary states of one closed string and one open string. All open-string–closed-string transitions occur through this operator, so it provides a tool for describing closed-string states in open-string field theory.

Because there are gauge symmetries in the description of string states, the operator $\check{\Upsilon}$ is uniquely specified by factorization only between physical states. Indeed, using the light-cone formalism, Goldstone[6] and Kaku and Kikkawa[7] evaluated a different transition operator $\Upsilon_\perp$ as a quantum-mechanical overlap of the open- and closed-string descriptions of a state.

Of course, these ancient texts do not treat the ghost degrees of freedom, and thus a major goal of this paper is to modernize the open-string–closed-string transition to a form Υ, which includes the Feynman-Faddeev-Popov ghosts and which respects the world-sheet Becchi-Rouet-Stora-Tyutin (BRST) invariance. If we describe the transition amplitude as a matrix element in independent closed- and open-string oscillators,

$$T_\Upsilon = \langle 0 \mid \Upsilon \mid \text{open} \rangle \mid \text{closed} \rangle \ ,$$

we construct Υ so that

$$\langle 0 \mid \Upsilon(Q^{\text{open}} + Q^{\text{closed}}) = 0 \ .$$

We make the ansatz that the new Υ should be given by the old operator multiplied by an exponential bilinear in b and c.

The world-sheet interpretation of $\check{\Upsilon}$ is less transparent than the light cone $\Upsilon_\perp$. In this article we concentrate on making the covariant overlap Υ_0, which is just $\Upsilon_\perp$ with the transverse oscillators replaced by covariant ones, into a BRST-invariant Υ. It is also possible to derive Υ directly as an overlap including bosonized ghosts. In fact, we will reverse that order in this paper, beginning, in Sec. II, with an overlap calculation of an operator Υ, done covariantly using bosonized ghosts and functional-integral methods. We would expect Υ to be BRST invariant because the BRST charge is an integral over a local density. In Sec. III, we give our original calculation, starting from the ansatz based on Υ_0 and fermionic ghosts. Thus we obtain a form for Υ which is BRST invariant by construction. In Sec. IV we show the equivalence between the bosonic and fermionic forms of Υ, thereby establishing the BRST invariance of the bosonized overlap.

The two forms, $\check{\Upsilon}$ and Υ, seem considerably different, yet they should give the same physical amplitudes. Using our new Υ, we construct the amplitudes for a closed-string tachyon or massless particle to scatter with N open tachyons. We show these amplitudes to be identical to the amplitudes calculated using the original $\check{\Upsilon}$, which was extracted by factorizing nonplanar loop graphs and therefore gives the correct answer. We also relate the coupling constant G, which multiplies Υ, to g, the three-open-string coupling. This involves determining the precise normalization factors which multiply the one-loop amplitude. We obtain these factors by a careful consideration of Feynman's "tree theorem."

The precise relation of G to g is important for the results of Sec. VI, in which we describe the Cremmer-Scherk[8] mechanism for generating a mass for the antisymmetric tensor closed-string state in our new formalism (using Υ in place of $\check{\Upsilon}$). We close this paper with four appendixes devoted to technical details: some binomial coefficient identities, the cancellation of cubic terms in the BRST invariance of Υ, an alternative evaluation of the overlap using Neumann functions, and the correct counting of diagrams using the Feynman tree theorem.

II. THE TRANSITION AS AN OVERLAP

In light-cone quantization, the open-string–closed-string transition amplitude is given[6,7] by an overlap operator $\Upsilon_\perp$. The transition corresponds to the closed-string breaking at a point σ_I, but otherwise unchanged. For convenience Υ's are defined with $\sigma_I=0$; as an interaction term an integral over σ_I needs to be inserted either into the interaction vertex or into the propagators. We shall construct the covariant analog:

$$\langle 0 | \Upsilon_0 | \psi_1\rangle_{\text{closed}} | \psi_2\rangle_{\text{open}} = \int \mathcal{D}X^\mu(\eta)\psi_1(X^\mu(\eta))\psi_2(X^\mu(\pi-\eta)) , \qquad (2.1)$$

where

$$\psi_i(X^\mu(\eta)) = \langle X^\mu(\eta) | \psi_i\rangle \qquad (2.2)$$

is just the Schrödinger-picture wave functional for the string state $|\psi_i\rangle$. In order to treat the two strings on an equal footing, we have expressed Eq. (2.1) as a bilinear form in ψ_1,ψ_2, rather than as a quantum-mechanical overlap in which one of the factors is complex conjugated, and which would be antilinear in that factor. If, as in string field theory[3] (suppressing ghosts), ψ_i satisfies the reality condition

$$\psi_i^*(X^\mu(\eta)) = \psi_i(X^\mu(\pi-\eta)) , \qquad (2.3)$$

then (2.1) coincides with the quantum-mechanical overlap. The reality condition (2.3) embodies the interpretation of a string state with reversed orientation as the charge conjugate of the first string. For a nonorientable string, ψ_i would, in addition, be invariant under $\eta\to\pi-\eta$, so (2.3) would imply that ψ_i would be real.

In this section we will use path-integral methods to construct the operator Υ, which represents the overlap of the wave functionals, including the bosonized ghosts. In most ways the bosonized ghost field $\phi(\sigma,\tau)$ acts just like another component $x^\mu(\sigma,\tau)$ of the ordinary space-time position field, so our calculation will parallel Refs. 6 and 7. There are differences, however. Firstly, on closed strings the left and right ghost numbers are not identified, so that the field ϕ may have a nonzero winding number n proportional to the difference of these ghost numbers. Then ϕ is not periodic but rather changes by $2\pi nR$ as σ goes once around the loop. The momentum values are also quantized. In addition, the ghost field couples to world-sheet curvature, so that ghost-number conservation is violated in a definite way by the world-sheet path integrals, and one must insert a factor

$$e^{i\beta\phi(\rho_I)} \quad \text{with } \beta=3/2\sqrt{2} \qquad (2.4)$$

at the point ρ_I where the closed string breaks to form an open string. Finally, unlike the Hermitian x^μ, the ghost field ϕ is anti-Hermitian. The problem of finding the operator expression for the overlap (2.1) is an algebraic issue,[9] however, and the ghosts are algebraically isomorphic to a compactified coordinate. We shall apply the path-integral methods treating ϕ as a Hermitian compactified coordinate. Once we reexpress Υ as a bilinear rather than sesquilinear operator, we have solved the algebraic problem, regardless of the Hermiticty properties of the field.

As stressed by Mandelstam,[10] path-integral techniques are extremely useful in evaluating overlap amplitudes of the type in Eq. (2.1). To see this let us write the right-hand side (RHS) of Eq. (2.1) as

$$\int \mathcal{D}Y\mathcal{D}Z \int \mathcal{D}X\langle X | e^{-T_cH_c} | Y\rangle\langle X^T | e^{-T_oH_o} | Z\rangle e^{i\beta X} \times\langle Y | e^{T_cH_c} | \psi_1\rangle\langle Z | e^{T_oH_o} | \psi_2\rangle , \qquad (2.5)$$

where we have defined the twisted functional

$$X^T(\eta) = X(\pi-\eta) . \qquad (2.6)$$

The propagators are real and invariant under twisting both states, so we can write

$$\langle X^T | e^{-T_oH_o} | Z\rangle = \langle Z | e^{-T_oH_o} | X^T\rangle = \langle Z^T | e^{-T_oH_o} | X\rangle . \qquad (2.7)$$

Thus (2.5) involves

$$\Omega(Z,Y) = \langle Z^T | e^{-T_oH_o}\Upsilon e^{-T_cH_c} | Y\rangle \equiv \int \mathcal{D}X(\eta)\langle Z^T | e^{-T_oH_o} | X\rangle e^{i\beta X(\eta_I)}\langle X | e^{-T_cH_c} | Y\rangle = \int \mathcal{D}X(\sigma,\tau)\exp\left[-\frac{1}{4\pi\alpha'}\int_{\mathcal{R}}(\nabla X)^2\right]e^{i\beta X(\rho_I)} . \qquad (2.8)$$

The region $\mathcal{R}$ is $(-T_c<\tau<T_o)\times(0\leq\sigma\leq\pi)$, as shown in Fig. 1(a). The path integral is over all trajectories satisfying the boundary conditions

$$\begin{aligned}&X(\sigma,-T_c)=Y(\sigma)\ ,\\&X(\sigma,T_o)=Z(\pi-\sigma)\ ,\\&X(\pi,\tau),\ \ X(0,\tau)\ \text{ unconstrained for } \tau>0\ ,\\&X(\pi,\tau)-X(0,\tau)=2\pi nR,\quad \tau<0\ .\end{aligned}\tag{2.9}$$

The last two conditions just reflect the fact that $H_o(H_c)$ is the Hamiltonian for the open string (closed string with winding number n). The representation (2.8) is obtained by simply writing each propagator as a functional integral. Clearly (2.8) can be interpreted physically as the amplitude for a closed string at Euclidean "time" $\tau=-T_c$ in the distant past to propagate freely until $\tau=0$, then make a sudden transition to an open string, and thereafter propagate as a free open string until $\tau=T_o$ in the distant future.

We shall actually evaluate the transition amplitude

$$\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)=\langle\mathcal{P}_o\,|\,e^{-T_oH}\Upsilon e^{-T_cH}\,|\,\mathcal{P}_c\rangle\ ,$$

where we specify initial and final states to be eigenstates of the momentum operators

$$\mathcal{P}_o(\sigma,t)=\frac{p_o}{\pi}+\frac{1}{\pi\sqrt{2}}\sum_{r\neq0}a_r\cos(r\sigma)e^{-irt}\ ,$$
$$\mathcal{P}_c(\sigma,t)=\frac{p_c}{\pi}+\frac{1}{\pi\sqrt{2}}\sum_{r\neq0}\left(A_re^{-2ir(t-\sigma)}+\tilde{A}_re^{-2ir(t+\sigma)}\right)\ .\tag{2.10}$$

(We have set $\alpha'=1$. We will reintroduce it only when necessary in Sec. V.) We define modes of the momentum, so at time $t=0$ we have

$$\mathcal{P}_o(\sigma)=\frac{p_o}{\pi}+\frac{1}{\pi}\sum_{r>0}p_r\cos(r\sigma)\ ,$$
$$\mathcal{P}_c(\sigma)=\frac{p_c}{\pi}+\frac{\sqrt{2}}{\pi}\sum_{r>0}[p_r^c\cos(2r\sigma)+p_r^s\sin(2r\sigma)]\ .\tag{2.11}$$

Notice that we have defined cosine and sine modes of the closed string p rather than introducing a $\tilde{p}$, because we want the p's to be Hermitian operators. We may define

$$A_r^c=\frac{1}{\sqrt{2}}(A_r+\tilde{A}_r),\quad A_r^s=\frac{i}{\sqrt{2}}(A_r-\tilde{A}_r)\epsilon(r)\ ,$$
$$p_r^c=\frac{1}{\sqrt{2}}(A_r^c+A_{-r}^c),\quad p_r^s=\frac{1}{\sqrt{2}}(A_r^s+A_{-r}^s)\ ,\tag{2.12}$$

with Hermiticities

$$A_r^\dagger=A_{-r},\quad \tilde{A}_r^\dagger=\tilde{A}_{-r},\quad A_r^{c\dagger}=A_{-r}^c,\quad A_r^{s\dagger}=A_{-r}^s\tag{2.13}$$

for the ordinary operators. (As we mentioned earlier, we may treat the ghosts as if they were Hermitian, as long as we express our final answer in a form in which the conjugates do not enter.) Now the (A^c,p^c) and (A^s,p^s) operators satisfy the same algebra as the open string (a,p). The momentum eigenstates are a direct product over modes. The nonzero modes are

$$|\,p_r\rangle=(r\pi)^{-1/4}\exp\left[-\frac{a_{-r}^2}{2r}+\sqrt{2}p_r\frac{a_{-r}}{r}-\frac{1}{2r}p_r^2\right]|\,0\rangle\tag{2.14}$$

which satisfy

$$\langle p_r\,|\,p_r'\rangle=\delta(p_r-p_r')\ .\tag{2.15}$$

When we evaluate the transition amplitude by the functional integral (2.8), we will find that the nonzero mode momenta enter in the form

$$\exp\left[-\frac{1}{2r}[p_r^2+(p_r^c)^2+(p_r^s)^2]+\beta_{ri,sj}\,p_r^ip_s^j+\gamma_{ri}\,p_r^i\right]\ ,$$

where i and j take on the values $\{\text{open},c,s\}$. To unify our notation we will also define $A_r^o=a_r$, and the energy scales $E_r^iT^i=(rT_o,2rT_c,2rT_c)$ for $i=\text{open},c,s$, respectively. The β's and γ's have factors which decay as $e^{-E_r^iT^i}$ for each p_r^i mode, to leading order in e^{-T}. We wish to show now that only this leading order is necessary in our evaluation of the transition amplitude.

The factor

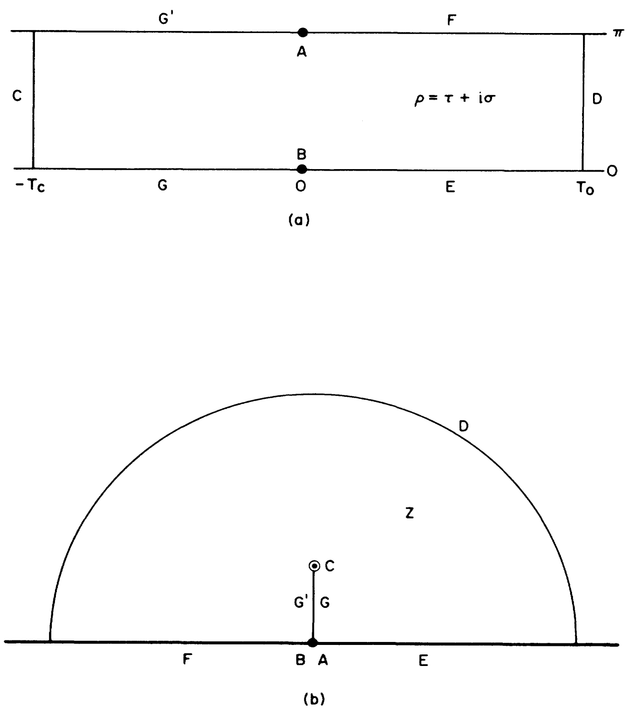

FIG. 1. The region for functional integration in the evaluation of Υ in Sec. II. In (a) the region $\mathcal{R}$ represents a closed string entering from the past at C. It propagates (with G and G' identified) as a free closed string until time 0. Then the string breaks at one point, which becomes the two ends, A and B, of the open string. Thereafter it propagates as an open string into the distant future, and at time T_o is specified along D. In (b) we map the region $\mathcal{R}$ into the UHP, with $z=(e^{2\rho}-1)^{1/2}$. The periodicity, or lack of it for the ghosts, is mapped into continuity or a fixed discontinuity on the interval $[0,1]i$.

$$\exp\left[-\sum \frac{1}{2r}[p_r{}^2+(p_r^c)^2+(p_r^s)^2]\right]$$

is, up to a normalization, just the momentum-space wave function for the simultaneous ground state of all the nonzero-mode oscillators. Thus the nonzero-mode part of the functional integral is proportional to the matrix element

$$\langle 0 \mid \exp(\beta_{ri,sj}\hat{p}\,^i_r\hat{p}\,^j_s+\gamma_{ri}\hat{p}\,^i_r) \mid p^s,p^c\rangle \mid p^o\rangle \ . \tag{2.16}$$

In the overlap, this factor will be multiplied by the nonzero modes of

$$\langle p^c,p^s \mid e^{T_cH_c} \mid \psi_1\rangle\langle p^o \mid e^{T_oH_o} \mid \psi_2\rangle$$

and integrated over all p's. The momentum eigenstates are complete, so we have

$$\langle 0 \mid \exp(\beta_{ri,sj}\hat{p}\,^i_r\hat{p}\,^j_s+\gamma_{ri}\hat{p}\,^i_r)e^{T_iH_i} \underset{T_i\to\infty}{\sim} \langle 0 \mid \exp\left[\tfrac{1}{2}\beta_{ri,sj}e^{T_iE^i_r+T_jE^j_s}A^i_rA^j_s + \frac{1}{\sqrt{2}}\gamma_{ri}e^{T_iE^i_r}A^i_r\right] \tag{2.17}$$

(for the nonzero modes), where we have used

$$e^{-T_iH_i}\hat{p}\,^i_r e^{T_iH_i} = \frac{1}{\sqrt{2}}(A^i_r e^{T_iE^i_r}+A^i_{-r}e^{-T_iE^i_r}) \underset{T_i\to\infty}{\sim} \frac{1}{\sqrt{2}}A^i_r e^{T_iE^i_r} \ . \tag{2.18}$$

For the zero-mode pieces, $e^{-T_oH_o}\Upsilon e^{-T_cH_c}$ has the same time dependence as $e^{-T_oH_o}e^{-T_cH_c}$, which exactly cancels the corresponding pieces of the e^{+TH} factors which multiply $\tilde{\Omega}$. Thus we see that in the limit $T_i\to\infty$, Υ approaches a constant determined by the leading behavior of β and γ. We need to evaluate $\tilde{\Omega}$ not only for the ordinary oscillator modes A^i_r, but also for ghost modes G_r. For the latter, all winding numbers in the closed-string sector need to be considered. Thus the position variable X conjugate to $\mathcal{P}_c$ may change by $2\pi Rn$, where n is an integer and R is the radius of compactification, which we will determine later. Thus

$$X(\tau,\sigma)=q-2ip_c\tau+2nR(\sigma-\tfrac{1}{2}\pi) + \frac{i}{\sqrt{2}}\sum_{m\neq 0}\left[\frac{A_m}{m}e^{-2m(\tau-i\sigma)}+\frac{\tilde{A}_m}{m}e^{-2m(\tau+i\sigma)}\right] . \tag{2.19}$$

The transition amplitude $\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)$ is evaluated as a Fourier transform of the transition function between position eigenstates $\Omega(Z,Y)$. Because we would like to consider the conjugate of the momentum to be the periodic piece of X, the Fourier transform is defined with the aperiodic piece subtracted off. Thus

$$\begin{aligned}\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c) &= \int \mathcal{D}Z(\sigma)\mathcal{D}Y(\sigma)\exp\left[i\int d\sigma\mathcal{P}_c(\sigma)[Y(\sigma)-2nR(\sigma-\tfrac{1}{2}\pi)]\right]\exp\left[i\int d\sigma\mathcal{P}_o(\sigma)Z(\sigma)\right]\Omega(Z,Y)\\ &= \int \mathcal{D}X(\sigma,\tau)\exp\left[-\frac{1}{4\pi}\int_{\mathcal{R}}(\nabla X)^2\right]\exp\left[\frac{i}{2}\beta[X(A)+X(B)]\right]\\ &\quad\times\exp\left\{i\int d\sigma\mathcal{P}_c(\sigma)\left[X(-T_c,\sigma)-2nR\left[\sigma-\frac{\pi}{2}\right]\right]\right\}\exp\left[i\int d\sigma\mathcal{P}_o(\pi-\sigma)X(T_o,\sigma)\right] ,\end{aligned} \tag{2.20}$$

where the integral now includes integration over X at the boundaries. X still satisfies $X(\tau,\sigma=\pi)=X(\tau,\sigma=0)+2\pi nR$ along G and G', and is unconstrained along E and E' (see Fig. 1). The interaction point for the ghost "insertion term" has been written symmetrically.[11] Notice that the D components X^μ of spacetime and the ghost field enter independently in the transition amplitude. The ordinary components behave just like the bosonized ghost field with n and β set to 0.

We evaluate the functional integral in $\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)$ in terms of a classical solution of the Poisson equation

$$\nabla^2\phi=-i\pi\beta[\delta^2(\rho-A)+\delta^2(\rho-B)] \tag{2.21}$$

subject to the boundary conditions

$$\begin{aligned}&\frac{\partial\phi}{\partial\tau}(\tau,\sigma)=-2\pi i\mathcal{P}_c(\sigma) \ \text{ on } C \ ,\\ &\frac{\partial\phi}{\partial\tau}(\tau,\sigma)=2\pi i\mathcal{P}_o(\pi-\sigma) \ \text{ on } D \ ,\\ &\frac{\partial\phi}{\partial\sigma}(\tau,\sigma)=0 \ \text{ on } E \text{ and } E' \ ,\\ &\phi(\tau,\pi)=\phi(\tau,0)+2\pi nR, \quad \frac{\partial\phi}{\partial\sigma}(\tau,\pi)=\frac{\partial\phi}{\partial\sigma}(\tau,0) \quad \text{for } \tau<0 \ .\end{aligned} \tag{2.22}$$

We may now write $X=\hat{X}+\phi$. The integral then becomes

$$\int \mathcal{D}\hat{X} \exp\left[\frac{-1}{4\pi}\int_{\mathcal{R}} (\nabla \hat{X})^2\right]$$

times an $\hat{X}$-independent factor which contains the ϕ dependence. As $\hat{X}$ is now periodic and has all dependence on the states and β removed, the functional integral is a function only of T_i, $N(T_i)$. Since we shall take $T_i \to \infty$, it will contribute a constant which can be absorbed into the coupling constant.[13] This will be determined in terms of the open-string coupling g by comparison to dual model calculations, which extract Υ from the nonplanar loop. What remains is

$$\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)=\exp\left\{-\frac{1}{4\pi}\int_{\mathcal{R}}(\nabla\phi)^2+\frac{i}{2}\beta[\phi(A)+\phi(B)]+i\int_D \phi\mathcal{P}_o+i\int_C \mathcal{P}_c\left[\phi-2nR\left[\sigma-\frac{\pi}{2}\right]\right]\right\} . \tag{2.23}$$

The $(\nabla\phi)^2$ term can be integrated by parts

$$-\frac{1}{4\pi}\int_{\mathcal{R}}(\nabla\phi)^2=-\frac{1}{4\pi}\int_{\delta\mathcal{R}}\phi n\cdot\nabla\phi-\frac{i}{4}\beta[\phi(A)+\phi(B)]$$
$$=-\frac{i}{2}\int_D \phi\mathcal{P}_o-\frac{i}{2}\int_C \phi\mathcal{P}_c-\frac{nR}{2}\int_{-T_c}^{0} d\tau\,\phi'(\tau,0)-\frac{i}{4}\beta[\phi(A)+\phi(B)] .$$

Thus we have the expression for $\tilde{\Omega}$ in terms of the classical solution ϕ:

$$\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)=\exp\left\{\frac{i}{4}\beta[\phi(A)+\phi(B)]+\frac{i}{2}\int_D \mathcal{P}_o(\pi-\sigma)\phi(\sigma)\right.$$
$$\left.+\frac{i}{2}\int_C \mathcal{P}_c(\sigma)\left[\phi(\sigma)-4nR\left[\sigma-\frac{\pi}{2}\right]\right]-\frac{nR}{2}\int_{-T_c}^{0} d\tau\,\phi'(\tau,0)\right\} . \tag{2.24}$$

All we need do now is solve the classical equation for ϕ. This is often done in terms of Neumann functions, and indeed we will sketch this approach in Appendix C. Here we use an alternate method.

Finding a solution of Laplace's equation in two dimensions in some specified region is simplified by the fact that the problem transforms covariantly under conformal transformations, and that the solution is the real part of an analytic function. We map our region $\mathcal{R}$ into the upper half-plane (UHP) [see Fig. 1(b)], with

$$z=(e^{2\rho}-1)^{1/2},\quad \rho=\tau+i\sigma . \tag{2.25}$$

As the times T_o and $T_c \to \infty$, the closed string is specified by a small circle around $z=i$, and the open string on a large semicircle at ∞. The source at the interaction point $z=0$ contributes a term

$$\phi_1=-2i\beta\,\mathrm{Re}\ln z , \tag{2.26}$$

which contributes the required flux of $-2\pi i\beta$ into the UHP at 0 and out again through D. The discontinuity $\phi(P')=\phi(P)+2\pi nR$ can be provided by a term

$$\phi_2=nR\,\mathrm{Im}\ln\frac{z-i}{z+i} . \tag{2.27}$$

Each of these has been chosen to satisfy the open-string boundary condition

$$\frac{\partial\phi}{\partial y}=0 \quad \text{on the real axis} , \tag{2.28}$$

so the remaining contribution is analytic in the UHP $-\{i\}$, and can be made symmetric under $y\to -y$. We could write this as a Laurent expansion about i, plus a point-charge term

$$\phi_3=\gamma\,\mathrm{Re}\ln(z^2+1) . \tag{2.29}$$

It is more convenient, however, to rearrange the basis of functions so that the singular piece is

$$\phi_4=\sum_{p=1}^{\infty}[e_p\,\mathrm{Re}(z^2+1)^{-p}+f_p\,\mathrm{Re}\,z(z^2+1)^{-p}] , \tag{2.30}$$

and the remaining analytic piece is

$$\phi_5=\sum_{p=0}^{\infty}[g_p\,\mathrm{Re}(z^2+1)^p+h_p\,\mathrm{Re}\,z(z^2+1)^p] . \tag{2.31}$$

In this form each piece satisfies (2.28) and the relation between the coefficients and the initial momentum components is simplified. The total ϕ is the sum of the above:

$$\phi=\phi_1+\phi_2+\phi_3+\phi_4+\phi_5 , \tag{2.32}$$

with the coefficients chosen so as to match the conditions on C and D.

To determine the coefficients, we examine $\partial\phi/\partial\tau$ at $\tau=-T_c$ and at $\tau=T_o$, and compare to Eq. (2.11). At $\tau=-T_c$, we note

$$z^2+1=e^{-2T_c+2i\sigma} ,$$
$$z=i\sum_{m=0}^{\infty}\begin{bmatrix}\frac{1}{2}\\ m\end{bmatrix}(-)^m e^{-2mT_c+2im\sigma} . \tag{2.33}$$

This allows us to express $\partial\phi/\partial\tau$ in terms of σ on the boundary C, and thus to relate the coefficients in ϕ_i to $\mathcal{P}_c$. The fact that z needs to be expanded does complicate this comparison, and the coefficients f_p are determined only through the combinations

$$q_r \equiv \sum_{m=0}^{\infty} f_{r+m} \begin{bmatrix} \frac{1}{2} \\ m \end{bmatrix} (-)^m . \tag{2.34}$$

The same expansion is necessary in evaluating $\partial\phi/\partial\tau$ at T_0, so we also define

$$j_r \equiv \sum_{m=0}^{\infty} h_{r+m} \begin{bmatrix} \frac{1}{2} \\ m \end{bmatrix} (-)^m . \tag{2.35}$$

These equations can be inverted:

$$f_r = \sum_{k=0}^{\infty} q_{r+k} \begin{bmatrix} -\frac{1}{2} \\ k \end{bmatrix} (-)^k ,$$
$$h_r = \sum_{k=0}^{\infty} j_{r+k} \begin{bmatrix} -\frac{1}{2} \\ k \end{bmatrix} (-)^k . \tag{2.36}$$

We shall have need of q_r and j_r for negative r as well. These are still given by Eqs. (2.34) and (2.35), with the understanding that

$$h_r = 0 \quad \text{for } r < 0, \quad f_r = 0 \quad \text{for } r \le 0 . \tag{2.37}$$

In combining terms proportional to $\sin 2r\sigma$ in $\phi_5(-T_c,\sigma)$, the expansion in powers runs the other way, and we need to evaluate

$$\sum_{m=0}^{r} h_{r-m} \begin{bmatrix} \frac{1}{2} \\ m \end{bmatrix} (-)^m = \sum_{s=0}^{\infty} j_s (-)^{r+s} \sum_{m=0}^{r} \begin{bmatrix} \frac{1}{2} \\ m \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ s-r+m \end{bmatrix}$$
$$= \sum_{s=0}^{\infty} \frac{(2s+1) j_s (-)^{r+s}}{2s-2r+1} \begin{bmatrix} -\frac{1}{2} \\ r \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ s \end{bmatrix} , \tag{2.38}$$

where in the last line we have used the identity (A4) from Appendix A. In the term proportional to $\sin 2r\sigma$, there is a piece coming from q_{-r}, which we evaluate, using (A2) from that appendix, as

$$q_{-r} = \sum_{m=r+1}^{\infty} \begin{bmatrix} \frac{1}{2} \\ m \end{bmatrix} (-)^m f_{m-r} = \sum_{s=1}^{\infty} q_s (-)^{r+s} \sum_{k=0}^{s-1} \begin{bmatrix} -\frac{1}{2} \\ k \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ s-k+r \end{bmatrix}$$
$$= -\sum_{s=1}^{\infty} \frac{s q_s}{s+r} (-)^{r+s} \begin{bmatrix} -\frac{1}{2} \\ r \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ s \end{bmatrix} . \tag{2.39}$$

With these tools, we can match the terms in $\partial\phi/\partial\tau(-T_c,\sigma)$ proportional to 1, $\cos 2r\sigma$, and $\sin 2r\sigma$, respectively, to find

$$2ip_c = -2\gamma ,$$
$$2\sqrt{2} i p_r^c = 2re_r e^{2rT_c} - 2i\beta e^{-2rT_c} - 2rg_r e^{-2rT_c} ,$$
$$2\sqrt{2} i p_r^s = 2rq_r e^{2rT_c} + \left[-2nR \begin{bmatrix} -\frac{1}{2} \\ r \end{bmatrix} (-)^r - \sum_{s=1}^{\infty} \frac{2rsq_s}{s+r} (-)^{r+s} \begin{bmatrix} -\frac{1}{2} \\ r \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ s \end{bmatrix} \right.$$
$$\left. + \sum_{s=0}^{\infty} \frac{2r(2s+1) j_s}{2s-2r+1} (-)^{r+s} \begin{bmatrix} -\frac{1}{2} \\ r \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ s \end{bmatrix} \right] e^{-2rT_c} . \tag{2.40}$$

A very similar analysis pertains at $\tau = T_o$. The expansion of the ϕ's uses

$$z^2 + 1 = e^{2T_o + 2i\sigma} ,$$
$$z = \sum_{m=0}^{\infty} \begin{bmatrix} \frac{1}{2} \\ m \end{bmatrix} (-)^m e^{(1-2m)(T_o + i\sigma)} . \tag{2.41}$$

Now matching the terms in $\partial\phi/\partial\tau(-T_c,\sigma)$ proportional to 1, $\cos 2r\sigma$, and $\cos(2r+1)\sigma$, respectively, gives

$$2ip_o=-2i\beta+2\gamma\ ,$$
$$2ip^o_{2r}=2rg_re^{2rT_o}-2i\beta e^{-2rT_o}-2re_re^{-2rT_o}\ , \tag{2.42}$$
$$-2ip^o_{2r+1}=(2r+1)j_re^{(2r+1)T_o}+\left[2nR\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}(-)^r-\sum_{s=1}^{\infty}\frac{(2r+1)2sq_s}{2s-2r-1}(-)^{s+r}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}\right.$$
$$\left.-\sum_{s=0}^{\infty}\frac{(2r+1)(2s+1)j_s}{2(r+s+1)}(-)^{r+s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}\right]e^{(-2r+1)T_o}\ .$$

We can now solve Eqs. (2.40) and (2.42) for the unknown coefficients, except for g_0 which is undetermined and irrelevant, provided momentum is conserved including the insertion

$$p_c+p_o+\beta=0\ ,\quad \gamma=-ip_c\ . \tag{2.43}$$

We see that to leading order, for positive r, g_r, and j_r are determined by (2.42), while e_r and q_r are determined by (2.40). Nonetheless each equation contributes to the next leading term, which is important. The solutions are

$$e_r=\sqrt{2}\frac{i}{r}p^c_re^{-2rT_c}+\frac{i\beta}{r}e^{-4rT_c}+\frac{i}{r}p_{2r}e^{-4rT_c-2rT_o}\ ,$$
$$g_r=\frac{i}{r}p_{2r}e^{-2rT_o}+\frac{i\beta}{r}e^{-4rT_o}+\sqrt{2}\frac{i}{r}p^c_re^{-4rT_o-2rT_c}\ ,$$
$$q_r=\sqrt{2}\frac{i}{r}p^s_re^{-2rT_c}+\frac{nR}{r}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}(-)^re^{-4rT_c}+\sqrt{2}i\sum_{s=1}^{\infty}\frac{(-)^{r+s}}{r+s}p^s_s\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-(4r+2s)T_c} \tag{2.44}$$
$$+2i\sum_{s=0}^{\infty}\frac{(-)^{r+s}}{2s-2r+1}p_{2s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-4rT_c-(2s+1)T_o}\ ,$$
$$j_r=\frac{-2i}{2r+1}p_{2r+1}e^{-(2r+1)T_o}-\frac{2nR}{2r+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}(-)^re^{-2(2r+1)T_o}$$
$$+2\sqrt{2}i\sum_{s=1}^{\infty}\frac{(-)^{s+r}}{2s-2r-1}p^s_s\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2sT_c-2(2r+1)T_o}-i\sum_{s=0}^{\infty}\frac{(-)^{s+r+1}}{s+r+1}p_{2s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-(2s+4r+3)T_o}\ .$$

We are now ready to insert our expression for ϕ into (2.24), to get $\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)$. We first note that there is one state-independent term $\frac{1}{2}\beta^2\ln 0$, which is singular but may be dropped. The rest of

$$\frac{i\beta}{4}[\phi(A)+\phi(B)]=-\frac{\beta}{\sqrt{2}}\sum_{r=1}^{\infty}\frac{p^c_r}{r}e^{-2rT_c}-\frac{\beta}{2}\sum_{r=1}^{\infty}\frac{p_{2r}}{r}e^{-2rT_o}\ . \tag{2.45}$$

The evaluation of the last term of (2.24) is simplified by the conformal mapping and the Cauchy-Riemann equations:

$$\int_{-T_c}^{0}d\tau\,\phi'(\tau,0)=\int_0^{\bar{y}}dy\frac{\partial}{\partial x}\phi(x=\epsilon,y)\ ,$$
$$\int_0^{\bar{y}}dy\frac{\partial}{\partial x}\mathrm{Re}f=\mathrm{Im}f\,|_0^{\bar{y}},\quad \int_0^{\bar{y}}dy\frac{\partial}{\partial x}\mathrm{Im}f=-\mathrm{Re}f\,|_0^{\bar{y}}\ ,$$

where $\bar{y}^2=1-e^{-2T_c}$. ϕ_1 and ϕ_3 do not contribute to this term. Applying these tricks to ϕ_2, ϕ_4, and ϕ_5, we find

$$-\frac{nR}{2}\int_{-T_c}^{0}d\tau\,\phi'(\tau,0)=-n^2R^2T_c-n^2R^2\ln 2-\frac{i}{\sqrt{2}}nR\sum_{r=1}^{\infty}\frac{p^s_r}{r}$$
$$+\frac{i}{\sqrt{2}}nR\sum_{r=1}^{\infty}\frac{p^s_r}{r}(-)^r\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}e^{-2rT_c}+inR\sum_{r=0}^{\infty}\frac{p_{2r+1}}{2r+1}(-)^r\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}e^{-2(r+1)T_o}\ . \tag{2.46}$$

The third term of (2.46) has an unexpected undamped behavior. We will soon see, however, that this is canceled by a contribution from the integration along the closed-string boundary. This is

$$\frac{i}{2}\int_C \mathcal{P}_c(\sigma)\left[\phi(\sigma)-4nR\left[\sigma-\frac{\pi}{2}\right]\right]=-p_c{}^2T_c-\tfrac{1}{2}\sum_{r=1}^{\infty}\left[\frac{p_r^{c2}}{r}+\frac{p_r^{s2}}{r}\right]+\frac{i}{\sqrt{2}}nR\sum_{r=1}^{\infty}\frac{p_r^s}{r}$$
$$+\frac{i}{\sqrt{2}}nR\sum_{r=1}^{\infty}\frac{p_r^s}{r}(-)^r\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}e^{-2rT_c}-\frac{\beta}{\sqrt{2}}\sum_{r=1}^{\infty}\frac{p_r^c}{r}e^{-2rT_c}$$
$$-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{p_r^sp_s^s}{r+s}(-)^{r+s}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2(r+s)T_c}-\frac{1}{\sqrt{2}}\sum_{r=1}^{\infty}\frac{p_r^cp_{2r}}{r}e^{-2r(T_o+T_c)}$$
$$-\sqrt{2}\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{p_r^sp_{2s+1}}{2s-2r+1}(-)^{r+s}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2rT_c-(2s+1)T_o}\ . \tag{2.47}$$

Finally the contribution from the open-string boundary is

$$\frac{i}{2}\int_D \mathcal{P}_o(\pi-\sigma)\phi(\sigma)=-p_o{}^2T_o-\tfrac{1}{2}\sum_{r=1}^{\infty}\frac{p_r{}^2}{r}-\frac{\beta}{2}\sum_{r=1}^{\infty}\frac{p_{2r}}{r}e^{-2rT_o}$$
$$-\frac{1}{\sqrt{2}}\sum_{r=1}^{\infty}\frac{p_r^cp_{2r}}{r}e^{-2r(T_c+T_o)}+inR\sum_{r=0}^{\infty}\frac{p_{2r+1}}{2r+1}(-)^r\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}e^{-(2r+1)T_o}$$
$$+\sqrt{2}\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\frac{p_s^sp_{2r+1}}{2s-2r-1}(-)^{r+s}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2sT_c-(2r+1)T_o}$$
$$+\tfrac{1}{2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{p_{2r+1}p_{2s+1}}{r+s+1}(-)^{r+s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2(r+s+1)T_o}\ . \tag{2.48}$$

Adding the four contributions, we find

$$\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)=\exp\Bigg[-(n^2R^2+p_c{}^2)T_c-p_o{}^2T_o-\tfrac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r}[(p_r^c)^2+(p_r^s)^2+(p_r)^2]$$
$$-n^2R^2\ln 2-\sqrt{2}\beta\sum_{r=1}^{\infty}\frac{p_r^c}{r}e^{-2rT_c}-\beta\sum_{r=1}^{\infty}\frac{p_{2r}}{r}e^{-2rT_o}$$
$$+inR\sum_{r=1}^{\infty}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}(-)^r\left[\sqrt{2}\frac{p_r^s}{r}e^{-2rT_c}+2\frac{p_{2r+1}}{2r+1}e^{-(2r+1)T_o}\right]$$
$$-\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-)^{r+s}}{r+s}p_r^sp_s^s\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2(r+s)T_c}$$
$$-2\sqrt{2}\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^{r+s}}{2s-2r+1}p_r^sp_{2s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2rT_c-(2s+1)T_o}$$
$$-\sqrt{2}\sum_{r=1}^{\infty}\frac{1}{r}p_{2r}p_r^ce^{-2r(T_c+T_o)}+\tfrac{1}{2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^{r+s}}{r+s+1}p_{2r+1}p_{2s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}e^{-2(r+s+1)T_o}\Bigg]\ . \tag{2.49}$$

Recalling the discussion leading to the expression (2.18) for Υ, we see that we may convert $\tilde{\Omega}(\mathcal{P}_o,\mathcal{P}_c)$ to Υ simply by replacing each $p_r^ie^{-E_r^iT^i}$ by $A_r^i/\sqrt{2}$, after dropping the $p_r{}^2/r$ terms, and noting that the terms linear in t cancel:

$$\Upsilon=\exp\left[-n^2R^2\ln 2-\beta\sum_{r=1}^{\infty}\frac{A_r^c}{r}-\frac{\beta}{\sqrt{2}}\sum_{r=1}^{\infty}\frac{a_{2r}}{r}+inR\sum_{r=1}^{\infty}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}(-)^r\left[\frac{A_r^s}{r}+\sqrt{2}\frac{a_{2r+1}}{2r+1}\right]\right.$$
$$-\tfrac{1}{2}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-)^{r+s}}{r+s}A_r^sA_s^s\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}-\sqrt{2}\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^{r+s}}{2s-2r+1}A_r^sa_{2s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}$$
$$\left.-\frac{1}{\sqrt{2}}\sum_{r=1}^{\infty}\frac{1}{r}A_r^ca_{2r}+\tfrac{1}{4}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^{r+s}}{r+s+1}a_{2r+1}a_{2s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}\right]. \tag{2.50}$$

We have evaluated the overlap Υ in (2.50) for arbitrary compactification radius R and insertion strength β for each component of a generalized X^μ, but our purpose for doing so is to apply it to bosonized ghosts. We must therefore find what the correct β and R are, and it will also be helpful to reexpress n in terms of ghost number operators.

We begin by reviewing the formal description of bosonized ghosts. We introduce bosonic ghost oscillators g_n, G_n, and $\tilde{G}_n$ obeying the usual commutation relations

$$[g_n,g_m]=[G_n,G_m]=[\tilde{G}_n,\tilde{G}_m]=n\delta_{n,-m}\ . \tag{2.51}$$

We adhere to the convention that lower case letters refer to the open string, upper case tilded and untilded letters refer to left- and right-moving modes of the closed string. The zero modes g_0, G_0, and $\tilde{G}_0$ are just the ghost numbers in these sectors. The precise connection between the bosonized and fermionic oscillators is

$$g_n=\sum_k \ddagger c_{-k}b_{k+n}\ddagger-\tfrac{1}{2}\delta_{n0}\ , \tag{2.52}$$

and the corresponding relations with $(g,c,b)\to(G,C,B)$, $(g,c,b)\to(\tilde{G},\tilde{C},\tilde{B})$ for the closed string. We use $\ddagger$ to imply normal ordering with respect to fermionic operators, and the usual : to refer to bosonic normal ordering. Note that the anti-Hermiticity of the ghosts implies an extra minus sign in (2.13) for $g,G,\tilde{G}$. Comparing Eqs. (2.19) and (2.11) to the standard forms

$$\mathcal{P}_\pm:=\mathcal{P}\pm\frac{1}{2\pi}\frac{\partial X}{\partial\sigma}=\frac{\sqrt{2}}{\pi}\sum_m\begin{bmatrix}\tilde{G}_me^{-2im\sigma}\\ G_me^{2im\sigma}\end{bmatrix}, \tag{2.53}$$

we learn that

$$\tilde{G}_0-G_0=nR^{\phi}\sqrt{2}\ . \tag{2.54}$$

By definition of the winding number, n must take on all integer values, so (2.52) and (2.54) imply that

$$R^{\phi}=\frac{1}{\sqrt{2}},\quad n=\tilde{G}_0-G_0\ . \tag{2.55}$$

We also note from (2.53) that

$$p_c^{\phi}=\frac{\tilde{G}_0+G_0}{\sqrt{2}}=\frac{2G_0+n}{\sqrt{2}}\ , \tag{2.56}$$

which implies that the plane wave $e^{i\phi p_c^{\phi}}$ transforms under $\phi\to\phi+2\pi R^{\phi}$ as

$$e^{i\phi p_c^{\phi}}\underset{\phi\to\phi+\pi\sqrt{2}}{\sim}-(-)^ne^{i\phi p_c^{\phi}}\ . \tag{2.57}$$

We must also determine the constant β. From (2.43) we find $\beta=-p_o-p_c=-(g_0+\tilde{G}_0+G_0)/\sqrt{2}$. To have the right ghost insertion to give a nonzero Υ on physical states, which have $g_0=\tilde{G}_0=G_0=-\frac{1}{2}$, we see that

$$\beta=\frac{3}{2\sqrt{2}} \tag{2.58}$$

as anticipated in Eq. (2.4).

Before substituting (2.55) and (2.58) into (2.50), we extend (2.12) to the ghosts,

$$G_r^c=\frac{1}{\sqrt{2}}(G_r+\tilde{G}_r),\quad G_r^s=\frac{i}{\sqrt{2}}(G_r-\tilde{G}_r),\quad r\ge 0\ . \tag{2.59}$$

We find that the terms in (2.50) linear in nR can be interpreted as zero-mode pieces of the following two terms for the ghost modes. Explicitly summing over ghost and ordinary oscillators, we find

$$\Upsilon=\exp\left[(G_0^s)^2\ln 2-\frac{3}{2\sqrt{2}}\sum_{r=1}^{\infty}\frac{G_r^c}{r}-\tfrac{3}{4}\sum_{r=1}^{\infty}\frac{g_{2r}}{r}-\tfrac{1}{2}\sum_{\substack{r=0,s=0\\ (r,s)\neq(0,0)}}^{\infty}\frac{(-)^{r+s}}{r+s}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}(A_r^s\cdot A_s^s+G_r^sG_s^s)\right.$$
$$-\sqrt{2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^{r+s}}{2s-2r+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}(A_r^s\cdot a_{2s+1}+G_r^sg_{2s+1})-\frac{1}{\sqrt{2}}\sum_{r=1}^{\infty}\frac{1}{r}(A_r^c\cdot a_{2r}+G_r^cg_{2r})$$
$$\left.+\tfrac{1}{4}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-)^{r+s}}{r+s+1}\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ s\end{bmatrix}(a_{2r+1}\cdot a_{2s+1}+g_{2r+1}g_{s+1})\right]. \tag{2.60}$$

This is our final form for Υ in bosonized form.

III. DERIVATION OF THE GHOST FACTOR FROM BRST INVARIANCE

We have seen how to obtain the ghost dependence of Υ using bosonized ghosts and path-integral methods. In the next section we shall discuss how to refermionize the ghosts to express Υ in terms of fermion ghost operators. However, since fermionization involves some choices of phases (Klein signs), which we shall not go through in every case, it is useful here to establish the fermion representation by starting with the transition amplitude without ghosts, and introducing ghosts by requiring BRST invariance.

The first step is to find the gauge identities satisfied by the coordinate- $(a,A,\tilde{A})$ dependent part of Υ, which we have called Υ_0. The reason these identities exist is that Υ_0 is an overlap and the L_n's can be expressed as integrals of a local density on the strings:

$$\left[\mathcal{P}\pm\frac{x'}{2\pi}\right]_c^2(\eta)=\frac{2}{\pi^2}\sum_m\begin{bmatrix}\tilde{L}_m\\ L_m\end{bmatrix}e^{\mp 2im\eta}\quad\text{closed string}\,, \tag{3.1}$$

$$\left[\mathcal{P}\pm\frac{x'}{2\pi}\right]_o^2(\eta)=\frac{1}{2\pi^2}\sum_m l_m e^{\mp im\eta}\quad\text{open string}\,. \tag{3.2}$$

We use η to label a point on a given single string. Recall that Υ_0 is an overlap which identifies the point η of one string with the point $\pi-\eta$ on the other. In the functional-integral representation of the overlap, the world-sheet coordinate σ is identified with η for the incoming and $\pi-\eta$ for the outgoing string. Thus one would naively expect to have an identity

$$\langle 0|\Upsilon_0\left[\mathcal{P}\pm\frac{x'}{2\pi}\right]_c^2(\eta)\approx\langle 0|\Upsilon_0\left[\mathcal{P}\pm\frac{x'}{2\pi}\right]_o^2(\pi-\eta)\,. \tag{3.3}$$

Equality (3.3) does not hold literally because of singular operator products at the interaction point. However, we will derive identities which are valid, modulo additive c numbers, by integrating (3.3) with a function $f(\eta)$ which vanishes there,[14] and is such that a single l_{-n}, L_{-n} or $\tilde{L}_{-n}$, with $n>0$, is related to a linear combination of l_n, L_n or $\tilde{L}_n$, with $n\geq 0$. This is the form of the Ward identity that allows one to assert that null states of the form $L_{-1}|\text{phys}\rangle$ or $(L_{-2}+\frac{3}{2}L_{-1}{}^2)|\text{phys}\rangle$ decouple from physical states. In this form Fock-state matrix elements of the Ward identity involve only a finite number of terms.

Let us first note that one can project out a particular $\tilde{L}_m$, L_m or l_m from (3.1) and (3.2) as follows:

$$\tilde{L}_m=\frac{\pi}{2}\int_0^\pi d\eta e^{2im\eta}\left[\mathcal{P}+\frac{x'}{2\pi}\right]_c^2(\eta)\,, \tag{3.4}$$

$$L_m=\frac{\pi}{2}\int_0^\pi d\eta e^{-2im\eta}\left[\mathcal{P}-\frac{x'}{2\pi}\right]_c^2(\eta)\,, \tag{3.5}$$

$$l_m=\pi\int_0^\pi d\eta e^{im\eta}\left[\mathcal{P}+\frac{x'}{2\pi}\right]_o^2(\eta)+\pi\int_0^\pi d\eta e^{-im\eta}\left[\mathcal{P}-\frac{x'}{2\pi}\right]_o^2(\eta)\,. \tag{3.6}$$

Now recall the mapping of the interacting string diagram to the upper half-plane,

$$z=\sqrt{\zeta^2-1}\quad\text{where }\zeta=e^{\rho}=e^{r+i\sigma}\,. \tag{3.7}$$

Then the closed string corresponds to the point $z=i$ and $\zeta=0$, and the open string is at $z=\infty$ and $\zeta=\infty$, while the interaction point is at $z=0$ and $\zeta=1$. If $f\sim z$ as $z\to 0$, the singular behavior near the interaction point will be softened. If $f(\zeta)$ can be expanded alternatively about $\zeta=0$ and $\zeta=\infty$, convergent up to $|\zeta|=1$, the integral will give a valid identity among $\tilde{L}$'s, L's, and l's. Suppose we have functions $f_\pm(\zeta)$ which vanish at the interaction point and which have alternate Laurent expansions about 0 and ∞ given by

$$f_\pm(\zeta)=\begin{cases}\sum_r k^c_{r\pm}\zeta^{2r} & \text{for }|\zeta|<1\,,\\ \sum_r k^o_{r\pm}\zeta^{-r} & \text{for }|\zeta|>1\,.\end{cases} \tag{3.8}$$

Then we expect

$$\langle 0|\Upsilon_0\int d\sigma\left[\left[\mathcal{P}+\frac{x'}{2\pi}\right]_c^2(\sigma)f_+(\sigma)+\left[\mathcal{P}-\frac{x'}{2\pi}\right]_c^2(\sigma)f_-(\pi-\sigma)\right]$$
$$\approx\langle 0|\Upsilon_0\int d\sigma\left[\left[\mathcal{P}+\frac{x'}{2\pi}\right]_o^2(\pi-\sigma)f_+(\sigma)+\left[\mathcal{P}-\frac{x'}{2\pi}\right]_o^2(\pi-\sigma)f_-(\pi-\sigma)\right] \tag{3.9}$$

or

$$\langle 0|\Upsilon_0\sum(k^c_{r+}\tilde{L}_r+k^c_{r-}L_r)\approx\tfrac{1}{2}\langle 0|\Upsilon_0\sum k^o_{r-}l_r\,, \tag{3.10}$$

provided $k^o_{r-}=(-)^r k^o_{r+}$.

We would first like to isolate a single L_{-n}, so we require an f with a single negative power in its Laurent expansion about 0. We can find an nth-order polynomial $P_n(\zeta^{-2})$ such that

$$f_n(\zeta)=P_n(\zeta^{-2})\sqrt{\zeta^2-1}\underset{\zeta\to 0}{\sim}\zeta^{-2n}+\text{const}\ . \tag{3.11}$$

Then $P_n(\zeta^{-2})\approx\zeta^{-2n}(\zeta^2-1)^{-1/2}+\text{const}$, so

$$P_n(\zeta^{-2})=\frac{-i}{\zeta^{2n}}\sum_{k=0}^{n-1}(-)^k\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\zeta^{2k}\ . \tag{3.12}$$

Then the expansions of f are given by

$$\begin{aligned}f_n(\zeta)&=\epsilon(\text{Im}\zeta)\frac{1}{\zeta^{2n}}\sum_{k=0}^{n-1}(-)^k\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\zeta^{2k}\sum_{l=0}^{\infty}(-)^l\begin{bmatrix}\frac{1}{2}\\ l\end{bmatrix}\zeta^{2l}\\ &=\epsilon(\text{Im}\zeta)\left[\frac{1}{\zeta^{2n}}+\sum_{l=0}^{\infty}\zeta^{2l}\frac{(-)^{l+n}}{2(l+n)}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n-1\end{bmatrix}\right]\end{aligned} \tag{3.13}$$

for $|\zeta|<1$ [with $\epsilon(y)=\text{sign of }y$], and

$$\begin{aligned}f_n(\zeta)&=\frac{-i}{\zeta^{2n}}\sum_{k=0}^{n-1}(-)^k\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\zeta^{2k}\sum_{l=0}^{\infty}(-)^l\begin{bmatrix}\frac{1}{2}\\ l\end{bmatrix}\zeta^{1-2l}\\ &=-i\sum_{l=1}^{\infty}\zeta^{1-2l}\frac{(-)^{l+n}}{2n+1-2l}\begin{bmatrix}-\frac{1}{2}\\ l-1\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n-1\end{bmatrix}\end{aligned} \tag{3.14}$$

for $|\zeta|>1$. We have used (A4) and (A2) from Appendix A to perform the sums. If we choose $f_+=f_n$, $f_-=-f_n$, Eq. (3.10) gives

$$\langle 0|\Upsilon_0\left[\tilde{L}_{-n}-L_{-n}+\sum_{l=0}^{\infty}(\tilde{L}_l-L_l)\frac{(-)^{l+n}}{2(l+n)}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n-1\end{bmatrix}\right]=\langle 0|\Upsilon_0\left[\frac{i}{2}\sum_{l=1}^{\infty}l_{2l-1}\frac{(-)^{l+n}}{2n+1-2l}\begin{bmatrix}-\frac{1}{2}\\ l-1\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n-1\end{bmatrix}\right]\ , \tag{3.15}$$

an identity of the form we were seeking. The even $\tilde{L}_{-n}+L_{-n}$ identity is obtained by choosing $f_+(\zeta)=f_-(\zeta)=\zeta^{-2n}-1$. As can be confirmed by evaluating simple matrix elements, it is necessary to add a c-number term due to the terms in L_{-n} quadratic in creation operators. The corresponding terms in (3.15) do not contribute because of the evenness of Υ_0. The correct identity is

$$\langle 0|\Upsilon_0\left[\tilde{L}_{-n}+L_{-n}-\tilde{L}_0-L_0-\tfrac{1}{2}l_{2n}+\tfrac{1}{2}l_0-\frac{D}{16}(n-1)\right]=0\ . \tag{3.16}$$

To obtain identities involving a single l_{-n}, we again make use of (3.7), but this time find an lth-order polynomial in ζ^2 such that

$$\hat{f}_l(\zeta)=\hat{P}_l(\zeta^2)\sqrt{\zeta^2-1}\underset{\zeta\to\infty}{\sim}\zeta^{2l+1}+\text{const}\ , \tag{3.17}$$

which implies

$$\hat{P}_l(\zeta^2)=\zeta^{2l}\sum_{k=0}^{l}(-)^k\left[\frac{1}{\zeta^2}\right]^k\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\ .$$

Then

$$\begin{aligned}\hat{f}_l(\zeta)&=\zeta^{2l}\sum_{k=0}^{l}(-)^k\left[\frac{1}{\zeta^2}\right]^k\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\sum_{m=0}^{\infty}(-)^m\begin{bmatrix}\frac{1}{2}\\ m\end{bmatrix}\zeta^{1-2m}\\ &=\zeta^{2l+1}+\sum_{m=0}^{\infty}\frac{(-)^{m+l+1}}{2(m+l+1)}\left[\frac{1}{\zeta}\right]^{2m+1}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ l\end{bmatrix}\end{aligned} \tag{3.18}$$

for $|\zeta|>1$, and

$$\hat{f}_l(\zeta)=\zeta^{2l}\sum_{k=0}^{l}(-)^k\left[\frac{1}{\zeta^2}\right]^k\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix} i\epsilon(\mathrm{Im}\zeta)\sum_{m=0}^{\infty}(-)^m\begin{bmatrix}\frac{1}{2}\\ m\end{bmatrix}\zeta^{2m}$$

$$=i\epsilon(\mathrm{Im}\zeta)\sum_{m=0}^{\infty}\zeta^{2m}\frac{(-)^{l+m}}{2l+1-2m}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ l\end{bmatrix} \tag{3.19}$$

for $|\zeta|<1$. Choosing $f_+=-f_-=\hat{f}_l$, (3.10) gives

$$\langle 0|\Upsilon_0\left[l_{-2l-1}+\sum_{m=0}^{\infty}\frac{(-)^{m+l+1}}{2(m+l+1)}l_{2m+1}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ l\end{bmatrix}+2i\sum_{m=0}^{\infty}(\tilde{L}_m-L_m)\frac{(-)^{l+m}}{2l+1-2m}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ l\end{bmatrix}\right]=0\,. \tag{3.20}$$

Finally $f_+=f_-=\zeta^{2l}-1$ gives

$$\langle 0|\Upsilon_0\left[l_{-2n}-l_0-\frac{D}{8}(n+1)-2(\tilde{L}_n+L_n)+2(\tilde{L}_0+L_0)\right]. \tag{3.21}$$

We can summarize all the identities in the form

$$\langle 0|\Upsilon_0\left[L^j_{-n}+\sum_{m=0}^{\infty}M^{jk}_{nm}L^k_m+k^j_n\right]=0\,, \tag{3.22}$$

where

$$L^+_n=\frac{1}{\sqrt{2}}(L_n+\tilde{L}_n),\quad L^-_n=\frac{1}{\sqrt{2}}(L_n-\tilde{L}_n),\quad L^0_n=l_n\,, \tag{3.23}$$

$$k^o_{2n}=-\frac{D}{8}(n+1),\quad k^o_{2n+1}=k^-_n=0\,,$$
$$k^+_n=-\frac{D}{16\sqrt{2}}(n-1)\,, \tag{3.24}$$

and the coefficients M^{jk}_{mn} are given in Table I.

We are now ready to add ghosts, in fermionic form, to make a BRST-invariant operator. We define b^i, c^i for $i=o,+,-$ by

$$c^+_n=\frac{1}{\sqrt{2}}(C_n+\tilde{C}_n),\quad c^-_n=\frac{1}{\sqrt{2}}(C_n-\tilde{C}_n),\quad c^o_n=c_n\,, \tag{3.25}$$

$$b^+_n=\frac{1}{\sqrt{2}}(B_n+\tilde{B}_n),\quad b^-_n=\frac{1}{\sqrt{2}}(B_n-\tilde{B}_n),\quad b^o_n=b_n\,,$$

and

$$Q=\sum_{n\neq 0}c^i_{-n}L^i_n+c^i_0(L^i_0-\alpha^i_0)-\tfrac{1}{2}\sum_{m,n}(m-n)\ddagger c^i_{-m}c^j_{-n}b^k_{m+n}\ddagger d_{ijk}\,, \tag{3.26}$$

where the mixing of modes described by d_{ijk} is due to our using cosine and sine modes rather than left and right ones.[15] The d_{ijk}'s are all 0 except for

$$d_{ooo}=1,\quad d_{+++}=d_{+--}=d_{-+-}=d_{--+}=\frac{1}{\sqrt{2}}\,.$$

The L_0's are shifted by

$$\alpha^o_0=1,\quad \alpha^+_0=\sqrt{2},\quad \alpha^-_0=0\,.$$

We make the ansatz that the full Υ is built from Υ_0 by inserting an exponential in $c\times b$:

TABLE I. The reflection coefficients M^{jk}_{mn} for the gauges, defined by $\langle 0|\Upsilon_0(L^j_{-n}+\sum_{m=0}^{\infty}M^{jk}_{nm}L^k_m+k^j_n)=0$ for $n>0$ in Sec. III.

M^{jk}_{mn}	$j=+$	$j=-$	$j=0,\ m=2k$	$j=0,\ m=2k+1$
$k=+$	$-\delta_{n0}$	0	$-2\sqrt{2}(\delta_{nk}-\delta_{n0})$	0
$k=-$	0	$\frac{(-)^{n+m}}{2(m+n)}\begin{bmatrix}-\frac{1}{2}\\ n\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ m-1\end{bmatrix}$	0	$-2\sqrt{2}i\frac{(-)^{k+n}}{2k+1-2n}\begin{bmatrix}-\frac{1}{2}\\ n\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ k\end{bmatrix}$
$k=o,\ n=0$	$\frac{1}{2\sqrt{2}}$	0	-1	0
$k=o,\ n=2l>0$	$-\frac{\delta_{lm}}{2\sqrt{2}}$	0	0	0
$k=o,\ n=2l+1$	0	$\frac{-i}{2\sqrt{2}}\frac{(-)^{l+m}}{2m-2l-1}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ m-1\end{bmatrix}$	0	$\frac{(-)^{k+l+1}}{2(k+l+1)}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ k\end{bmatrix}$

$$\langle 0 | \Upsilon = \langle 0 | \Upsilon_0 \exp\left[\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_n^j b_m^k M_{nm}^{jk}\right], \qquad (3.27)$$

where $\langle 0|$ includes a $\langle \uparrow\uparrow\uparrow |$ for the ghost oscillators. It is not difficult to see that the ansatz (3.27), with the same M as in (3.22), is necessary in order that all the terms in $\langle 0 | \Upsilon Q$ linear in the L_n^i's cancel. In fact

$$\langle 0 | \Upsilon Q = 0 \qquad (3.28)$$

implies

$$0 = \langle 0 | \Upsilon_0 \sum_{n=1}^{\infty} c_n^i \left[\sum_{m=0}^{\infty} M_{nm}^{ij} L_m^j - M_{n0}^{ij}\alpha_0^j + L^i_{-n} + \sum_{m=0}^{n-1}(n+m)M_{n-m,m}^{jk} d_{kij}\right] + \langle 0 | \Upsilon_0 \sum_{l,m=1}^{\infty}\sum_{n=0}^{\infty} E_{lmn}^{ijk} c_l^i c_m^j b_n^k . \qquad (3.29)$$

The two terms in this equation are independent and must separately vanish. Making use of (3.22), we see that the first term will vanish if

$$-k_n^i - \sum_j M_{n0}^{ij}\alpha_0^j + \sum_{jk}\sum_{m=0}^{n-1}(n+m)M_{n-m,m}^{jk} d_{kij} = 0 \qquad (3.30)$$

for each i, $n>0$. Equation (3.30) is trivially satisfied for $i=-$, all n, and for $i=o$, odd n. The remaining cases are

$$\frac{D}{8}(l+1)+1-4+\sum_{m=0}^{2l-1}(2l+m)M_{2l-m,m}^{oo}=0 , \qquad i=o, \quad n=2l ,$$

$$\frac{D}{16\sqrt{2}}(n-1)-\frac{1}{2\sqrt{2}}+\sqrt{2}+\frac{1}{\sqrt{2}}\sum_{m=0}^{n-1}(n+m)(M_{n-m,m}^{++}+M_{n-m,m}^{--})=0, \quad i=+$$

or

$$\frac{D}{8}(l+1)-3+\sum_{r=0}^{l-1}\frac{2l+2r+1}{2l}(-)^l\begin{bmatrix}-\frac{1}{2}\\ r\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ l-r-1\end{bmatrix}-2l=0 , \qquad (3.31)$$

$$\frac{D}{8}(n-1)+3+\sum_{m=0}^{n-1}\frac{n+m}{n}(-)^n\begin{bmatrix}-\frac{1}{2}\\ n\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n-m-1\end{bmatrix}-2n=0 , \qquad (3.32)$$

where we have consulted Table I.

Using the identities from Appendix A, the LHS of these equations reduce to

$$\frac{D-26}{8}(l+1), \quad \frac{D-26}{8}(n-1) \qquad (3.33)$$

so (3.30) is true if $D=26$. The proof that $E_{lmn}^{abc}=0$ is considerably more tedious, and we shall sketch it in Appendix B. We note here, though, that the definition of Υ as an overlap should arrange BRST invariance up to possible "quantum" fluctuation effects, i.e., up to $O(\hbar)$. The term involving the E's does not involve such fluctuations, and it should therefore not obstruct BRST invariance. It is the first term of (3.29) that explicitly involves $O(\hbar)$ effects, and we have seen that they do, except for $D=26$, cause a violation of BRST invariance. We have collected the coefficients M_{nm}^{jk} in Table I, and conclude by quoting the final fermionic form for $\langle 0 | \Upsilon$, which we reexpress in terms of left and right modes as

$$\langle 0 | \Upsilon_F = \langle 0 | \exp(W_F)$$

with

$$\begin{aligned} W_F = {} & \frac{1}{2}\sum_{k,l=1}^{\infty}\frac{(-)^{k+l}}{2(k+l)}\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}(A_k-\tilde{A}_k)\cdot(A_l-\tilde{A}_l)+\tfrac{1}{2}\sum_{k,l=0}^{\infty}\frac{(-)^{k+l}}{2(k+l+1)}\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}a_{2k+1}\cdot a_{2l+1} \\ & -\sum_{k=1}^{\infty}\frac{1}{2k}(A_k+\tilde{A}_k)\cdot a_{2k}+i\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-)^{k+l}}{2l-2k-1}\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}a_{2k+1}\cdot(A_l-\tilde{A}_l) \\ & +\frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(C_n-\tilde{C}_n)(B_m-\tilde{B}_m)\frac{(-)^{m+n}}{2(m+n)}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n-1\end{bmatrix}-\frac{1}{2}\sum_{n=1}^{\infty}(C_n+\tilde{C}_n)(B_0+\tilde{B}_0) \\ & +\sum_{n,m=0}^{\infty}c_{2n+1}b_{2m+1}\frac{(-)^{m+n+1}}{2(m+n+1)}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n\end{bmatrix}-\sum_{n=1}^{\infty}c_{2n}b_0 \end{aligned}$$

$$+\frac{i}{4}\sum_{n,m}(C_n-\widetilde{C}_n)b_{2m+1}\frac{(-)^{m+n+1}}{2n-1-2m}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n-1\end{bmatrix}+\frac{1}{4}\sum_{n=1}^{\infty}(C_n+\widetilde{C}_n)b_0$$

$$-2i\sum_{n,m=0}^{\infty}c_{2n+1}(B_m-\widetilde{B}_m)\frac{(-)^{m+n}}{2n+1-2m}\begin{bmatrix}-\frac{1}{2}\\ m\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\ n\end{bmatrix}$$

$$+2\sum_{n=1}^{\infty}c_{2n}(B_0+\widetilde{B}_0)-\frac{1}{4}\sum_{n=1}^{\infty}(C_n+\widetilde{C}_n)b_{2n}-2\sum_{n=1}^{\infty}c_{2n}(B_n+\widetilde{B}_n)\ . \tag{3.34}$$

IV. RELATION BETWEEN BOSONIZED AND FERMIONIZED GHOSTS

The easiest way to make contact between the results of Secs. II and III is to compare simple matrix elements. For example, using the fermionized ghost form of Υ from the last section, we find

$$\langle 0\,|\,\Upsilon_F\ddagger C^j(y)B^k(z)\ddagger\,|\downarrow\downarrow\downarrow\rangle=\sum_{n,m}M^{kj}_{nm}y^m z^n=f^{kj}(z,y)\ , \tag{4.1}$$

where

$$C^j(y)=\sum_n y^{-n}c^j_n\ , \tag{4.2}$$

$$B^k(z)=\sum_n z^{-n}b^k_n\ . \tag{4.3}$$

We have collected the $f^{kj}(v,u)$ in Table II.

The standard bosonization formulas[16,17] express a system of bosonic oscillators as a bilinear in fermionic oscillators. For the world-sheet Feynman-Faddeev-Popov ghosts of the open string satisfying

$$\{b_n,c_m\}=\delta_{n,-m} \tag{4.4}$$

this formula reads

$$g_n=\sum_k\ddagger c_{-k}b_{k+n}\ddagger-\tfrac{1}{2}\delta_{n0}\ . \tag{4.5}$$

It follows (with due attention paid to operator ordering) that

$$[g_n,g_m]=n\delta_{n,-m}\ , \tag{4.6}$$

$$g_n^\dagger=-g_{-n}\ . \tag{4.7}$$

For completeness we also quote

$$L_n^{\rm gh}=\sum_k(k-n)\ddagger c_{-k}b_{k+n}\ddagger$$
$$=\tfrac{1}{2}\sum_k{:}g_{-k}g_{k+n}{:}-\tfrac{3}{2}ng_n-\frac{\delta_{n0}}{8}\ , \tag{4.8}$$

Note that g_0 is just the ghost-number operator. Equations (4.4) and (4.5) imply that

$$[g_n,c_l]=c_{n+l},\quad [g_n,b_l]=-b_{n+l}$$

from which it follows[17] that

$$b(z)=\sum_k z^{-k}b_k=z^{L_0}{:}\exp\left[\sum_n\left[\frac{g_n}{n}\right]\right]{:}z^{-L_0}$$
$$={:}\exp\left[\sum_{n\neq0}\frac{g_nz^{-n}}{n}\right]{:}U_0z^{(1-2g_0)/2}\ , \tag{4.9}$$

$$c(z)=\sum_k z^{-k}c_k=z^{L_0}{:}\exp\left[\sum_n\left[-\frac{g_n}{n}\right]\right]{:}z^{-L_0}$$
$$={:}\exp\left[-\sum_{n\neq0}\frac{g_nz^{-n}}{n}\right]{:}U_0{}^{-1}z^{(1+2g_0)/2}\ , \tag{4.10}$$

where $U_0=$ "$e^{g_0/0}$" is an operator that destroys one unit of ghost number:

$$[g_0,U_0]=-U_0\ .$$

If we label the vacua of the nonzero modes by ghost number, we can define a standard U_0 by

TABLE II. The generating functions for the reflection coefficients M^{kj}_{mn}, $f^{kj}(v,u)=\sum_{n,m}M^{kj}_{nm}v^nu^m$.

$f^{kj}(v,u)$	$j=+$	$j=-$	$j=o$
$k=+$	$-\frac{v}{1-v}$	0	$\frac{1}{2\sqrt{2}}\left[\frac{v}{1-v}-\frac{u^2v}{1-u^2v}\right]$
$k=-$	0	$\frac{v}{u-v}\left[\left[\frac{1-u}{1-v}\right]^{1/2}-1\right]$	$\frac{i}{2\sqrt{2}}\frac{uv}{1-vu^2}\left[\frac{1-u^2}{1-v}\right]^{1/2}$
$k=o$	$2\sqrt{2}\left[\frac{v^2}{1-v^2}-\frac{uv^2}{1-uv^2}\right]$	$-2i\sqrt{2}\frac{v}{1-uv^2}\left[\frac{1-u}{1-v^2}\right]^{1/2}$	$\frac{uv}{u^2-v^2}\left[\left[\frac{1-u^2}{1-v^2}\right]^{1/2}-1\right]-\frac{v^2}{1-v^2}$

$$U_0 \,|\, 0,g\rangle = |\, 0,g-1\rangle \ .$$

For definiteness we shall identify the $g=-\frac{1}{2}$ vacuum $|\downarrow\rangle$ of the c's and b's with $|\,0,-\frac{1}{2}\rangle$.

One can clearly construct analogous bosonization formulas for the closed-string pairs B,C and $\tilde{B},\tilde{C}$. With this construction b,c will commute with B,C and $\tilde{B},\tilde{C}$, and B,C will commute with $\tilde{B},\tilde{C}$. As is well known, these operators can be made mutually anticommuting with the device of a Klein transformation. In addition, we will need to make canonical scale changes oppositely in the B's and the C's. Before we incorporate such factors, we will call the closed-string analogs of (4.9) and (4.10) B^b and C^b. Once a choice of Klein signs is made, the relative phase between Υ_F and Υ_B is fixed.

Now our task is to evaluate (4.1) in bosonized form. For this purpose it is better to work in terms of left and right modes, so we reexpress (2.60) as

$$\begin{aligned}\langle 0\,|\,\Upsilon_B = \exp\Bigg[& -\frac{(\tilde{G}_0-G_0)^2}{2}\ln 2-\frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k}(g_{2k}+G_k+\tilde{G}_k)\\ & +\frac{1}{2}\sum_{k,l=0}^{\infty}\frac{(-)^{k+l}}{2(k+l+1)}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{l}(a_{2k+1}\cdot a_{2l+1}+g_{2k+1}g_{2l+1})-\sum_{k=1}^{\infty}\frac{1}{2k}[(A_k+\tilde{A}_k)\cdot a_{2k}+(G_k+\tilde{G}_k)g_{2k}]\\ & +\frac{1}{2}\sum_{(k,l)\neq(0,0)}\frac{(-)^{k+l}}{2(k+l)}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{l}[(A_k-\tilde{A}_k)\cdot(A_l-\tilde{A}_l)+(G_k-\tilde{G}_k)(G_l-\tilde{G}_l)]\\ & +i\sum_{k,l=0}^{\infty}\frac{(-)^{k+1}}{2l-2k-1}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{l}[a_{2k+1}\cdot(A_l-\tilde{A}_l)+g_{2k+1}(G_l-\tilde{G}_l)]\Bigg] \ . \end{aligned} \tag{4.11}$$

In Eq. (4.11) the bra $\langle 0\,|$ is the sum of nonzero-mode vacua coupling to all kets of zero total momentum and total ghost number $-\frac{3}{2}$. That is,

$$\langle 0\,|\,0,p,P,g_0,G_0,\tilde{G}_0\rangle \equiv \delta(p+P)\delta_{g_0+G_0+\tilde{G}_0,-3/2} \ .$$

We evaluate the bosonized version of (4.1) by taking the bracket of (4.11) with the bosonized ghost analog of the kets $\ddagger C^jB^k\ddagger\,|\downarrow\downarrow\downarrow\rangle$, obtained by applying (4.9), (4.10), and their closed-string analogs to $|\,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle$. We will then need to check that we get the same expression as for fermions, using (4.1) directly. The evaluation is carried out in the usual way by moving raising operators to the left and lowering operators to the right. We then encounter the following sums:

$$\begin{aligned} g_{11}(u,v) &= \frac{1}{4}\sum_{k,l=1}^{\infty}(-)^{k+l}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{l}u^kv^l\frac{1}{k+l}\\ &= \frac{1}{2}\ln\frac{[1+(1-u)^{1/2}][1+(1-v)^{1/2}]}{2[(1-u)^{1/2}+(1-v)^{1/2}]} \end{aligned} \tag{4.12}$$

$$\begin{aligned} g_{21}(u,v) &= \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}(-)^{k+l}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{l}u^{2k+1}v^l\frac{1}{2l-2k-1}\\ &= \frac{1}{2i}\ln\frac{(1-v)^{1/2}+i(u^{-2}-1)^{1/2}}{(1-v)^{1/2}-i(u^{-2}-1)^{1/2}}\,\frac{1-i(u^{-2}-1)^{1/2}}{1+i(u^{-2}-1)^{1/2}} \ , \end{aligned} \tag{4.13}$$

$$\begin{aligned} g_{22}(u,v) &= \frac{1}{4}\sum_{k,l=0}^{\infty}(-)^{k+l}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{l}u^{2k+1}v^{2l+1}\frac{1}{k+l+1}\\ &= -\frac{1}{2}\ln\frac{v(1-u^2)^{1/2}+u(1-v^2)^{1/2}}{u+v} \ . \end{aligned} \tag{4.14}$$

The easiest way to prove these sums is to apply $[u(\partial/\partial u)+v(\partial/\partial v)]$ to (4.12) and (4.14) and $[u(\partial/\partial u)-2v(\partial/\partial v)]$ to (4.13). The integration constant could depend on u/v or u^2v, respectively, but such dependence is incompatible with the expansions, so integrating back is unique.

We clearly have to deal with several cases separately. A relatively simple case is to take both fields in (4.1) to be left modes of the closed string. The normal ordering in (4.1) may be effected by subtracting the singular contribution as $z\rightarrow y$:

$$\langle 0|\Upsilon\left[C^b(y)B^b(z)-\frac{z}{y-z}\right]|-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{z}{y-z}\left[\left[\frac{1-y}{1-z}\right]^{3/4}\exp[g_{11}(y,y)+g_{11}(z,z)-2g_{11}(y,z)]-1\right]$$
$$=\frac{z}{y-z}\left[\frac{(1-y)^{1/2}}{1-z}\frac{(1-y)^{1/2}+(1-z)^{1/2}}{2}-1\right]. \tag{4.15}$$

With fermionized ghosts, (4.1) gives the linear combination

$$\tfrac{1}{2}(f^{++}+f^{+-}+f^{-+}+f^{--})(z,y)=\frac{1}{2}\left[\frac{z}{y-z}\left[\frac{(1-y)^{1/2}}{(1-z)^{1/2}}-1\right]-\frac{z}{1-z}\right]$$

which is identical to (4.15).

The matrix element involving $:\tilde{C}\,\tilde{B}:$ is identical to (4.15) and need not be checked separately. The one involving $C\tilde{B}$ is more interesting since it creates nonzero winding number, $\tilde{G}_0-G_0=-2$. Evaluating this using bosonized ghosts gives

$$z(\tfrac{1}{2})^{(\tilde{G}_0-G_0)^2/2}\left[\frac{1-y}{1-z}\right]^{3/4}\exp\left[g_{11}(y,y)+g_{11}(z,z)+2g_{11}(y,z)+(G_0-\tilde{G}_0)\sum_{k=1}^{\infty}\frac{(-)^k}{2k}\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}(z^k+y^k)\right]$$
$$=\frac{1}{2}\frac{(1-y)^{1/2}}{1-z}z\frac{(1-y)^{1/2}-(1-z)^{1/2}}{z-y} \tag{4.16}$$

to be compared with the appropriate linear combination of (4.1):

$$\tfrac{1}{2}(f^{++}-f^{-+}+f^{+-}-f^{--})=-\frac{1}{2}\frac{z}{y-z}\left[\frac{y-1}{1-z}+\left[\frac{1-y}{1-z}\right]^{1/2}\right]$$

which is the same as (4.16) except for a minus sign. Note the essential role of the winding-number contributions in getting the full magnitude correctly. Again the matrix elements involving $\tilde{C}B$ behave identically to $C\tilde{B}$ and need not be checked separately.

The next case we turn to is with both fields in (4.1) acting as open-string fields. Again the normal ordering is effected by subtracting the c-number contraction term:

$$\langle 0|\Upsilon\left[c(y)b(z)-\frac{z}{y-z}\right]|-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{z}{y-z}\left[\exp[g_{22}(y,y)+g_{22}(z,z)-2g_{22}(y,z)]\left[\frac{1-y^2}{1-z^2}\right]^{3/4}-1\right]$$
$$=\frac{z}{y-z}\left[\frac{y(1-z^2)^{1/2}+z(1-y)^2)^{1/2}}{(y+z)(1-y^2)^{1/4}(1-z^2)^{1/4}}\left[\frac{1-y^2}{1-z^2}\right]^{3/4}-1\right]=f^{oo}(z,y)\ , \tag{4.17}$$

which is what we found with (4.1) for the fermionic ghosts.

The last case to check involves both open- and closed-string ghosts. In this case we shall find agreement only up to a canonical scaling of the closed-string fields relative to the open-string fields. There are two cases to consider. The first is the bosonized calculation of

$$\langle 0|\Upsilon_B c(y)B^b(z)|-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle=(\tfrac{1}{2})^{(\tilde{G}_0-G_0)^2/2}z\left[\frac{1-y^2}{1-z}\right]^{3/4}(1-y^2z)^{-1/2}$$
$$\times\exp\left[g_{22}(y,y)+g_{11}(z,z)-ig_{21}(y,z)+(G_0-\tilde{G}_0)\ln\frac{1+(1-z)^{1/2}}{2}\right.$$
$$\left.+(G_0-\tilde{G}_0)\ln[(1-y^2)^{1/2}-iy]\right]$$
$$=\frac{z}{\sqrt{2}}\left[\frac{1-y^2}{1-z}\right]^{3/4}(1-y^2z)^{-1/2}\frac{1+\left[1-\frac{1}{z}\right]^{1/2}}{2(1-y^2)^{1/4}(1-z)^{1/4}}$$
$$\times\left[\frac{(1-z)^{1/2}-i(y^{-2}-1)^{1/2}}{(1-z)^{1/2}+i(y^{-2}-1)^{1/2}}\frac{1+i(y^{-2}-1)^{1/2}}{1-i(y^{-2}+1)^{1/2}}\right]^{1/2}\frac{2}{1+(1-z)^{1/2}}\frac{1}{(1-y^2)^{1/2}-iy}$$
$$=\frac{1}{\sqrt{2}}\left[\frac{z}{1-z}-\frac{y^2z}{1-y^2z}+\frac{iyz}{1-y^2z}\left[\frac{1-y^2}{1-z}\right]^{1/2}\right]. \tag{4.18}$$

The appropriate linear combination of (4.1) to get the corresponding fermionic expression is $(1/\sqrt{2})(f^{+o}+f^{-o})(z,y)$, which is smaller than (4.18) by a factor of $2\sqrt{2}$. This is the first indication that we need to do a canonical rescaling of the B's and C's. The corresponding factor for C is found by a similar calculation which gives

$$\langle 0 \mid \Upsilon_B C^b(y) b(z) \mid -\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{i}{2\sqrt{2}}\left[\frac{f^{o+}+f^{o-}}{\sqrt{2}}\right] . \tag{4.19}$$

Both the phase and the $2\sqrt{2}$ are unwanted. The unwanted $2\sqrt{2}$ factors can be removed by a canonical transformation

$$C=2\sqrt{2}C^b,\quad B=\frac{1}{2\sqrt{2}}B^b ,$$
$$\tilde{C}=2\sqrt{2}\tilde{C}^b,\quad \tilde{B}=\frac{1}{2\sqrt{2}}\tilde{B}^b . \tag{4.20}$$

That the left-handed modes need the same canonical rescaling as the right-handed ones can be verified by comparing the bosonized evaluation of $\langle 0 \mid \Upsilon\tilde{C}^b b \mid -\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle$ and $\langle 0 \mid \Upsilon c\tilde{B} \mid -\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\rangle$ with $(f^{o+}-f^{o-})/\sqrt{2}$ and $(f^{+o}-f^{-o})/\sqrt{2}$, respectively. We could have anticipated the need for (4.20) by inspection of the ghost part of the fermionized vertex (3.34), where one sees that the terms in $c_nB_m, c_n\tilde{B}_m$ are multiplied by 2 but those in $C_nb_m, \tilde{C}_nb_m$ are multiplied by $\tfrac{1}{4}$. The renormalization (4.20) makes both these coefficients $1/\sqrt{2}$ if expressed in terms of $C_1^b, \tilde{C}_1^b$, etc. If we had not made the canonical transformation (4.20), we would still have a vertex annihilated by $Q=Q^{\text{closed}}+Q^{\text{open}}$, but the expression for Q^{closed} in terms of fermionic oscillators would need an unexpected factor of $2\sqrt{2}$.

The bosonized matrix elements we have evaluated are obviously sufficient to determine the form of the fermionized ghost factor up to signs and the overall field renormalization as in (4.20). So aside from these factors, we have obtained the fermion form of the vertex in two different ways.

The boson-fermion equivalence holds for states created by any number of excitations on the vacuum. To see this, one should compare matrix elements of arbitrary monomials of fermion ghost fields in the two languages and relate them. We shall illustrate this procedure for the special case where the open string is in the ghost number $-\tfrac{1}{2}$ vacuum state

$$\mid \downarrow\rangle = \mid 0,-\tfrac{1}{2}\rangle . \tag{4.21}$$

Thus we shall find the precise relation of the two languages, including phase factors, within purely closed-string excitations which have overall ghost number -1, since the vertex vanishes unless

$$g_0+G_0+\tilde{G}_0=-\tfrac{3}{2} . \tag{4.22}$$

This is enough to test the consistency of bosonization in the presence of winding number, and illustrates the essential method. We shall calculate

$$\langle 0 \mid \Upsilon C(u_N)C(u_{N-1}) \cdots C(u_1)\tilde{B}(v_1) \cdots \tilde{B}(v_N) \mid \downarrow\downarrow\downarrow\rangle \tag{4.23}$$

in each language and establish the relation between them.

We start by recalling the fermionic result Eq. (4.1):

$$\begin{aligned}\langle 0 \mid \Upsilon_F C(u)\tilde{B}(v) \mid \downarrow\downarrow\downarrow\rangle &= \tfrac{1}{2}\langle 0 \mid \Upsilon(C^+ + C^-)(B^+ - B^-) \mid \downarrow\downarrow\rangle \\ &= \tfrac{1}{2}[f^{++}(v,u)-f^{--}(v,u)] \\ &= \frac{v(1-u)^{1/2}}{2(1-v)}\left[\frac{(1-u)^{1/2}-(1-v)^{1/2}}{u-v}\right] \\ &\equiv \frac{vU}{2V^2}\left[\frac{U-V}{u-v}\right] ,\end{aligned} \tag{4.24}$$

where to save writing we have introduced the notation

$$U\equiv(1-u)^{1/2},\quad V\equiv(1-v)^{1/2} . \tag{4.25}$$

Because Υ is an exponential of a bilinear in $\tilde{C}$ and B, Eq. (4.23) can be evaluated as a Wick expansion with (4.24) as the propagator and relative signs determined by the Grassmann odd character of C and $\tilde{B}$.

In terms of bosonized ghosts, we can use (4.9) and (4.10) to assert that (4.23) is proportional to

$$\begin{aligned}&\langle 0 \mid \Upsilon_B :\exp\left[-\sum_n G_n u_N^{-n}/n\right]:u_N^{1/2+G_0} \cdots :\exp\left[-\sum_n G_n u_1^{-n}/n\right]:u_1^{1/2+G_0} \\ &\quad\times:\exp\left[\sum_n \tilde{G}_n v_1^{-n}/n\right]:v_1^{1/2-\tilde{G}_0} \cdots :\exp\left[\sum_n \tilde{G}_n v_N^{-n}/n\right]:v_N^{1/2-\tilde{G}_0} \mid 0,-\tfrac{1}{2}\rangle \mid 0,-\tfrac{1}{2}\rangle .\end{aligned} \tag{4.26}$$

To evaluate (4.26) first move all positive moded G's and $\tilde{G}$'s to the right to obtain

$$\prod_{i=1}^N v_i \prod_{i<j}(v_i-v_j)(u_j-u_i)\langle 0 \mid \Upsilon_B \exp\left[\sum_{n=1}^\infty \frac{G_{-n}}{n}\sum_i u_i^n - \sum_{n=1}^\infty \frac{\tilde{G}_{-n}}{n}\sum_i v_i^n\right] \mid 0,-\tfrac{1}{2}+N\rangle \mid 0,-\tfrac{1}{2}-N\rangle . \tag{4.27}$$

Then (4.26) becomes

$$2^{-2N^2}\prod_i v_i \prod_{i<j}(v_i-v_j)(u_j-u_i)\prod_i\left[\frac{1-u_i}{1-v_i}\right]^{3/4}\prod_{i,j}\left[\frac{(1+U_i)(1+U_j)}{2(U_i+U_j)}\right]^{1/2}\prod_{i,j}\left[\frac{(1+V_i)(1+V_j)}{2(V_i+V_j)}\right]^{1/2}$$
$$\times\prod_{i,j}\left[\frac{(1+U_i)(1+V_j)}{2(U_i+V_j)}\right]\prod_i\left[\frac{1+U_i}{2}\right]^{-2N}\prod_i\left[\frac{1+V_i}{2}\right]^{-2N}$$
$$=2^{-2N^2}\prod_i\left[\frac{v_iU_i}{V_i^2}\right]\prod_{i<j}(v_i-v_j)(u_j-u_i)\prod_{i<j}\frac{2}{U_i+U_j}\prod_{i<j}\frac{2}{V_i+V_j}\prod_{i,j}\frac{2}{U_i+V_j}\ , \quad (4.28)$$

where in the last line we have done some rearranging of factors. Next notice that

$$\frac{1}{U_i+U_j}=\frac{U_i-U_j}{U_i^2-U_j^2}=\frac{U_i-U_j}{u_j-u_i}\ ,$$

etc., so that (4.28) becomes

$$\prod_i\left[\frac{v_iU_i}{2V_i^2}\right]\prod_{i<j}(U_i-U_j)(V_j-V_i)\prod_{i,j}\left[\frac{V_j-U_i}{u_i-v_j}\right]. \quad (4.29)$$

Now notice the common factors $\frac{1}{2}vUV^{-2}$ in (4.29) and the contraction (4.24). Thus the ratio of fermion evaluation to boson one is

$$\frac{F}{B}=\sum_{\text{contractions}}(\pm)\prod\frac{U-V}{u-v}\Big/\prod_{i<j}(U_i-U_j)(V_j-V_i)\prod_{i,j}\left[\frac{V_j-U_i}{u_i-v_j}\right]. \quad (4.30)$$

The contraction

$$\frac{U_1-V_1}{u_1-v_1}\frac{U_2-V_2}{u_2-v_2}\cdots\frac{U_N-V_N}{u_N-v_N}=\prod_i\frac{U_i-V_i}{u_i-v_i}=(-)^N\prod_{i,j}\left[\frac{V_j-U_i}{u_i-v_j}\right]\prod_{i\neq j}(U_i+V_j)\ , \quad (4.31)$$

where we have used the identity

$$1=\frac{V_j-U_i}{u_i-v_j}(V_j+U_i)\ . \quad (4.32)$$

Each other contraction can be associated with one of the permutations $P(i)$ of the indices of U, thus

$$\frac{F}{B}=(-)^N\frac{\sum_P(-)^P\prod_{i\neq j}(U_{Pi}+V_j)}{\prod_{i<j}(U_i-U_j)(V_j-V_i)}\ . \quad (4.33)$$

The numerator can be written

$$\prod_{i,j}(U_i+V_j)\sum_P(-)^P\prod_i\frac{1}{U_{Pi}+V_i}$$
$$=\det\left[\frac{1}{U_i+V_j}\right]\prod_{k,l}(U_k+V_l)\ . \quad (4.34)$$

According to Cauchy[18] the determinant can be evaluated as

$$\det\left[\frac{1}{U_i+V_j}\right]=\prod_{i,j}\frac{1}{U_i+V_j}\prod_{i<j}(U_i-U_j)(V_i-V_j)\ . \quad (4.35)$$

Thus

$$\frac{F}{B}=(-)^N\prod_{i<j}(-)=(-)^{N+(N^2-N)/2}=(-)^{(N^2+N)/2}\ . \quad (4.36)$$

Thus we have established the precise relation between (4.23) and (4.26).

To fix the relative phase of Υ_B and Υ_F in this sector, we must specify the Klein factors which convert $c,b,C^b,B^b,\tilde{C}^b,\tilde{B}^b$ into mutually anticommuting fields. One choice is

$$C=2\sqrt{2}(-)^{g_0-1/2}iC^b\ ,$$
$$B=\frac{1}{2\sqrt{2}}(-)^{g_0+1/2}iB^b\ ,$$
$$\tilde{C}=2\sqrt{2}(-)^{G_0+g_0}\tilde{C}^b\ ,$$
$$\tilde{B}=2\sqrt{2}(-)^{G_0+g_0}\tilde{B}^b\ , \quad (4.37)$$

where $c,b,C^b,B^b,\tilde{C}^b,\tilde{B}^b$ are defined by (4.9) and (4.10) and their analogs. Then comparison of Eq. (4.36) with (4.37) enables us to infer that

$$\langle 0|\Upsilon_F|g_0=-\tfrac{1}{2}\rangle=\langle 0|i^{n^2/4}\Upsilon_B|g_0=-\tfrac{1}{2}\rangle\ , \quad (4.38)$$

where $n=\tilde{G}_0-G_0$ is the winding number. The extension of this formula to the case $g_0\neq-\frac{1}{2}$ is tedious but straightforward. We conjecture that it is given by replacing $n^2/4$

by a general quadratic polynomial in n and $g_0+\frac{1}{2}$. Then if this ansatz is correct, the calculations sketched in the first part of this section determine the coefficients of the polynomial uniquely, with the result that

$$i^{n^2/4}\to i^{[n^2-(g_0+1/2)^2]/4+n(g_0+1/2)/2-3(g_0+1/2)/2}$$

in Eq. (4.38). Since Υ_F has been unambiguously determined from BRST invariance and is all we need in what follows, we shall not endeavor to prove our ansatz here.

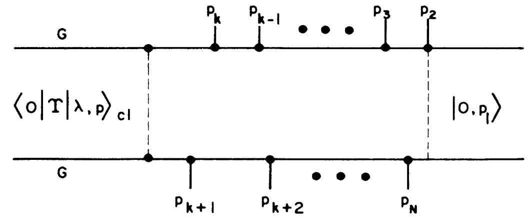

FIG. 2. Interacting string diagram for the process closed-string$\to N$ open-string tachyons. Tachyons $2,3,\ldots,N$ are attached to the diagram by the vertex operator $V_l=g:\exp[ip_lX(\sigma,\tau)]:$, where $\sigma_l=0$ or π and the τ_l are integrated respecting the indicated time ordering.

V. SCATTERING AMPLITUDE FOR ONE CLOSED STRING AND N OPEN STRINGS

In this section we shall evaluate the scattering amplitudes for one closed string and N open strings. For simplicity we shall take each of the open strings in its (tachyonic) ground state, and restrict the closed string to the ground state and the massless first excited states. Then we shall compare our results to the calculations[4,5] based on factorization of nonplanar open-string loop graphs. The result is that our amplitudes provide the same answers for physical states, once we fix the proportionality constant between the open-closed coupling G and the three-open-string coupling g.

To describe open strings we shall use the vertex operator approach. The Fubini-Veneziano vertex for the emission of an open-string tachyon of momentum p from an open string is

$$\begin{Bmatrix} V_0 \\ V_0^T \end{Bmatrix}=g:\exp\left[ip\cdot X\begin{bmatrix}\sigma=0\\ \sigma=\pi\end{bmatrix}\right]:$$
$$=ge^{ip\cdot q_0}:\exp\left[\sqrt{2\alpha'}p\cdot\sum_{n\neq 0}(\pm)^n\frac{a_{-n}}{n}\right]:, \quad (5.1)$$

where here and in the following we have restored the dependence on α'. To make V BRST invariant, one simply multiplies V_0 by a factor $\sum_{n=-\infty}^{\infty}(\pm)^n c_n$:

$$V=g\sum_n c_n(\pm)^n e^{ip\cdot q_0}:\exp\left[\sqrt{2\alpha'}p\cdot\sum_{n\neq 0}(\pm)^n\frac{a_{-n}}{n}\right]:, \quad (5.2)$$

so that

$$\{Q,V(p)\}=0\ . \quad (5.3)$$

The standard bosonic dual amplitude for N tachyons is then

$$\langle\downarrow,p_N\,|\,V(p_{N-1})\frac{\alpha' b_0}{\mathcal{L}_0}V(p_{N-2})\cdots\frac{\alpha' b_0}{\mathcal{L}_0}V(p_2)\,|\downarrow,p_1\rangle=\langle 0,p_N\,|\,V_0(p_{N-1})\frac{\alpha'}{L_0-1}V_0(p_{N-2})\cdots\frac{\alpha'}{L_0-1}V_0(p_2)\,|\,0,p_1\rangle$$
$$=\int dy_i\prod_{i<j}|y_i-y_j|^{2\alpha' p_i\cdot p_j}(g\alpha')^{N-2}\frac{1}{\alpha'}\ , \quad (5.4)$$

where the dependence on ghost operators is trivial because only the c_0 term in V gives a nonvanishing contribution.

To couple the closed-string state $|\lambda,\text{closed}\rangle$ to N open strings we use our Υ to obtain

$$\langle 0\,|\,G\Upsilon\,|\,\lambda,\text{closed}\rangle\frac{\alpha' b_0}{\mathcal{L}_0}V(p_N)\frac{\alpha' b_0}{\mathcal{L}_0}\cdots\frac{\alpha' b_0}{\mathcal{L}_0}V(p_2)\,|\downarrow,p_1\rangle\ . \quad (5.5)$$

Again because of the simple form of the ghost dependence of V, the open-string ghost operators may be dropped so (5.5) reduces to

$$\langle 0\,|\,G\Upsilon\,|\,\lambda,\text{closed}\rangle\frac{\alpha'}{L_0-1}V_0(p_N)\frac{\alpha'}{L_0-1}\cdots\frac{\alpha'}{L_0-1}V_0(p_2)\,|\downarrow,p_1\rangle$$
$$=G(g\alpha')^{N-1}\int dy_2\cdots dy_N\prod_{i<j}|y_j-y_k|^{2\alpha' p_i\cdot p_j}\langle 0\,|\,\Upsilon\,|\,\lambda,\text{closed}\rangle\exp\left[\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{a_{-n}}{n}\cdot\sum_i y_i^n p_i\right]|\downarrow\rangle\ , \quad (5.6)$$

where the integration region is

$$0=|y_1|\le|y_2|\le\cdots\le|y_N|\le 1 \quad (5.7)$$

and the sign of y is positive for a vertex at $\sigma=0$ and negative at $\sigma=\pi$. A given dual diagram, which corresponds to a given cyclic ordering $(123\cdots N)$, is a sum of (5.5) over all arrangements of vertices in Fig. 2 with that cyclic order, summing over k and all relative time orderings of vertices on the top with those on the bottom. In fact, only this sum will be BRST invariant.

The part of the matrix element in (5.6) involving open-string oscillators involves only Υ_0 and may be evaluated immediately:

$$\langle 0|\Upsilon_0 \exp\left[\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{a_{-n}}{n}\sum_i y_i^n p_i\right]|0\rangle = \exp\left[2\alpha'\sum_{k,l}p_k p_l g_{22}(y_k,y_l)\right]$$
$$\times\langle 0|\exp\left[\tfrac{1}{2}\sum_{k,l=1}^{\infty}\begin{bmatrix}-\frac{1}{2}\\k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\l\end{bmatrix}\frac{(-)^{k+l}}{2(k+l)}(A_k-\tilde{A}_k)\cdot(A_l-\tilde{A}_l)\right.$$
$$-\sqrt{2\alpha'}\sum_{k=1}^{\infty}\frac{1}{2k}(A_k+\tilde{A}_k)\cdot\sum_i p_i y_i^{2k}$$
$$\left.+i\sqrt{2\alpha'}\sum_{k,l=0}^{\infty}\frac{(-)^{k+l}}{2l-2k-1}\begin{bmatrix}-\frac{1}{2}\\k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\l\end{bmatrix}(A_l-\tilde{A}_l)\cdot\sum_i p_i y_i^{2k+1}\right]$$
$$\equiv\exp\left[2\alpha'\sum_{k,l}p_k\cdot p_l g_{22}(y_k,y_l)\right]\langle 0|\mathcal{V}(p,y)|\lambda,\text{closed}\rangle\ . \tag{5.8}$$

Referring to (4.14) we see that the first exponential on the RHS of (5.8) is just

$$\prod_{i,j=2}^{N}\left|\frac{y_i+y_j}{y_1\sqrt{1-y_j^2}+y_j\sqrt{1-y_i^2}}\right|^{\alpha' p_i\cdot p_j}=\prod_{i=2}^{N}(1-y_i^2)^{-1/2}\prod_{i\neq j}|y_i\sqrt{1-y_j^2}-y_j\sqrt{1-y_i^2}|^{\alpha' p_j\cdot p_j}\prod_{i\neq j}|y_i-y_j|^{-\alpha' p_i\cdot p_j}\ . \tag{5.9}$$

Using (5.8) and (5.9), we see that (5.6) can be rewritten

$$G(g\alpha')^{N-1}\sum_k\int_{R_k}dy_2\cdots dy_N\prod_{i=2}^{N}|1-y_i^2|^{-1/2}\prod_{i\neq j}|y_i\sqrt{1-y_j^2}-y_j\sqrt{1-y_i^2}|^{\alpha' p_i\cdot p_j}\langle 0|\mathcal{V}(p,y)|\lambda,\text{closed}\rangle\ . \tag{5.10}$$

The sum over all relative time orderings between top and bottom of Fig. 2, for a fixed k, means that R_k in (5.10) is specified by

$$R_k: -1<y_{k+1}<\cdots<y_N<0<y_2<\cdots<y_k<1\ . \tag{5.11}$$

We can simplify (5.10) by changing variables to

$$y_i=\sin\theta_i/2,\quad -\pi<\theta_i<\pi\ .$$

Notice that the vertices on the upper edge are mapped into $0<\theta_i<\pi$, while those on the lower go into $-\pi<\theta_i<0$.

In this region we may define $\mathcal{V}$ as a function of θ, $\hat{\mathcal{V}}(\theta)=\mathcal{V}(\sin\theta/2)$ by

$$\hat{\mathcal{V}}(p,\theta)=\exp\left[\tfrac{1}{2}\sum_{k,l=1}^{\infty}\begin{bmatrix}-\frac{1}{2}\\k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\l\end{bmatrix}\frac{(-)^{k+l}}{2(k+l)}(A_k-\tilde{A}_k)\cdot(A_l-\tilde{A}_l)\right.$$
$$\left.-\sqrt{2\alpha'}\sum_{k=1}^{\infty}\frac{1}{2k}(A_k+\tilde{A}_k)\cdot\sum_i p_i\sin^{2k}\theta_i/2+i\sqrt{2\alpha'}\sum_{l=1}^{\infty}(A_l-\tilde{A}_l)\cdot\sum_i p_i\omega_l(\theta_i)\right]\ , \tag{5.12}$$

where $\omega_l(\theta)$ is the coefficient of v^l in the Taylor expansion of

$$g_{21}(\sin\theta/2,v)=\frac{1}{2i}\ln\left[\frac{(1-v)^{1/2}+i\cot(\theta/2)}{(1-v)^{1/2}-i\cot(\theta/2)}\,\frac{1-i\cot(\theta/2)}{1+i\cot(\theta/2)}\right]\ . \tag{5.13}$$

Noting that the derivatives with respect to v remove any cuts, we see that $\hat{\mathcal{V}}$ in this form can be extended analytically outside the range $-\pi<\theta<\pi$ to be a periodic function under $\theta\to\theta+2\pi$. Thus $\mathcal{V}$ may be regarded as a periodic function of θ_i, and the sum over k may be taken by extending the integration region to

$$0=\theta_1<\theta_2<\cdots<\theta_N<2\pi \tag{5.14}$$

without restricting how many lie in $[0,\pi]$. Our amplitude becomes simply

$$G\left[\frac{g\alpha'}{2}\right]^{N-1}\int_0^{2\pi}d\theta_2\cdots d\theta_N\prod_{i\neq j}\left|\sin\left[\frac{\theta_i-\theta_j}{2}\right]\right|^{\alpha' p_i\cdot p_j}\langle 0\,|\,\hat{\mathcal{V}}(p,\theta)\,|\,\lambda,\text{closed}\rangle\ . \tag{5.15}$$

At this point we need to point out that this integral representation for the scattering amplitude converges only for all kinematic variables below their thresholds. In particular, there are poles in $(\sum p_i)^2$ at the positions of open-string particles. As usual, one can extract amplitudes at higher (mass)^2 by analytically continuing around these poles. This is required, for example, in the case of massless closed-string couplings.

We note the particular cases of (5.12) for the tachyon and massless levels

$$\langle 0\,|\,\mathcal{V}\,|\,0\rangle=1\ , \tag{5.16}$$

$$\langle 0\,|\,\mathcal{V}A^{\mu}{}_{-1}\tilde{A}^{\nu}{}_{-1}\,|\,0\rangle=-\frac{g^{\mu\nu}}{16}+\left[-\frac{\sqrt{2\alpha'}}{2}\sum_i p_i^{\mu}\sin^2\theta_i/2+i\sqrt{2\alpha'}\sum_i p_i^{\mu}\omega_1(\theta_i)\right]\times\left[-\frac{\sqrt{2\alpha'}}{2}\sum_j p_j^{\nu}\sin^2\theta_j/2-i\sqrt{2\alpha'}\sum_j p_j^{\nu}\omega_1(\theta_j)\right]\ . \tag{5.17}$$

From $\omega_1(\theta)=(\partial/\partial v)g_{21}(\sin\theta/2,v)\,|_{v=0}$,

$$\omega_1(\theta)=\tfrac{1}{2}\sin(\theta/2)\cos(\theta/2)$$

so

$$\begin{aligned}\langle 0\,|\,\mathcal{V}A^{\mu}{}_{-1}\tilde{A}^{\nu}{}_{-1}\,|\,0\rangle&=-\frac{g^{\mu\nu}}{16}+\frac{\alpha'}{2}\sum_i p_i^{\mu}\sin(\theta_i/2)e^{i\theta_i/2}\sum_j p_j^{\nu}\sin(\theta_j/2)e^{-i\theta_j/2}\\&=-\frac{g^{\mu\nu}}{16}+\frac{\alpha'}{8}\left[\sum_{i,j}p_i^{\mu}p_j^{\nu}e^{i(\theta_i-\theta_j)}+p^{\mu}p^{\nu}-p^{\nu}\sum_i p_i^{\mu}e^{i\theta_i}-p^{\mu}\sum_i p_i^{\nu}e^{-i\theta_i}\right]\ .\end{aligned} \tag{5.18}$$

Of course, the physical components of $A^{\mu}{}_{-1}A^{\nu}{}_{-1}\,|\,0\rangle$ are

$$\epsilon\cdot A_{-1}\tilde{\epsilon}\cdot\tilde{A}_{-1}\,|\,0\rangle\ ,$$

where

$$p\cdot\epsilon=p\cdot\tilde{\epsilon}=0\ , \tag{5.19}$$

so for these

$$\langle 0\,|\,\mathcal{V}\epsilon\cdot A_{-1}\tilde{\epsilon}\cdot\tilde{A}_{-1}\,|\,0\rangle=-\frac{\epsilon\cdot\tilde{\epsilon}}{16}+\frac{\alpha'}{8}\sum_{i,j}\epsilon\cdot p_i\tilde{\epsilon}\cdot p_j e^{i(\theta_i-\theta_j)}\ . \tag{5.20}$$

Our calculations of the coupling of a closed string to N open strings may be compared to the results of old calculations found by factorizing the singularities of nonplanar open-string loop graphs. This provides not only a check that the apparently different operators Υ and $\check{\Upsilon}$ give the same physical amplitudes, but it also enables us to find the proportionality constant between G and the coupling g of three open strings. We have checked that the form of the tachyon and massless physical states agree with Refs. 4 and 5. To extract the proportionality constant from those papers we must be careful to supply missing normalization factors. We would also like to reinsert the dependence on α' which has usually been set to $\frac{1}{2}$. Also the old references use a propagator

$$\frac{1}{L_0-1}=\frac{1}{\frac{1}{2}p^2+\sum a_{-n}a_n-1} \tag{5.21}$$

which does not have the proper normalization; the propagator should be

$$\frac{\alpha'}{L_0-1}\underset{\alpha'\to 1/2}{\longrightarrow}\frac{1}{2(L_0-1)}\neq\frac{1}{L_0-1}\ . \tag{5.22}$$

Next, the loop amplitude quoted in Ref. 4 is the coefficient of $\delta(\sum p_i)$ rather than $(2\pi)^D\delta(\sum p_i)$, so to get the usual Feynman amplitude [which has no $(2\pi)^D$ in trees] we should include a factor of $(2\pi)^{-26}$ to compensate for this. So the new factors to supply in a loop amplitude with $N+M$ particles are

$$(\alpha')^{N+M}\frac{1}{(2\pi)^{26}}\frac{1}{(2\alpha')^{13}}\underset{\alpha'\to 1/2}{\longrightarrow}\left[\frac{1}{2}\right]^{N+M}\frac{1}{(2\pi)^{26}} \tag{5.23}$$

together with the substitution $p\to\sqrt{2\alpha'}p$ to restore the α' dependence. In addition there is a missing factor of $1/\pi$ on the RHS of Eq. (2.6) in Ref. 4. This has caused an erroneous factor of π^{p^2} to appear in some of their equations [e.g., their Eq. (2.12)]. Finally we shall use a spacelike metric

$$g^{\mu\nu}=\text{diag}(-1,+1,\ldots,+1)\ . \tag{5.24}$$

Inserting all these factors and correcting mistakes, Eq. (2.13) of Cremmer and Scherk for the one-loop nonplanar amplitude with r particles inside and s particles outside becomes

$$F_{r,s}^{\text{Feynman}}=(\alpha' g)^{r+s}2^{2\alpha' p^2}\frac{1}{(2\pi)^{26}}\left[\frac{1}{2\alpha'}\right]^{13}\int_0^1 dq\, q^{(\alpha'/2)p^2-3}f(q^2)^{-24}$$
$$\times\int_0^\pi d\theta\prod_{i=2}^{r}\prod_{l=2}^{s}d\theta_i d\phi_l\prod_{i<j}|\bar\psi(\theta_j-\theta_i)|^{2\alpha' p_i\cdot p_j}$$
$$\times\prod_{l<m}\bar\psi(\phi_m-\phi_l)^{2\alpha' k_m\cdot k_l}\prod_{i,l}|\bar\psi_T(\theta+\theta_i+\phi_l)|^{2\alpha' p_i\cdot k_l} \quad (5.25)$$

with[19]

$$0=\theta_1\le\theta_2\le\cdots\le\theta_r\le\pi\,,$$
$$0=\phi_1\le\phi_2\le\cdots\le\phi_s\le\pi \quad (5.26)$$

and

$$\bar\psi(\theta)=\sin\theta\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos2\theta+q^{4n}}{(1-q^{2n})^2}\,, \quad (5.27)$$

$$\bar\psi_T(\theta)=\prod_{n=1}^{\infty}\frac{1-2q^{2n-1}\cos2\theta+q^{4n-2}}{(1-q^{2n})^2}\,. \quad (5.28)$$

To relate G to g, it is enough to examine (5.25) near the closed-string tachyon pole $\alpha' p^2\sim 4$:

$$F_{r,s}\underset{\alpha' p^2\to 4}{\sim}\pi\frac{2^8}{(2\pi)^{26}}(\alpha' g)^{r+s}\left[\frac{1}{2\alpha'}\right]^{13}\frac{1}{\frac{1}{2}\alpha' p^2-2}\int d\theta\prod_{i<j}|\sin(\theta_i-\theta_j)|^{2\alpha' p_i\cdot p_j}\int d\theta\prod_{l<m}|\sin(\phi_l-\phi_m)|^{2\alpha' k_l\cdot k_m}\,. \quad (5.29)$$

This amplitude, evaluated directly from factoring the nonplanar loop, must be compared with (5.15) and (5.16) for $N=r,s$, which gives

$$F_{r,s}\sim\frac{G^2}{p^2-\dfrac{4}{\alpha'}}\int_0^\pi d\theta_i\prod_{i<j}|\sin(\theta_i-\theta_j)|^{2\alpha' p_i\cdot p_j}\int_0^\pi d\phi_l\prod_{l<m}|\sin(\phi_l-\phi_m)|^{2\alpha' k_l\cdot k_m}\,.$$

Thus we require

$$G^2=\frac{\pi g^2}{16\alpha'^{12}(2\pi)^{26}}=\frac{g^2T_0^{12}}{32(2\pi)^{13}}\,. \quad (5.30)$$

Using (5.30), it is then easy to confirm that the singularities at zero mass agree with (5.17) modulo the gauge terms. This confirms that Υ does give the correct dual amplitudes. In interpreting (5.30) it is important to bear in mind how g and G enter physical scattering amplitudes. At the tree level each tachyon emission vertex includes a factor of g and the propagators are (5.22). The complete Feynman tree amplitude for N tachyons is then the sum over all cyclically distinct permutations (*including* anticyclic ones). Similiarly, the complete one-loop amplitude $F_{r,s}$ for $r\neq s$ and fixed sets of particles r,s, is given by summing over all cyclically distinct permutations of the r particles among themselves and (separately) of the s particles among themselves. In this way we see that (5.15) must be summed over cyclically distinct permutations of the N tachyons (including anticyclic ones). The consistency of these rules is based on a careful application of the Feynman tree theorem.

VI. CLOSED-STRING–OPEN-STRING MIXING

The Υ operator is well suited to the description of the important physical effect of closed-string–open-string mixing: it directly gives the transition amplitude from one type of string to the other. If there is mass degeneracy between open- and closed-string states the Υ operator completely controls the mixing phenomenon.

Suppose there are a set of open-string states b degenerate at mass M with closed-string states B and a transition matrix GR_{bB}. Then the propagator for open or closed strings can be developed near mass shell:

$$D_{ab}=\frac{\delta_{ab}}{p^2+M^2}+G^2\sum_A\frac{R_{aA}R_{Ab}}{(p^2+M^2)^3}+G^4\sum_{A,B,c}\frac{R_{aA}R_{Ac}R_{cB}R_{Bb}}{(p^2+M^2)^5}+\cdots$$
$$=\frac{1}{p^2+M^2}\left[I-\frac{G^2}{(p^2+M^2)^2}RR^T\right]^{-1}_{ab}$$
$$=\left[p^2+M^2-\frac{G^2}{p^2+M^2}RR^T\right]^{-1}_{ab}\,. \quad (6.1)$$

Let the eigenvalues of $(RR^T)_{ab}$ be λ_α:

$$(RR^T)_{ab}V_\alpha^b=\lambda_\alpha V_\alpha^a\,. \quad (6.2)$$

Then

$$D_{ab} V_\alpha^b = \frac{1}{p^2+M^2-\dfrac{G^2}{p^2+M^2}\lambda_\alpha} V_\alpha^a \tag{6.3}$$

has poles at

$$p^2+M^2=\pm G\sqrt{\lambda_\alpha}$$

or

$$-p^2=M^2\mp G\sqrt{\lambda_\alpha}\ . \tag{6.4}$$

These (mass)^2 shifts are $O(G)$ and thus larger than other (mass)^2 shifts which are of $O(G^2)$.

What we have just described is the completely standard mixing of degenerate levels. However, in a gauge theory there is another well-known example of "mixing" in which physical states from one system "mix" with unphysical states of the other system. The classic example of this is the Higgs mechanism in which a physical massless scalar mixes with a longitudinal (unphysical) photon to produce a massive zero helicity state which completes the $O(D-1)$ representation so the photon can (and does) become a massive vector particle.

In the context of string theory, Cremmer and Scherk[8] have demonstrated this mechanism in the closed-string mixing at the massless level. The closed-string antisymmetric tensor state devours the open-string photon to become a massive antisymmetric tensor field. In our language of mixing, a longitudinal antisymmetric tensor (unphysical) state mixes with the photon. The resulting (mass)^2 is $O(G^2)$, unlike the above example of mixing between physical states. Since string theory possesses a gauge invariance for massive levels as well as massless ones, this Higgs-Cremmer-Scherk mechanism could well be operative at higher levels.[20]

Later we shall explain how to describe the Cremmer-Scherk phenomenon using our Υ operator. The original Cremmer-Scherk discussion used $\check{\Upsilon}$, which differs from Υ in a significant respect: Υ contains no gradient couplings. With $\check{\Upsilon}$ there is a gradient coupling between the photon and the longitudinal $B_{\mu\nu}$ and the discussion proceeds exactly as with the Schwinger model. With Υ there is no such gradient coupling: the mixing involves world-sheet Feynman-Faddeev-Popov ghost states in an essential way.

Before we turn to the detailed discussion of the Cremmer-Scherk mechanism, we make some general remarks about closed-string–open-string mixing. Let us first notice where the degeneracies are. The closed-string (mass)^2 spectrum is given by

$$\alpha'(M^2)=2(N+\tilde{N}-2)=4N-4 \quad \text{for } N=0,1,2,\ldots \tag{6.5}$$

because $N=\tilde{N}$. The open-string (mass)^2 spectrum is given by

$$\alpha'(m^2)=n-1 \quad \text{for } n=0,1,2,\ldots\ . \tag{6.6}$$

Thus degeneracy occurs when

$$n=4N-3,\quad N=1,2,3,\ldots\ . \tag{6.7}$$

Thus all closed-string levels other than the tachyon can be involved, but only open-string levels with $n=1$ (mod4) are involved in the mixing phenomenon. Notice that they are all odd twist open-string states.

For the problem of mixing between physical states it is also of interest to know how many states of each system have a given set of quantum numbers. General methods for answering this question for an arbitrary mass level have been developed in a recent series of papers.[21,22] Here we shall content ourselves with a statement of the situation at the first massive bosonic closed-string level $N=\tilde{N}=2$ degenerate with the open-string level $n=5$. Referring to Ref. 21 we see that the $n=5$ level possesses the following irreducible representations (irreps) of O(25):

$$\text{open string } n=5:\ \yng(5)+\yng(3,1)+\yng(3)+\yng(2,1)+\yng(1,1)+\yng(1)\ . \tag{6.8}$$

Either a straightforward analysis of the $N=\tilde{N}=2$ closed-string level, or an application of the more systematic analysis of Ref. 22 gives the O(25) irreps:

$$\text{closed string } N=\tilde{N}=2:\ \yng(4)+\yng(3,1)+\yng(2,2)+\yng(2)+\yng(1,1)+\mathbb{1}\ , \tag{6.9}$$

where $\mathbb{1}$ means a scalar. In both (6.8) and (6.9) each irrep appears exactly once at this level.

Comparing (6.8) and (6.9), we see that Lorentz invariance will only allow the mixing of states with symmetry patterns

$$\yng(3,1) \quad \text{and} \quad \yng(1,1)\ . \tag{6.10}$$

Since each pattern is nondegenerate in each system, the mixing at this level does not require diagonalizing a matrix, the shifts are simply given by

$$(\Delta M)^2_{\yng(3,1)}=\pm|G\langle 0|\Upsilon|\yng(3,1)\rangle_c|\yng(3,1)\rangle_o| \tag{6.11}$$

and

$$(\Delta M)^2_{\yng(1,1)}=\pm|G\langle 0|\Upsilon|\yng(1,1)\rangle_c|\yng(1,1)\rangle_o|\ . \tag{6.12}$$

(Of course finding the physical states $|\yng(1,1)\rangle$ and $|\yng(3,1)\rangle$ involves some technical gymnastics.)

It is amusing to note that there are no open-string scalars at levels $n=5$ and 9, so the first opportunity for mixing of scalar states is at the $n=13$ level for which

$$\alpha' {M_0}^2=n-1=12\ . \tag{6.13}$$

Finally, we come to the promised analysis of the Cremmer-Scherk mechanism using our Υ operator in place of their $\check{\Upsilon}$. Let us first recall the usual analysis due to Schwinger of the Higgs mechanism. One first invokes gauge invariance to assert that the vacuum-polarization tensor $\Pi_{\mu\nu}(q)$ has the tensor structure

$$\Pi_{\mu\nu}(q)=(q^2 g_{\mu\nu}-q_\mu q_\nu)\Pi(q^2)\ , \tag{6.14}$$

where we have written (6.14) in a way to stress that the normal situation is for $\Pi(q^2)$ not to have a pole at $q^2=0$, so that (6.14) would imply that the self-mass of the photon is zero. A pole in $\Pi(q^2)$ at $q^2=0$ would give rise to a gauge-invariant photon (mass)^2:

$${m_\gamma}^2=\lim_{q^2\to 0}|q^2\Pi(q^2)|\ . \tag{6.15}$$

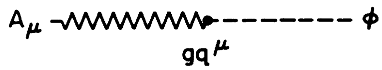

FIG. 3. Feynman diagram for the photon (A_μ)–Higgs-field (ϕ) vertex.

FIG. 4. Diagram contributing to the open-string self-energy.

In the Schwinger model (QED in a 2 space-time dimensions) $\Pi(q^2)$ has a pole for dimensional reasons. Of course in QED in higher dimensions $\Pi(q^2)$ does not have a massless pole. In spontaneously broken gauge theories there is a massless scalar (Goldstone boson) which produces a massless pole in $\Pi(q^2)$ via the coupling in Fig. 3 and the coupling g determines the mass of the gauge particle. The explicit (and difficult) calculation of the (mass)^2, i.e., the coefficient of $g_{\mu\nu}$, may be traded for a knowledge of how massless states couple to each other [the $q_\mu q_\nu$ term in (6.14)]: gauge invariance links the two pieces of information.

Coming back to the problem at hand, we begin by first finding the constraints BRST invariance places on self-energy matrices

$$\langle 0|\sigma\ , \tag{6.16}$$

$$\langle 0|\Sigma \tag{6.17}$$

for the open- and closed-string massless levels, respectively:

$$\langle 0|\sigma(Q_{\text{open}}+Q'_{\text{open}})=0\ , \tag{6.18}$$

$$\langle 0|\Sigma(Q_{\text{closed}}+Q'_{\text{closed}})=0\ , \tag{6.19}$$

where primed and unprimed operators refer to the two external legs of the self-energy. Equations (6.18) and (6.19) are imposed off shell and, as we shall see, require $\langle 0|\sigma$ and $\langle 0|\Sigma$ to have massless poles.

Let us first consider Eq. (6.18). For a photon, the self-energy $\langle 0|\sigma$ must contain a term

$$\langle 0|a_1\cdot a_1'$$

which is the analog of the $g_{\mu\nu}$ of Eq. (6.14). One simple solution to (6.18) is

$$\langle 0|\left[a_1\cdot a_1'-\frac{a_0\cdot a_1 a_0\cdot a_1'}{a_0^2}\right] \tag{6.20}$$

which is completely analogous to the ansatz (6.14). It will not do for our analysis since Υ has no gradient couplings, so it cannot produce $a_0\cdot a_1$ terms. Fortunately there is another solution to (6.18) given by

$$\langle 0|\left[a_1\cdot a_1'+c_1b_1'+c_1'b_1-\frac{4}{a_0^2}c_1c_1'b_0b_0'\right] \tag{6.21}$$

which is, in fact, the most general solution without gradient couplings, acting on the $g_0+g_0'=-1$, $n=n'=1$ states of the open string.

The photon self-energy $\langle 0|\sigma$, to order g^2, contains several contributions difficult to calculate, but only the massless closed-string intermediate states of Fig. 4 can contribute to the massless pole piece of (6.21). Thus we can determine the strength of the pole by analyzing the contribution of two Υ's between states with $N=\tilde{N}=n=1$, connected by a closed-string propagator. Because the transition amplitude is actually an integral over the interaction point σ_I on the closed string at which the joining or splitting takes place, the propagator contains a projector onto $\mathcal{L}_0=\tilde{\mathcal{L}}_0$. As we are working in the form where the interaction vertices are bras, we represent the propagator as a ket identifying two closed strings,

$$\Delta=\frac{\alpha'\delta_{\mathcal{L}_0,\tilde{\mathcal{L}}_0}}{2(\mathcal{L}_0+\tilde{\mathcal{L}}_0)}|\psi\rangle\ , \tag{6.22}$$

where $|\psi\rangle$ is a state describing the overlap of two closed strings, X and X':

$$\begin{aligned}&[X(\sigma)-X'(\pi-\sigma)]|\psi\rangle=0\ ,\\&[P(\sigma)+P'(\pi-\sigma)]|\psi\rangle=0\ ,\\&[C(\sigma)+C'(\pi-\sigma)]|\psi\rangle=0\ ,\\&[B(\sigma)-B'(\pi-\sigma)]|\psi\rangle=0\ .\end{aligned} \tag{6.23}$$

Thus $|\psi\rangle$ is given by

$$\exp\left[-\sum_n\frac{A_{-n}\cdot A'_{-n}}{n}-\sum_n\frac{\tilde{A}_{-n}\cdot\tilde{A}'_{-n}}{n}+\sum_n(C_{-n}B'_{-n}+\tilde{C}_{-n}\tilde{B}'_{-n}+C'_{-n}B_{-n}+\tilde{C}'_{-n}\tilde{B}'_{-n})\right]|\downarrow\downarrow\times\downarrow\downarrow\rangle\ , \tag{6.24}$$

where we are working in Siegel gauge, so all the B_0's annihilate $|\psi\rangle$. For the intermediate states corresponding to the tachyon and massless levels, the propagator gives

$$\begin{aligned}\Delta=\frac{1}{p^2-4/\alpha'}|\downarrow\downarrow\times\downarrow\downarrow\rangle+\frac{1}{p^2}&(-A_{-1}\cdot A'_{-1}+C_{-1}B'_{-1}+C'_{-1}B_{-1})\\&\times(-\tilde{A}_{-1}\cdot\tilde{A}'_{-1}+\tilde{C}_{-1}\tilde{B}'_{-1}+\tilde{C}'_{-1}\tilde{B}_{-1})|\downarrow\downarrow\times\downarrow\downarrow\rangle+\cdots\ .\end{aligned} \tag{6.25}$$

The expression for the contribution of $\langle 0 | \Upsilon$ on the latter state is simplified by noting that W_F must act twice, creating one unit of excitation in each of N, $\tilde{N}$, and n, and must give no closed string B_0's. Then the effective part is

$$\langle 0 | \Upsilon = \langle 0 | + \langle 0 | [\tilde{C}_1 a_1 \cdot A_1 - C_1 a_1 \cdot \tilde{A}_1 + \tilde{C}_1 C_1 b_1 + 2c_1 (C_1 \tilde{B}_1 - \tilde{C}_1 B_1)] b_0 + \cdots \; . \tag{6.26}$$

Thus we find

$$G^2 \langle 0 | \Upsilon \Upsilon' \Delta = \frac{G^2}{p^2 - \dfrac{4}{\alpha'}} \langle 0 | + \frac{G^2}{8p^2} \langle 0 | c_1 c_1' b_0 b_0' + \text{nonsingular terms} \; . \tag{6.27}$$

Comparing with (6.21), we see that the term $\langle 0 | a_1 \cdot a_1'$ must contribute to (6.27) as

$$\frac{G^2}{p^2 - \dfrac{4}{\alpha'}} \langle 0 | - \frac{G^2}{8p^2} \frac{a_0{}^2}{4} \langle 0 | \left[a_1 \cdot a_1' + c_1 b_1' + c_1' b_1 - \frac{4}{a_0{}^2} c_1 c_1' b_0 b_0' \right] = \frac{G^2}{p^2 - \dfrac{4}{\alpha'}} \langle 0 | - \frac{\alpha' G^2}{16} \langle 0 | a_1 \cdot a_1' + \text{other terms} \; . \tag{6.28}$$

To interpret this equation we note that (6.28) multiplied by two open-string propagators is a correction to the free open-string propagator:

$$\frac{\alpha'}{\mathcal{L}_0} + \frac{\alpha'^2 G^2}{\mathcal{L}_0{}^2} \left[\frac{|0\rangle\langle 0|}{p^2 - 4/\alpha'} - \frac{\alpha'}{16} a^{\mu}{}_{-1} | 0 \rangle \langle 0 | a_{1\mu} + \cdots \right] . \tag{6.29}$$

The corrected photon propagator reads

$$\frac{g_{\mu\nu}}{p^2} - \frac{\alpha' G^2}{16} \frac{g_{\mu\nu}}{(p^2)^2} \approx \frac{g_{\mu\nu}}{p^2 + \dfrac{\alpha' G^2}{16}} \; , \tag{6.30}$$

i.e., the photon gets a mass

$$m_\gamma{}^2 = + \frac{\alpha' G^2}{16} = \frac{G^2}{32\pi T_0} \; . \tag{6.31}$$

A simple check on the sign is to note that the closed tachyon contribution to the open tachyon propagator in (6.29) has the sign required by unitarity in field theory. The result agrees with Cremmer and Scherk after the factor of π mistake is corrected. Indeed, it is not hard to evaluate the contribution of $B_{\mu\nu}$ to Eq. (5.25) near $p^2 = 0$ by analytic continuation, with the result

$$F_{r,s}^{\text{tree}} + F_{r,s}^{\text{Feynman}} \underset{p^2 \to 0}{\sim} - \frac{1}{p^2} A_r^{\mu} g_{\mu\nu} A^{\nu} + \frac{\alpha' G^2}{16} \left[\frac{1}{p^2} \right]^2 A_r^{\mu} \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] A_s^{\nu} \; ,$$

where A_r^{λ} is the coupling of the open-string photon to r open-string tachyons.

As a consistency check on our calculation we should find that the closed-string antisymmetric tensor field gets the same mass (6.31) when we repeat the analysis for the closed-string self-energy. We would like to calculate the process depicted in Fig. 5 and compare it to the general solution of Eq. (6.19). We shall only find solutions of Eq. (6.19) without gradient couplings, since we know Υ provides no such couplings. By definition $\langle 0 | \Sigma$ must contain terms of the type

FIG. 5. Diagram contributing to the closed-string self-energy.

$$\langle 0 | (A_1 \cdot A_1')(\tilde{A}_1 \cdot \tilde{A}_1') \; , \tag{6.32}$$

$$\langle 0 | (A_1 \cdot \tilde{A}_1')(\tilde{A}_1 \cdot A_1') \; , \tag{6.33}$$

$$\langle 0 | (A_1 \cdot \tilde{A}_1)(A_1' \cdot \tilde{A}_1') \; , \tag{6.34}$$

with $\langle 0 | C_0 = \langle 0 | \tilde{C}_0 = \langle 0 | C_0' = \langle 0 | \tilde{C}_0' = 0$, where for simplicity we have restricted these choices to the sector with $N = \tilde{N} = N' = \tilde{N}' = 1$, since $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ for physical closed-string states. This restriction to $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ accounts for the integration over the breaking point σ_I over which Υ should be integrated. As we shall see, we shall also need to restrict to $B_0 = \tilde{B}_0$.

Each of the states (6.32)–(6.34) can be made BRST invariant via the substitutions

$$A_1 \cdot A_1' \to A_1 \cdot A_1' + C_1 B_1' + C_1' B_1 - \frac{4}{A_0{}^2} C_1 C_1' B_0 B_0' \; , \tag{6.35}$$

$$\tilde{A}_1\cdot\tilde{A}'_1 \to \tilde{A}_1\cdot\tilde{A}'_1+\tilde{C}_1\tilde{B}'_1+\tilde{C}'_1\tilde{B}_1-\frac{4}{A_0{}^2}\tilde{C}_1\tilde{C}'_1\tilde{B}_0\tilde{B}'_0\ , \tag{6.36}$$

$$A_1\cdot\tilde{A}'_1 \to A_1\cdot\tilde{A}'_1+C_1\tilde{B}'_1+\tilde{C}'_1B_1-\frac{4}{A_0{}^2}C_1\tilde{C}'_1B_0\tilde{B}'_0\ , \tag{6.37}$$

$$\tilde{A}_1\cdot A'_1 \to \tilde{A}_1\cdot A'_1+\tilde{C}_1B'_1+C'_1\tilde{B}_1-\frac{4}{A_0{}^2}\tilde{C}_1C'_1\tilde{B}_0B'_0\ , \tag{6.38}$$

$$A_1\cdot\tilde{A}_1 \to A_1\cdot\tilde{A}_1+C_1\tilde{B}_1+\tilde{C}_1B_1-\frac{4}{A_0{}^2}C_1\tilde{C}_1B_0\tilde{B}_0\ , \tag{6.39}$$

$$A'_1\cdot\tilde{A}'_1 \to A'_1\cdot\tilde{A}'_1+C'_1\tilde{B}'_1+\tilde{C}'_1B'_1-\frac{4}{A_0{}^2}C'_1\tilde{C}'_1B'_0\tilde{B}'_0\ , \tag{6.40}$$

and in fact a linear combination of (6.32)–(6.34) so modified is the most general solution of Eq. (6.19) without gradient couplings. Since Fig. 5 will never produce a singularity

$$\frac{1}{A_0{}^4} \tag{6.41}$$

the linear combination is immediately constrained by the requirement that the terms of type (6.41) cancel.

Next we must evaluate Fig. 5. The open-string propagator can be taken in the form

$$\frac{\alpha'}{\mathcal{L}_0}\exp\sum_n\left[(-)^n\left[-\frac{a_{-n}\cdot a'_{-n}}{n}+c_{-n}b'_{-n}+c'_{-n}b_{-n}\right]\right]|\downarrow\times\downarrow\rangle\ , \tag{6.42}$$

with $b_0|\downarrow\times\downarrow\rangle=b'_0|\downarrow\times\downarrow\rangle=0$.

It is fairly easy to see that the dependence on the B_0's in the singular terms of Eqs. (6.35)–(6.40) will not be exactly reproduced. For example, the term involving $A_1\cdot A'_1\tilde{C}_1\tilde{C}'_1/A_0{}^2$ comes out proportional to

$$\frac{1}{A_0{}^2}\langle 0|A_1\cdot A'_1\tilde{C}_1\tilde{C}'_1(B'_0+3\tilde{B}'_0)(B_0+3\tilde{B}_0)$$

rather than the corresponding term in the product of (6.35) with (6.36). Such discrepancies do disappear on states annihilated by $B_0-\tilde{B}_0$ and $B'_0-\tilde{B}'_0$. We must restrict our discussion of a BRST-invariant mass term to this sector as well as restricting $\mathcal{L}_0-\tilde{\mathcal{L}}_0$ and $\mathcal{L}'_0-\tilde{\mathcal{L}}'_0$ to be zero. This is because

$$\begin{aligned}\{B_0-\tilde{B}_0,Q\}&=\mathcal{L}_0-\tilde{\mathcal{L}}_0\ ,\\ \{B'_0-\tilde{B}'_0,Q\}&=\mathcal{L}'_0-\tilde{\mathcal{L}}'_0\ .\end{aligned} \tag{6.43}$$

In the following we shall work in this restricted state space.

On the restricted subspace, each $B_0,\tilde{B}_0$ in Eqs. (6.35)–(6.40) may be replaced by

$$\tilde{B}_0,B_0\to\frac{B_0+\tilde{B}_0}{2},\quad \tilde{B}'_0,B'_0\to\frac{B'_0+\tilde{B}'_0}{2}\ .$$

Making use of the same restriction, the evaluation of Fig. 5 reduces to

$$\begin{aligned}&\frac{G^2}{p^2-\frac{1}{\alpha'}}\langle 0|+\frac{G^2}{16p^2}\langle 0|[(\tilde{C}_1A_1-C_1\tilde{A}_1)\cdot(\tilde{C}'_1A'_1-C'_1\tilde{A}'_1)\\ &\qquad -2(C_1\tilde{B}_1-\tilde{C}_1B_1)\tilde{C}'_1C'_1+2\tilde{C}_1C_1(C'_1\tilde{B}'_1-\tilde{C}'_1B'_1)](B_0+\tilde{B}_0)(B'_0+\tilde{B}'_0)\\ &=\frac{G^2}{p^2-\frac{1}{\alpha'}}\langle 0|+\frac{G^2}{16p^2}\langle 0|[\tilde{C}_1\tilde{C}'_1(A_1\cdot A'_1+C'_1B_1+C_1B'_1)+C_1C'_1(\tilde{A}_1\cdot\tilde{A}'_1+\tilde{C}_1\tilde{B}'_1+\tilde{C}'_1\tilde{B}_1)\\ &\qquad -C_1\tilde{C}'_1(\tilde{A}_1\cdot A'_1+\tilde{C}_1B'_1+C'_1\tilde{B}_1)-\tilde{C}_1C'_1(\tilde{A}'_1\cdot A_1+\tilde{C}'_1B_1+C_1\tilde{B}'_1)]\\ &\qquad \times(B_0+\tilde{B}_0)(B'_0+\tilde{B}'_0)\ .\end{aligned}$$

Comparing this to the modified forms of (6.35)–(6.40) allows us to infer that the closed-string mass terms couple as

$$\frac{G^2}{p^2-\frac{1}{\alpha'}}\langle 0|-\frac{\alpha'G^2}{32}\langle 0|(A_1\cdot A'_1\tilde{A}_1\cdot\tilde{A}'_1-A_1\cdot\tilde{A}'_1\tilde{A}_1\cdot A'_1)+\cdots \tag{6.44}$$

which implies a mass for $B_{\mu\nu}$ of

$$m_B{}^2=\frac{\alpha'G^2}{16}$$

in agreement with (6.31).

The difficulty we encountered with $B_0\neq\tilde{B}_0$ is probably inevitible. It suggests that the gauge-fixed theory involving closed strings should be explicitly restricted to the subspace with $\mathcal{L}_0=\tilde{\mathcal{L}}_0$ and $B_0=\tilde{B}_0$, a conclusion reached in several[1,23] other contexts. One way to implement this restriction in our formalism would be to multiply $\langle 0 | \Upsilon$ on the right by a projector onto $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ states and a factor $2^{-1/2}(B_0 - \tilde{B}_0)$. This form of the transition amplitude is

$$\langle 0 | \Upsilon P_{\mathcal{L}_0 = \tilde{\mathcal{L}}_0} \frac{B_0 - \tilde{B}_0}{\sqrt{2}} . \tag{6.45}$$

Because of the $B_0 - \tilde{B}_0$ insertion, (6.45) would couple open-string states of ghost number $-\frac{1}{2}$ to closed-string states of ghost number 0. Clearly (6.45) is BRST invariant. In all interaction terms in closed-string theory on the world sheet, there is a choice of interpreting the required integration over σ, the interaction point, as part of the propagator or as part of the interaction. Equation (6.45) corresponds to including this integral in the interaction. Closed-string propagators then include a factor of $B_0 + \tilde{B}_0$ only, rather than $B_0 \tilde{B}_0$. This is the factor one would expect from functional integrals to go along with a $(\mathcal{L}_0 + \tilde{\mathcal{L}}_0)^{-1}$, which corresponds to integration only over the length of a tube, and hence a single Teichmuller parameter with zero mode $B_0 + \tilde{B}_0$. If a factor of $B_0 - \tilde{B}_0$ appears on the closed-string side of all vertices, however, the closed-string propagator must also include a factor of $\frac{1}{2}(C_0 - \tilde{C}_0)$ as well, in agreement with Ref. 23.

VII. CONCLUSIONS AND OUTLOOK

In this article, we have endeavored to develop the insight from dual models that a theory of interacting open strings should provide a consistent description of closed strings. To this end we have constructed a BRST-invariant operator Υ whose matrix elements are the amplitudes for transitions from a closed to an open string or vice versa.

The coordinate part of our Υ differs in detail from $\check{\Upsilon}$, the transition operator obtained in the early 1970s by factorizing the nonplanar one-loop dual amplitudes. One can understand the essential differences between the two by noting that they are expressed in different conformal frames. Υ may be evaluated as a functional integral over a cylinder cut from $\tau = 0$ to $\tau = \infty$. On the other hand, the domain associated with $\check{\Upsilon}$ is an annulus with one circle being the open-string boundary.

As a test of our formalism, and indeed of the old light-cone vertex, we have shown that our Υ leads to the correct scattering amplitudes involving a single closed string and any number of open strings, at least if the open-string interactions are described by BRST-invariant vertex operator insertions at the boundary of the open string. In addition, we have given a new discussion of the Cremmer-Scherk mechanism wherein the closed-string antisymmetric tensor state combines with the open-string photon to produce a massive tensor state. This discussion uses BRST invariance in a novel way, which might also be useful in gauge field theory.

An underlying motivation for this work was to provide a solid foundation for thinking about closed strings in the field theory of open strings. In this context one would like to evaluate in string field theory the nonplanar one-loop correction to the one open-string irreducible two-point function. This diagram should have poles at the closed-string masses whose residues would factor into two closed-string–open-string transition amplitudes similar to our $\langle 0 | \Upsilon | \text{open} \rangle | \text{closed} \rangle$. In view of the close parallel between our Υ and that of light-cone gauge, we would expect our Υ to be appropriate to the string field theories developed in Refs. 14 and 24. The evaluation of the nonplanar loop for Witten's string field theory is in progress. It gives yet a different $\hat{\Upsilon}$ for the transition amplitude, corresponding to the difference in conformal frames. For the on-shell closed-string tachyon T, it is found that

$$\langle 0 | \hat{\Upsilon} | T,p \rangle = \langle \uparrow | \exp \left[-\frac{1}{2} \sum_{n=1}^{\infty} (-)^n \frac{a_n{}^2}{n} + \sqrt{2} p \sum_{l=1}^{\infty} (-)^l \frac{a_{2l}}{l} - \sum_{k=1}^{\infty} (-)^k c_k b_k \right] , \tag{7.1}$$

and further this is BRST invariant:

$$\langle 0 | \hat{\Upsilon} | T,p \rangle Q = 0 . \tag{7.2}$$

The close similarity of (7.1) to the semiclassical results of Ref. 25 must be interpreted carefully. $\langle 0 | \hat{\Upsilon} | T,p \rangle$ is irreducible with respect to open strings and is therefore not a matrix element of the quantized open-string field between the second-quantized vacuum and a single closed-string state. In particular, it is a bra of ghost number $+\frac{1}{2}$; since it is designed to give nonzero brackets with physical open-string kets of ghost number $-\frac{1}{2}$. Rather, the semiclassical field of Ref. 25 should be related to

$$\Psi = G^{-1} \langle\langle \text{vac} \| \Phi \| 1 \text{ closed string} \rangle\rangle \approx \langle 0 | \hat{\Upsilon} | T,p \rangle \frac{1}{\mathcal{L}_0} | \psi \rangle , \tag{7.3}$$

where the states in doubled brackets are in the second-quantized state space, and $\mathcal{L}_0{}^{-1} | \psi \rangle$ is the open-string propagator [see Eq. (6.42)]. Equation (7.2) is the condition for BRST invariance and should not be confused with a field equation. If we apply Q to (7.3) and make use of (7.2), we obtain the semiclassical field equation

$$Q\Psi = \langle 0 | \hat{\Upsilon} | T,p \rangle c_0' | \psi \rangle , \tag{7.4}$$

from which we see that $\langle 0 | \hat{\Upsilon} | T,p \rangle$ is properly interpreted as the *source* of the semiclassical string field, not the field itself.

Work remaining to be done includes completing the factorization of the irreducible two-point function in Witten's string field theory, which would lead to $\hat{\Upsilon}$ for arbitrary closed-string states. We stress that this calculation yields, in principle, all the absolute coupling strengths of each of these states to arbitrary open strings.

The theory of closed strings residing in Witten's string field theory can be developed further by calculating the irreducible higher point functions. For example, the nonplanar irreducible three-point function at order g^5 contains the three-closed-string vertex function. The knowledge of all irreducible N point functions determines the effective action. Of course the ultimate goal of these studies is to abstract from the open-string field theory a theory of closed strings alone. The effective action might provide a useful tool to aid in this venture. The main reason for believing this goal should be achievable is that it has already been achieved in the light-cone gauge.[7] The main obstacle in Witten's version of string field theory is that, unlike in light-cone gauge, there are not separate fields for open and closed strings. One is therefore confronted with the problem of disentangling in an off-shell context the closed- and open-string dynamics. That this is possible in principle is clear from the fact that one can vary the Chan-Paton factors which couple only to open strings. The extent to which this disentangling is practically feasible remains an open problem.

Note added in proof. The vanishing of the cubic terms in Eq. (3.29) is, in fact, a direct consequence of the mutual consistency of the Ward identities, Eq. (3.22). This can be shown by using the Ward identities and the Virasoro algebra to reduce

$$0=\langle 0\,|\,\Upsilon([L^k_{-m},L^l_{-n}]-(n-m)\delta^{kl}L^k_{-m-n})$$

to a linear combination of positive-model L's. This leads to the vanishing of the RHS of Eq. (B1), so the direct calculation of Appendix B is not really necessary. This general analysis shows that using the ansatz Eq. (3.27) with the M's corresponding to $\check{\Upsilon}$ will make $\check{\Upsilon}$ BRST invariant, since the vanishing of the linear terms for $D=26$ can be checked directly without difficulty. The appropriate M's can be obtained from Ref. 27, after correcting a sign mistake $[p!\to(-)^p p!]$ in the last equation of Sec. III.

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APPENDIX A: SUMS OF BINOMIAL COEFFICIENTS

We give here some identities involving sums of products of binomial coefficients which are needed at several places in the text. First

$$\sum_{k=0}^{r}\begin{bmatrix}a\\k\end{bmatrix}\begin{bmatrix}b\\r-k\end{bmatrix}=\begin{bmatrix}a+b\\r\end{bmatrix}. \tag{A1}$$

This is a well-known identity which is an immediate consequence of $(1+x)^a(1+x)^b=(1+x)^{a+b}$. In the special case, when $a=-b$, the sum vanishes unless $r=0$, which is used in proving Eq. (2.36).

A more difficult sum is

$$C_{nr}:=\sum_{k=0}^{r}\begin{bmatrix}-\frac{1}{2}\\k\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\n-k\end{bmatrix}=\begin{cases}\dfrac{1}{2n}\begin{bmatrix}-\frac{1}{2}\\n-r-1\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\r\end{bmatrix} & \text{for } n\neq 0\,,\\ 1 \quad \text{for } n=0 \quad \text{all } r\geq 0\,.\end{cases} \tag{A2}$$

The proof for $n\neq 0$ follows from showing that

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}C_{n,r}x^n y^r=\frac{\partial}{\partial x}\left[\frac{(1+x)^{1/2}(1+xy)^{-1/2}}{1-y}\right]$$
$$=\tfrac{1}{2}(1+x)^{-1/2}(1+xy)^{-3/2}$$

and expanding. It is also useful to transform the right-hand side by using

$$\begin{bmatrix}-\frac{3}{2}\\s-1\end{bmatrix}=-2s\begin{bmatrix}-\frac{1}{2}\\s\end{bmatrix},\quad \begin{bmatrix}-\frac{3}{2}\\s\end{bmatrix}=(2s+1)\begin{bmatrix}-\frac{1}{2}\\s\end{bmatrix}. \tag{A3}$$

We also need the similar identity

$$D_{n,r}:=\sum_{k=0}^{r}\begin{bmatrix}\frac{1}{2}\\k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\k+n\end{bmatrix}$$
$$=\frac{1}{2n+1}\begin{bmatrix}-\frac{1}{2}\\r\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\n+r\end{bmatrix}. \tag{A4}$$

This is proven by noting that

$$\sum_{n=-\infty}^{\infty}\sum_{r=0}^{\infty}D_{n,r}x^{n+1/2}y^r=\left[\frac{y+x}{1+x}\right]^{1/2}\frac{1}{1-y}\,,$$

which can be differentiated with respect to x and expanded to give (A4). Our proof applies for all integer n; for negative n it is helpful to rewrite it using $l=-n>0$, $s=r+n$, $p=k-l$, so that

$$\sum_{p=0}^{s}\begin{bmatrix}-\frac{1}{2}\\p\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\p+l\end{bmatrix}=-\frac{1}{2l-1}\begin{bmatrix}-\frac{1}{2}\\l+s\end{bmatrix}\begin{bmatrix}-\frac{3}{2}\\s\end{bmatrix}. \tag{A5}$$

APPENDIX B: DETAILS OF THE PROOF OF BRST INVARIANCE

In order to complete the proof of BRST invariance, we must show that the E^{abc}_{klm} of (3.29) are zero. The first step is to express them in terms of the M's by

$$2E^{abc}_{klm} = -d_{dbc}(m+2l)M^{ad}_{k,m+l} - d_{abd}(k-l)M^{dc}_{k+l,m} + d_{dac}(m+2k)M^{bd}_{l,m+k} + \sum_{r=0}^{l-1} d_{dbe}(r+l)M^{ad}_{kr}M^{ec}_{l-r,m} - \sum_{r=0}^{k-1} d_{dae}(r+k)M^{bd}_{lr}M^{ec}_{k-r,m} + \sum_{r=0}^{m} d_{dec}(m-2r)M^{ae}_{kr}M^{bd}_{l,m-r} \ . \tag{B1}$$

Next, we define a generating function

$$\hat{E}^{abc}(x,y,z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=0}^{\infty} x^k y^l z^m E^{abc}_{klm} \ . \tag{B2}$$

We then see we can express $\hat{E}$ in terms of the f^{ab}, the generating functions for the M^{ab}, which are given in Table II. We find

$$\begin{aligned}2\hat{E}^{def}(x,y,z) = & -d_{gef}\left[z\frac{\partial}{\partial z} + 2y\frac{\partial}{\partial y}\right]\frac{y}{z-y}[f^{dg}(x,z) - f^{dg}(x,y)] \\ & + d_{gdf}\left[z\frac{\partial}{\partial z} + 2x\frac{\partial}{\partial x}\right]\frac{x}{z-x}[f^{eg}(y,z) - f^{eg}(y,x)] \\ & - d_{deg}\left[x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right]\frac{1}{x-y}[yf^{gf}(x,z) - xf^{gf}(y,z)] \\ & + d_{aeg}\left[2f^{gf}(y,z)y\frac{\partial}{\partial y}f^{da}(x,y) + f^{da}(x,y)y\frac{\partial}{\partial y}f^{gf}(y,z)\right] \\ & - d_{adg}\left[2f^{gf}(x,z)x\frac{\partial}{\partial x}f^{ea}(y,x) + f^{ea}(y,x)x\frac{\partial}{\partial x}f^{gf}(x,z)\right] \\ & + d_{gaf}\left[f^{da}(x,z)z\frac{\partial}{\partial z}f^{eg}(y,z) - f^{eg}(y,z)z\frac{\partial}{\partial z}f^{da}(x,z)\right] \ . \end{aligned} \tag{B3}$$

The evaluation of $\hat{E}$ is straightforward but extremely tedious. We checked the E^{000} pieces by hand, but used MACSYMA to verify that the other components of $\hat{E}$ do indeed vanish, proving that Υ is invariant under the BRST charge [Eq. (3.28)].

APPENDIX C: CALCULATION OF THE OVERLAP WITH NEUMANN FUNCTIONS

The solution to the classical equations (2.22) may be expressed in terms of the Neumann functions for the region $\mathcal{R}$ in Fig. 1(a). These Neumann functions are defined to satisfy[10]

$$\left[\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right]N(\rho,\rho') = \left[\frac{\partial^2}{\partial\tau'^2} + \frac{\partial^2}{\partial\sigma'^2}\right]N(\rho,\rho') = 2\pi\delta^2(\rho-\rho') \ , \tag{C1}$$

$$N(\rho,\rho'), \frac{\partial N}{\partial\sigma}, \frac{\partial N}{\partial\sigma'} \ \text{periodic in } \sigma(\sigma') \ \text{for } \tau<0(\tau'<0) \ , \tag{C2}$$

$$\frac{\partial N}{\partial\sigma} = 0 \ \left[\frac{\partial N}{\partial\sigma'} = 0\right] \ \text{for } \sigma=0,\pi \text{ and } \tau>0 \ (\sigma'=0,\pi \text{ and } \tau'>0) \ , \tag{C3}$$

$$\frac{\partial N}{\partial\tau}(\rho,\rho') \ \left[\frac{\partial N}{\partial\tau'}(\rho,\rho')\right] \ \text{independent of } \sigma,\rho'(\sigma',\rho) \ \text{ at } \tau=-T_c,T_o(\tau'=-T_c,T_o) \ . \tag{C4}$$

Conditions (C4) allow flux to escape through the boundaries at the ends of $\mathcal{R}$.

Next define the periodic piece ϕ_P of ϕ by

$$\phi = \phi_P + 2nR\left[\sigma - \frac{\pi}{2}\right] \tag{C5}$$

so that ϕ_P is periodic for $\tau<0$ and satisfies $\phi_P' = -2nR$ for $\sigma=0,\pi,\tau>0$. Then ϕ_P may be expressed in terms of N, with the help of Green's theorem:

$$\phi_P(\rho') = -i\beta N(0,\rho') + \frac{nR}{\pi}\int_0^T d\tau [N(\tau,\sigma;\rho')]_{\sigma=0}^{\pi}$$
$$-i\int_0^\pi d\sigma\, p_2(\pi-\sigma)N(T_o,\sigma;\rho') - i\int_0^\pi d\sigma\, p_1(\sigma)N(-T_c,\sigma;\rho') + \frac{1}{2\pi}\int d\sigma\, \phi_P(\sigma,\tau)\frac{\partial N}{\partial \tau}(\sigma,\tau;\rho')\Big|_{-T_c}^{T_o} . \quad \text{(C6)}$$

Inserting (C5) into (2.24) and using (C6), one can show that

$$\ln\Omega(p_o,p_c) = \frac{\beta^2}{2}N(0,0) - n^2R^2T_c - \frac{n^2R^2}{2\pi}\int_{-T_c}^{0} d\tau' \int_0^{T_o} d\tau \frac{\partial}{\partial\sigma'}[N(\rho,\rho')\,|_{\sigma=0}^{\pi}]\Big|_{\sigma'=0}$$
$$+\beta\int_0^\pi d\sigma\, p_2(\pi-\sigma)N(\rho_o;0) + \beta\int_0^\pi d\sigma\, p_1(\sigma)N(\rho_c;0)$$
$$+\tfrac{1}{2}\int_0^\pi d\sigma\, d\sigma'\, p_1(\sigma)p_1(\sigma')N(\rho_c,\rho_c') + \tfrac{1}{2}\int_0^\pi d\sigma\, d\sigma'\, p_2(\pi-\sigma)p_2(\pi-\sigma')N(\rho_o,\rho_o')$$
$$+\int_0^\pi d\sigma\, d\sigma'\, p_1(\sigma)p_2(\pi-\sigma')N(\rho_c;\rho_o') + \frac{inR}{\pi}\int_0^\pi d\sigma'\, p_1(\sigma')\int_0^{T_o} d\tau\, N(\rho;\rho_c')\Big|_{\sigma=0}^{\pi}$$
$$+inR\int_0^\pi d\sigma'\, p_2(\pi-\sigma')\int_{-T_c}^{0} d\tau \frac{\partial}{\partial\sigma}N(\rho;\rho_o')\Big|_{\sigma=0} . \quad \text{(C7)}$$

The result (C7) involved integrating some of the τ integrals by parts. When $\tau'>0$, we performed the maneuver

$$\int_0^{T_o} d\tau\, N(\rho,\rho')\Big|_0^\pi = \int_{-T_c}^{T_o}\left[\sigma-\frac{\pi}{2}\right]\frac{\partial N}{\partial\sigma}\Big|_0^\pi - \int_{-T_c}^{T_o} d\tau \int_0^\pi d\sigma \left[\sigma-\frac{\pi}{2}\right]\frac{\partial^2 N}{\partial\sigma^2}$$
$$=\pi\int_{-T_c}^{0}\frac{\partial N}{\partial\sigma}\Big|_{\sigma=0} - 2\pi\left[\sigma'-\frac{\pi}{2}\right] + \int_0^\pi d\sigma\left[\sigma-\frac{\pi}{2}\right]\frac{\partial N}{\partial\tau}\Big|_{-T_c}^{T_o}$$
$$=\pi\int_{-T_c}^{0}\frac{\partial N}{\partial\sigma}\Big|_{\sigma=0} - 2\pi\left[\sigma'-\frac{\pi}{2}\right] , \quad \text{(C8)}$$

where we used (C1)–(C4). Similarly for $\tau<0$ we rewrite

$$\int_{-T_c}^{0}\frac{\partial}{\partial\sigma'}N(\rho,\rho')\Big|_{\sigma'=0} = \frac{1}{\pi}\int_0^{T_o} N(\rho,\rho')\Big|_{\sigma'=0}^{\pi} + 2\left[\sigma-\frac{\pi}{2}\right] . \quad \text{(C9)}$$

These maneuvers serve to make the $T_i\to\infty$ of (C7) well behaved. In view of the discussion at the beginning of Sec. II, we only need the leading behavior of each coefficient as $T_i\to\infty$, which means we can replace the limits on the τ integration by $\pm\infty$.

The Neumann function for the region $\mathcal{R}_\infty$ obtained from $\mathcal{R}$ by taking $T_i\to\infty$ can be immediately written by conformally mapping to the upper half-plane:

$$N_\infty(\rho,\rho') = \ln|(e^{2\rho}-1)^{1/2}-(e^{2\rho'}-1)^{1/2}| + \ln|(e^{2\rho}-1)^{1/2}-[(e^{2\rho'}-1)^{1/2}]^*| .$$

We can replace N by N_∞ when both arguments are far away from T_o, $-T_c$, as $T_i\to\infty$. This can clearly be done in the third term of (C7):

$$-\frac{n^2R^2}{2\pi}\int_{-T_c}^{0} d\tau' \int_0^{T_o} d\tau\frac{\partial}{\partial\sigma'}[N(\rho,\rho')\,|_{\sigma=0}^{\pi}]\Big|_{\sigma'=0}$$
$$\approx -\frac{n^2R^2}{2\pi}\int_{-\infty}^{0} d\tau'\int_0^\infty d\tau\, 2\frac{\partial}{\partial\sigma'}\left[\mathrm{Re}\ln\frac{(e^{2\tau}-1)^{1/2}+(e^{2\rho'}-1)^{1/2}}{(e^{2\tau}-1)^{1/2}-(e^{2\rho'}-1)^{1/2}}\right]\Big|_{\sigma'=0}$$
$$\approx \frac{n^2R^2 i}{\pi}\int_0^\infty d\tau \ln\frac{(e^{2\rho}-1)^{1/2}+i}{(e^{2\rho}-1)^{1/2}-i} = -\frac{n^2R^2}{\pi}\int_0^\infty dx\frac{\ln(1+x^2)}{1+x^2} = -n^2R^2\ln 2 ,$$

where we used the Cauchy-Riemann equation to do the τ' integration.

The (divergent term) $(\beta^2/2)N(0,0)$ may be absorbed into a redefinition of the coupling constant. The remaining terms in (C7) have one or both of the arguments of N at T_o or $-T_c$. For these one cannot simply replace N by N_∞. However, since only the leading term in the coefficient of each mode p_n^i need be retained, one can simply relate the contribution of a given mode to the corresponding contribution with N replaced by N_∞. This procedure has been carried through in Ref. 7 for the terms in (C7) bilinear in the p_i, with results in agreement with Eq. (2.50). We shall therefore restrict attention to the terms linear in n and β.

Consider first the terms linear in β. We expand

$$N_\infty(\rho,0)=2\ln|e^{2\rho}-1|=\begin{cases}2\tau-\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}e^{-2n\tau}\cos 2n(\pi-\sigma), & \tau>0\ ,\\ -\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}e^{+2n\tau}\cos 2n(\sigma), & \tau<0\ .\end{cases}$$

On the other hand, $N(\rho,0)$ will have the expansion

$$N(\rho,0)=\begin{cases}2\tau-\displaystyle\sum_{n=1}^{\infty}(e^{-2n\tau}+e^{-2n(2T_o-\tau)})\cos 2n(\pi-\sigma)N_n^+(T_i), & \tau>0\ ,\\ -\displaystyle\sum_{n=1}^{\infty}(e^{+2n\tau}+e^{-2n(2T_c+\tau)})\cos 2n\sigma N_n^-(T_i), & \tau<0\ ,\end{cases}$$

where for fixed τ,σ agreement with N_∞ for $T_i\to+\infty$ implies

$$N_n^\pm(T_i)\underset{T_i\to\infty}{\sim}\frac{1}{n}\ .$$

Evaluating $N(\rho,0)$ at $\tau=T_o,-T_c$ gives

$$N(\rho,0)=\begin{cases}2T_o-2\displaystyle\sum_{n=1}^{\infty}N_n^+(T_i)e^{-2nT_o}\cos 2n(\pi-\sigma), & \tau=T_o\ ,\\ -2\displaystyle\sum_{n=1}^{\infty}N_n^-(T_i)e^{-2nT_c}\cos 2n\sigma, & \tau=-T_c\ .\end{cases}$$

Since we only need the leading term in each mode we can replace $N_n^\pm$ by $1/n$. Using this and setting $\beta=3/2\sqrt{2}$ yields for the linear terms in β:

$$+\frac{3T_o}{\sqrt{2}}p^o-\frac{3}{2\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n}(e^{-2nT_o}p_{2n}^o+e^{-2nT_c}p_n^c)\ .$$

The first term combines with linear terms in T_o,T_c coming from the bilinear terms in p as

$$\begin{aligned}-(p^o)^2T_o-(p^c)^2T_c+2T_op^o(p^o+p^c)+2\beta T_op^o&=-(p^o)^2T_o-(p^c)^2T_c+2T_op^o(p^o+p^c+\beta)\\&=-[(p^o)^2T_o+(p^c)^2T_c]\end{aligned}$$

by momentum conservation, and these together with the term $-T_cn^2R^2$ contribute the proper time dependence due to the zero-mode pieces of the Hamiltonians.

Finally, we consider the terms linear in winding number, first computing for $\tau'<0$,

$$\begin{aligned}\int_0^\infty d\tau N_\infty(\rho,\rho')\Big|_{\sigma=0}^{\pi}&=2\int_0^\infty d\tau\ln\left|\frac{(e^{2\rho'}-1)^{1/2}+(e^{2\tau}-1)^{1/2}}{(e^{2\rho'}-1)^{1/2}-(e^{2\tau}-1)^{1/2}}\right|\\&=2\pi\,\mathrm{Re}\{i\ln[1+(1-e^{2\rho'})^{1/2}]\}\\&=2\pi\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\begin{pmatrix}-\frac{1}{2}\\ n\end{pmatrix}e^{2n\tau'}\sin 2n\sigma'\end{aligned}$$

and, for $\tau'>0$,

$$\begin{aligned}\int_{-\infty}^{0}d\tau\frac{\partial}{\partial\sigma}N_\infty(\rho,\rho')\Big|_{\sigma=0}&=\mathrm{Re}\int_{-\infty}^{0}d\tau\frac{\partial}{\partial\sigma}\{\ln[i(1-e^{2\rho})^{1/2}-(e^{2\rho'}-1)^{1/2}]+\ln[i(1-e^{2\rho})^{1/2}-(e^{2\rho'^*}-1)^{1/2}]\}\\&=-\mathrm{Re}\,i\ln[(e^{2\rho'}-1)^{1/2}-i][(e^{2\rho'^*}-1)^{1/2}-i]\\&=-2\sum_{k=0}^{\infty}\frac{(-)^k}{2k+1}\begin{pmatrix}-\frac{1}{2}\\ k\end{pmatrix}e^{-(2k+1)\tau'}\cos(2k+1)\sigma'\ .\end{aligned}$$

By exactly the same argument given for the terms linear in β the effect of replacing N_∞ by N and evaluating these expansions at $\tau' = -T_c, T_o$, respectively, is to multiply the RHS of each expression by 2. Inserting this information in (C7) we find agreement with the corresponding terms in Eq. (2.50).

APPENDIX D: THE FEYNMAN TREE THEOREM

Perturbative unitarity fixes, in principle, the weight of multipoint multiloop amplitudes in terms of trees with a smaller number of points and loops. A very useful form of these constraints is the Feynman tree theorem.[26] Let us first note the correct relative weights for N tachyon dual resonance amplitudes in the tree approximation. This is determined by simple factorization to be

$$\mathcal{M}^{\text{tree}} = \sum_P A(k_{p1}, k_{p2}, \ldots, k_{pN}) ,$$

where the sum is over all cyclically symmetric dual amplitudes

$$A(k_1, \ldots, k_N) = \langle 0, k_N \,|\, gV(k_{N-1}) \frac{\alpha'}{L_0 - 1} \times gV(k_{N-2}) \cdots gV(k_1) \,|\, 0, k_1 \rangle .$$

It is important that anticyclic permutations are included in the sum. Factorization does not, of course, determine the value of g. For our purposes in Secs. V and VI, we need the precise normalization of the one-loop corrections to $\mathcal{M}^{\text{tree}}$. Feynman's tree theorem determines the normalization of the integrand of the loop momentum integral. Define

$$\mathcal{M}^{1\text{ loop}} = \int \frac{d^D p}{(2\pi)^D} \mathcal{I}(p) .$$

Consider the tree amplitude obtained from $\mathcal{I}$ by cutting one line of the diagram in all ways that leave the diagram connected. Each cut gives rise to two contributions corresponding to the choice of which line is taken incoming and which outgoing (depending on the sign of the energy). The tree theorem states that the sum of all these contributions gives the complete tree amplitude involving $N+2$ particles, the two extra particles having momenta p and $-p$. One can easily check in examples that this theorem accounts for the rule that the contribution to the amplitude from a diagram with symmetry must be divided by the symmetry number of the diagram. For example, a tadpole in ϕ^3 field theory should be divided by 2. This follows from the tree theorem because cutting the one internal line gives two identical three-point functions.

Now consider the application of the tree theorem to one-loop open-string dual amplitudes. These may be drawn as an annulus with external tachyon legs attached to the inner or outer circles. The relevant cuts are those that convert the annulus to a strip. A given dual one-loop diagram is specified by splitting the N particles into groups of r and $N-r$ particles, and specifying a cyclic ordering for each group. For definiteness attach the group containing particle 1 to the inner boundary.

The simplest way to evaluate a given dual diagram, however, is to break up the integration region into different cells corresponding to the relative time ordering of particles emitted from the inner and outer boundaries. Each such cell contributes to $\mathcal{I}$ in the form

$$\operatorname{Tr}\left\{ gV\frac{\alpha'(\pm)^R}{L_0-1} gV\frac{\alpha'(\pm)^R}{L_0-1} \cdots gV\frac{\alpha'(\pm)^R}{L_0-1} \right\} , \qquad \text{(D1)}$$

where the choice $(-)^R$ is taken whenever the propagator separates two vertices on opposite boundaries. With the contribution of each cell defined as in (D1), the tree theorem determines the weight of a given diagram to be 1. That is, for each break up r, $N-r$, one sums over cyclically inequivalent permutations of each group with no extra combinatoric factors. Again, just as with trees it is essential that anticyclic permutations are included in these sums.

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