Complete bosonization of two-dimensional QCD in the path-integral framework

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We complete the bosonization of two-dimensional QCD within the path-integral approach. We compute the contribution of the gauge field sector, which was considered as a background in previous works. This enables us to obtain an effective Lagrangian completely written in terms of bosons. The current algebra which arises in this bosonic context is also discussed.

Since Witten[1] showed how to extend two-dimensional bosonization techniques to the case of non-Abelian internal symmetries there has been much interest in the application of this procedure to the solution of various two-dimensional models. This, together with the exact evaluation of the fermion determinant[2,3] allowed the development of a path-integral approach to bosonization[4–9] which showed to be very fruitful in the study of interacting fermion models such as two-dimensional QCD (QCD_2), the chiral Gross-Neveu model, etc.

In the approach of Ref. 2, fermions are decoupled by means of a chiral change in the fermionic path-integral variables. Associated with this transformation one has a Jacobian whose computation is directly connected with the evaluation of the fermion determinant and originates a topological Wess-Zumino term contributing to the effective action. Following this method the fermion determinant for QCD_2 was exactly computed[2] and physical properties of the model were discussed in terms of the resulting effective Lagrangian.[5]

In this paper we complete the path-integral bosonization of QCD_2 by properly taking into account the role that the gauge field sector plays in the final effective Lagrangian. Indeed, in previous studies the gauge field was considered as a background field and therefore the physical picture of the model has remained so far partially understood.

We start by briefly sketching how the fermion determinant can be computed using the technique developed in Ref. 2 (see also Ref. 10). The functional integral for QCD_2 with massless fermions (in Euclidean space) reads

$$Z=\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp\left[-\int d^2x(\bar{\psi}\not{D}\psi+\tfrac{1}{4}\,\mathrm{tr}F_{\mu\nu}{}^2)\right], \tag{1}$$

where $\not{D}=\gamma^\mu D_\mu=i\not{\partial}+g\not{A}$, and A_μ takes values in the Lie algebra of $SU(N_C)$. The massless fermions are taken in the fundamental representation of $SU(N_C)$ with the generators t_a normalized so that $\mathrm{tr}t_a t_b=\frac{1}{2}\delta_{ab}$ and the quadratic Casimir operator in the adjoint representation given by $f_{abc}f_{abd}=C(G)\delta_{cd}, C(G)=N_C$.

Exactly as it happens in the Abelian case (the Schwinger model) there exists a change in the fermion variables which completely decouples fermions from gauge fields, at the classical level. Although this is very simply done in the so-called decoupling gauge,[2,11] we shall work in an arbitrary gauge[10] which will be more suitable to our present purposes (later on we shall choose the light-cone gauge $A_+=A_0+i(A_1=0)$.

Then we can write[11]

$$\not{A}=-\frac{i}{g}(\not{\partial}e^{\gamma_5\phi+i\eta})e^{-\gamma_5\phi-i\eta}, \tag{2}$$

where $\phi=\phi^a t^a$ and $\eta=\eta^a t^a$ are scalar fields taking values in the Lie algebra of $SU(N_C)$. [Notice that Eq. (2) becomes $A_\mu=-(1/g)\epsilon_{\mu\nu}\partial_\nu\phi+(1/g)\partial_\mu\eta$ in the Abelian case, giving the usual decomposition of A_μ into longitudinal and transverse parts.] It is straightforward to check that in terms of the new fermion variables

$$\begin{aligned}\chi(x)&=e^{-\gamma_5\phi(x)-i\eta(x)}\psi(x)\,,\\ \bar{\chi}(x)&=\bar{\psi}(x)e^{\gamma_5\phi(x)+i\eta(x)}\,,\end{aligned} \tag{3}$$

the fermion Lagrangian becomes completely decoupled from gauge fields:

$$\mathcal{L}_F=\bar{\psi}\not{D}\psi=\bar{\chi}i\not{\partial}\chi\ . \tag{4}$$

Now quantum effects are taken into account by the change in the fermionic measure under transformation (3):

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi=J_F\mathcal{D}\bar{\chi}\mathcal{D}\chi\ , \tag{5}$$

where J_F is the fermionic non-Abelian Jacobian. The functional integral (1) then reads

$$\begin{aligned}Z&=\int \mathcal{D}A_\mu \exp\left[-\tfrac{1}{4}\int d^2x\ \mathrm{tr}F_{\mu\nu}{}^2\right]\det\not{D}\\ &=\int \mathcal{D}A_\mu \exp\left[-\tfrac{1}{4}\int d^2x\ \mathrm{tr}F_{\mu\nu}{}^2\right]J_F\det i\not{\partial}\ .\end{aligned} \tag{6}$$

In order to compute J_F one introduces an extended transformation depending on a parameter t ($t \in [0,1]$) that allows to build up the whole change (3) varying t from 0 to 1 (Ref. 2). The result is

$$\ln J_F = W = \frac{g^2}{2\pi} \int d^2x \, \mathrm{tr} A_\mu A^\mu - \frac{g^2}{2\pi} \int d^2x \int_0^1 dt \, \mathrm{tr} \gamma_5 \mathbf{A}_t \phi \mathbf{A}_t - \frac{g}{\pi} \int d^2x \int_0^1 dt \, \mathrm{tr} A_t^\mu \mathcal{D}_\mu^t \eta \, , \tag{7}$$

where $\mathcal{D}_\mu^t = \partial_\mu - ig[A_\mu^t, \;]$ and with

$$\mathbf{A}^t = -\frac{i}{g}(\not\partial e^{t(\gamma_5\phi + i\eta)}) e^{-t(\gamma_5\phi+i\eta)} , \quad \mathbf{A} = \mathbf{A}^{t=1} .$$

This result coincides with that of Ref. 10. Making $\eta = 0$ one gets the fermion determinant in the decoupling gauge, as evaluated in Ref. 2. One can also easily obtain the expression for the light-cone gauge $A_+ = A_0 + iA_1 = 0$, $A_- = A_0 - iA_1 = (i/g) U \partial_- U^{-1}$:

$$W[A_-, A_+ = 0] = W[U] = \frac{-1}{8\pi} \mathrm{tr} \int d^2x \, \partial_\mu U \partial_\mu U^{-1} + \frac{i}{4\pi} \epsilon_{\mu\nu} \mathrm{tr} \int_0^1 dt \int d^2x \, U_t^{-1} \partial_t U_t U_t^{-1} \partial_\mu U_t U_t^{-1} \partial_\gamma U_t \, , \tag{8}$$

where

$$U_t = e^{2t\phi}, \quad U_{t=1} = U \, . \tag{9}$$

With this last gauge choice the kinetic gluon term takes the simple form

$$\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} = \tfrac{1}{4} (\partial_+ A_-)^2 = -\frac{1}{4g^2} [\partial_+ (U \partial_- U^{-1})]^2 \tag{10}$$

($\partial_\pm = \partial_0 \pm i\partial_1$) and the generating functional reads

$$Z = \int \mathcal{D}A_- \exp\left[\frac{1}{4g^2} \mathrm{tr} \int d^2x [\partial_+ (U\partial_- U^{-1})]^2 \right] e^{-W[U]} \, , \tag{11}$$

where the (decoupled) free fermion determinant has been disregarded since it plays no role in the present discussion (although it has to be kept if fermionic correlation functions are computed by adding adequate sources).

We have now arrived to a last and central step in our analysis: If we compute the Jacobian J_A connected with the nontrivial change in the bosonic measure

$$\mathcal{D}A_- = J_A \mathcal{D}U \, , \tag{12}$$

we shall then get a complete description of the original model in terms of the bosonic field U. In other words, we shall get the complete bosonized form of QCD_2.

In a different context, a change of variables similar to (11) was considered in Ref. 3 and more recently in Refs. 12–14. We shall briefly describe how J_A can be easily calculated by considering an infinitesimal variation $\delta\phi$ of U as defined in (9) and then evaluating $\delta A_- / \delta\phi$, thus giving

$$\frac{\delta A_-}{\delta \phi} = \frac{1}{g} D_-^{\mathrm{ad}} \, , \tag{13}$$

where D^{ad} is the covariant derivative in the adjoint representation:

$$D_-^{\mathrm{ad}} = \partial_- - ig[A_-, \;] \, . \tag{14}$$

We can then write

$$J_A(U) = \det D_-^{\mathrm{ad}} \tag{15}$$

and then express this determinant as a path integral over fermions in the adjoint representation:

$$J_A(U) = \int \mathcal{D}\bar{\chi} \mathcal{D}\chi e \exp\left(\int d^2x \, \bar{\chi} D_-^{\mathrm{ad}} \chi \right) . \tag{16}$$

This determinant can be related to the determinant in the fundamental representation given by Eq. (8) just by noting that if one considers an infinitesimal change of the fermionic variables this leads to the Jacobian [see, for example, Eq. (2.30) of Ref. 2]

$$\delta \ln J[\phi \, dt] = \frac{g}{4\pi} \epsilon_{\mu\nu} \int d^2x \, \mathrm{tr} \phi F_{\mu\nu}^{dt} \, , \tag{17}$$

where $F_{\mu\nu}^{dt}$ is the curvature associated to A_μ^{dt} as defined above. The Jacobian for a finite transformation is obtained from (17) by integration over t. Now, (17) is valid for all fermion representations and hence allows us to relate the result in the adjoint and the fundamental ones. Our convention is $\mathrm{tr} t_a t_b = N \delta_{ab}$ with $N = \frac{1}{2}$ and $f_{apq} f_{bpq} = C(G) \delta_{ab}$ with $C(G) = N_C$. We then have

$$\delta \ln J \,|_{\mathrm{ad}} = \frac{C(G)}{N} \delta \ln J \Big|_{\mathrm{fund}} = 2N_C \delta \ln J \,|_{\mathrm{fund}} \, , \tag{18}$$

and the same relation then holds for the full Jacobian (16):

$$\ln J_A = \ln J \,|_{\mathrm{ad}} = -2N_C W[U] \, . \tag{19}$$

Thus, replacing (19) in (10) one gets the completely bosonized version of the QCD_2 generating functional

$$Z=\int \mathcal{D}U\, e^{-S_B[U]}\ , \tag{20}$$

with

$$S_B[U]=(2N_C+1)W[U]-\frac{1}{4g^2}\,\mathrm{tr}[\partial_+(U\partial_-U^{-1})]^2\ . \tag{21}$$

This action corresponds to a nonlinear σ model with a Wess-Zumino term with a shifted value for the level k, $k=1+2N_C$, plus the contribution of the $F_{\mu\nu}F_{\nu\mu}$ term which crucially changes the effective dynamics. Indeed, the model defined by (21) corresponds to a higher-derivative Lagrangian this being at the origin of the massiveness of the boson fields. Indeed, consider an expansion

$$U=1+2\phi^a t^a+O(\phi^2)\ . \tag{22}$$

The bosonic effective Lagrangian then reads

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}=&-\frac{1}{g^2}\,\mathrm{tr}\left[\phi\left[\Box\Box-\frac{kg^2}{2\pi}\Box\right]\phi\right]\\&+\frac{4i}{g^2}\epsilon_{\mu\nu}\,\mathrm{tr}\left[\phi(\partial_\mu\Box\phi)\partial_\nu\phi-\frac{kg^2}{6\pi}\phi\partial_\mu\phi\partial_\nu\phi\right]\\&+\text{higher-order terms}\ ,\end{aligned} \tag{23}$$

with $k=1+2N_C$. Now our previous comment becomes apparent. As usual in the path-integral approach to bosonization[3], one gets an effective Lagrangian with higher-order derivative terms. Its free part corresponds to $N_C{}^2-1$ massive scalars [with mass $m=g(k/2\pi)^{1/2}$] and the same number of massless gauge excitations.[15]

In contrast[15,16] with QED_2 where the massive field is free, here a self-interaction is present, given by the Wess-Zumino term and the $F_{\mu\nu}{}^2$ term the whole leading to a Skyrme-type effective Lagrangian. Note that because of the nontriviality of J_A [Eq. (19)] the scalar masses are multiplied by the factor $2N_C+1$. Hence, in the large-N_C limit (where as usual[17] one defines $g=g_0/\sqrt{N_C}$) one gets $m=g_0/\sqrt{\pi}$ exactly as in the Abelian case. (Cf. Ref. 5 where only the fermionic contribution was analyzed and Ref. 18 where the J_A contribution was disregarded.)

Coming back to the bosonized action S_B defined by Eq. (21) one can repeat the analysis of Knizhnik and Zamolodchikov[19] to study the fermionic currents in terms of the U field by noting that the transformation

$$U\to U'=A(x_+)UB(x_-)\ , \tag{24}$$

with A and B arbitrary $SU(N_C)$-valued matrices is an invariance of S_B,

$$S_B[U']=S_B[U]\ , \tag{25}$$

this leading to the conserved currents[9]

$$\begin{aligned}J_+=&\frac{ik}{2\pi}[(\partial_+U^{-1})U]\\&+\frac{i}{g^2}[[\partial_++(\partial_+U^{-1})U],(\partial_+U^{-1})U]\ ,\\J_-=&-\frac{-ik}{2\pi}[(\partial_-U)U^{-1}]\\&+\frac{i}{g^2}[\partial_-+(\partial_-U)U^{-1},(\partial_-U)U^{-1}]\ ,\end{aligned} \tag{26}$$

which satisfy a Kac-Moody algebra with level $k=1+2N_C$.

At this point it is interesting to discuss how the free fermionic model results can be derived within this framework. Let us first note that in order to compute vacuum expectation values (VEV's) of currents one has to add external sources s_μ in the generating functional (1),

$$Z=\int \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}A_\mu\times\exp\left[-\int d^2x[\bar\psi(\not{D}+\not{s})\psi+\tfrac{1}{4}F_{\mu\nu}{}^2]\right]\ , \tag{27}$$

so that, for example,

$$\langle J^a_\mu(x)J^b_\nu(y)\rangle=\frac{1}{Z}\frac{\delta^2 Z}{\delta s^a_\mu(x)\delta s^b_\nu(y)}\bigg|_{s=0}\ . \tag{28}$$

Then instead of Eq. (6) one has

$$Z[s]=\int \mathcal{D}A_\mu\exp\left[-\tfrac{1}{4}\int d^2x\,\mathrm{tr}F_{\mu\nu}{}^2\right]\det(\not{D}+\not{s})\ . \tag{29}$$

In the noninteracting case ($g\to 0$) the path integral over A factors out and $Z[s]$ reduces to

$$Z[s]=\mathcal{N}\det(i\not{\partial}+\not{s})\ . \tag{30}$$

Suppose one just wants to consider the right-handed currents $J_+{}^2$. One then puts $s_+=0$, $s_-=ig^{-1}\partial_-g$ (the left-handed case can be treated analogously) and uses $W[s_-,s_+=0]$ from Eq. (8).

Now, in Ref. 9 we have computed the current algebra for a model of $SU(N)$ fermions in a general background s_μ by computing the current-current correlation function from (30) and then taking the Bjorken-Johnson-Low limit. Using this technique and making $s_\mu=0$ at the end of the computations one gets

$$\begin{aligned}[J^a_+(x_-),J^b_+(y_-)]=&if^{abc}J^c_+(x_-)\delta(x_--y_-)\\&+\frac{i}{2\pi}\delta^{ab}\delta^1(x_--y_-)\ ,\end{aligned} \tag{31}$$

that is, J_+ satisfies a Kac-Moody algebra with level $k=1$ as is to be expected for free fermions.[1]

It is also very simple to consider the case of fermions with flavor $N_F\neq 1$. Indeed, $W[U]$ in (10) has to be replaced by $N_F\times W[U]$ but J_A is still given by (17). One then has, for the bosonized action S_B, instead of (21),

$$S_B=(2N_C+N_F)W[U]-\frac{1}{4g^2}\,\mathrm{tr}[\partial_+(U\partial_-U^{-1})]^2\ . \tag{32}$$

The scalar mass becomes then

$$m = g[(2N_C + N_F)/2\pi]^{1/2}$$

and in the $g \to 0$ limit one has also a Kac-Moody algebra but this time with level $k = N_F$.

In summary we have presented the complete bosonization of QCD_2 in the path-integral framework by considering the contribution of both the fermionic and the gauge-field sectors. To this end, we have computed the fermionic determinant and the bosonic Jacobian J_A which emerges when one writes down the path-integral measure as a Haar measure over the U fields which naturally appear in the decoupling of the fermion sector. We have shown that the resulting effective model contains massive self-interacting scalars with mass

$$m = g[(2N_C + 1)/2\pi]^{1/2}$$

for an $SU(N_C)$ gauge group.

We have also discussed the current algebra in terms of the bosonic fields. In order to go further into the understanding of the model, studying the fermionic correlation functions in terms of bosons at long distances (the decoupling implies fermions are free at short distances) one has to consider the complete action S_B, for instance, by means of a $1/N$ expansion. Also, the bosonization of QCD_2 with massive fermions can be envisaged in this approach following the lines discussed in Ref. 6. We hope to report on all these aspects in a forthcoming work.

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