Is the nonlinear σ model the $m_\sigma \to \infty$ limit of the linear σ model?

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(Received 12 August 1986; revised manuscript received 8 September 1987)

We propose to investigate whether the SO(N) nonlinear σ model is equivalent to the $m_\sigma \to \infty$ limit of the linear σ model by comparing the corresponding one-loop effective-action expansions up to the four-derivative terms and including the symmetry-breaking term. For this purpose we use a new background-field method to calculate the effective-action expansion directly. In the case of the linear σ model, the renormalization procedure is implemented carefully before the $m_\sigma \to \infty$ limit is taken. For the nonlinear σ model we introduce a new and intuitive covariant treatment for the perturbation calculation of the field theory with nonlinear constraint. We do not find any noninvariant terms in either case. We show that the divergent parts of the effective Lagrangians due to $m_\pi \to 0$, $m_\sigma \to \infty$, or $N \to \infty$ are equivalent in the two models. However, the nonleading finite parts of the effective Lagrangians are different. Therefore, the two operations, taking the $m_\sigma \to \infty$ limit and calculating the quantum corrections, do not commute. The origin of this difference may be a violation of decoupling.

I. INTRODUCTION

The σ model has been extremely useful to provide an explicit realization of spontaneously broken symmetry in quantum field theory.[1,2] In the strong-interaction sector, chiral symmetry is spontaneously broken. The pions emerge naturally as $I=1$ Goldstone bosons, while the mass of the $I=0$ σ meson can remain arbitrarily high. The dynamics of this renormalizable model with quantum corrections has been studied in detail.[3] However, even though this model realizes the SU(2)$\times$SU(2) current algebra, the partial conservation of axial-vector current, and therefore the corresponding low-energy theorem, the correct soft-pion limits of the physical amplitudes can only be obtained through the cancellation of many terms. This problem has been circumvented by the introduction of the minimal nonlinear σ model.[2] The role of the σ meson is thus eliminated.

Recently interest in the nonlinear σ model has been renewed because of the prospect of describing baryons as solitons in the Skyrme model.[4] There is also the possibility of a simplified effective dynamics of a gauged nonlinear σ model resulting from the strong-interacting Higgs sector in the standard electroweak model.[5,6] Supersymmetric nonlinear σ models have also played an important role in recent theoretical investigations.

The minimal nonlinear σ model is conventionally regarded as the formal limit of the linear σ at $m_\sigma \to \infty$. While this idea may be correct at the tree level, there is no compelling reason that the nonlinear σ model is completely equivalent to the linear σ model at $m_\sigma \to \infty$ when quantum corrections are included. More precisely one should integrate out the heavy σ field to obtain a chiral-invariant effective action. Such a task turns out to be nontrivial and has not been successfully carried out. Since the decoupling theorem is not applicable in this model, there will be observable consequences in the light-meson sector at low energy from the one-loop correction due to the heavy-σ-meson loop. The resulting nonlinear σ model is nonrenormalizable.[7]

In the SU(2)$\times$SU(2) σ model, Appelquist and Bernard have reconstructed the effective Langrangian using symmetry principles from the one-loop Feynman diagram and have demonstrated explicitly at leading order of the asymptotic expansion for large m_σ that the one-loop effective Lagrangian of the linear and the nonlinear σ model are equivalent provided that one makes the identification $1/\epsilon \to \ln m_\sigma$ (Ref. 5). Akhoury and Yao have generalized this result to the SO(N) σ model.[8] A surprising and disturbing result is the existence of apparently noninvariant divergent terms in these calculations. These terms do not contribute on the mass shell. The suggestion that these terms may be eliminated order by order by a field redefinition involving space-time derivatives is not very satisfactory, since such redefinition would generate new terms arising from the Jacobian in the functional integration. A recent calculation by Aitchison and Fraser[9] also contains the same noninvariant terms, but they also point out the difficulty with the effective-action expansion from the infrared problem as $m_\pi \to 0$. However, a covariant approach to evaluate the divergent part of the effective action of the nonlinear σ model does not show any noninvariant terms.[10]

Is the nonlinear σ model the $m_\sigma \to \infty$ limit of the linear σ model? Are the noninvariant terms necessarily the by-product of the $m_\sigma \to \infty$ limit of the linear σ model? Does the infrared problem prevent a meaningful derivative expansion of the σ model? These questions can be answered more satisfactorily if a more complete calculation can be performed. The recent progress on the evaluation of the effective action expansion has made such calculation possible.[11-18] In this paper we shall present the complete calculation up to the four-derivative terms of the SO(N) effective action expansion of the linear σ model before taking the $m_\sigma \to \infty$ limit. Similar to the calculation of the effective potential, the calculation will be performed without any prior assumption on the symmetry of the vacuum.[19,20] The $m_\sigma \to \infty$ will then be taken carefully and the effect of the counterterms and the renormalization will also be included. The calculation contains all one-loop diagrams instead of a small subset of Feynman diagrams. It includes all finite terms as well as the divergent terms. An explicit partially conserved axial-vector current type of symmetry-breaking term has also been included in the Lagrangian.

Perturbative calculations for field theories with nonlinear constraints are nontrivial. The best treatment for the noncovariant method requires the use of the covariant constraint equation to eliminate a certain component of the irreducible representation in a noncovariant manner. This asymmetrical procedure is compensated by using a field-dependent metric $g(\pi_{ij})$ in curved space. In Sec. V we present a new manifestly covariant formulation on the perturbation calculation of the nonlinear σ model.

We shall show that the divergent terms corresponding to $1/\epsilon \to \ln m_\sigma \to \infty$ and also $N \to \infty$ are equivalent between the linear and the nonlinear σ models. However the finite terms are completely different. We shall also give a brief discussion on the insight we obtain from this calculation on the infrared problem.

II. COVARIANT DERIVATIVE EXPANSION

In this section we shall briefly review recent results of the covariant derivative expansion.[18] It will be generalized slightly so that it can be applied directly to the nonlinear σ model. In most cases the one-boson-loop contribution to the effective action can be cast into the following form in D-dimensional Euclidean space:

$$\mathcal{L}_{1\,\text{loop}} = \frac{i}{2}\text{Tr}Q(X)\ln[\Pi^2 + U(X)] , \tag{2.1}$$

where $U(x)$ is a function of background fields and is a matrix in coordinate space and internal-symmetry space,

$$(\Pi_\mu)_{ij} = \delta_{ij}P_\mu - V^a_\mu(X)t^a_{ij} \tag{2.2}$$

is the generalized momentum, and V^a_μ are the gauge fields while t^a_{ij} are the generators of the gauge group. The coordinate matrix elements are

$$\langle x \mid U_{ij}(X) \mid y \rangle = U_{ij}(X)\delta^D(x-y) \tag{2.3}$$

and

$$\langle x \mid P^\mu \mid y \rangle = \frac{1}{i}\partial^\mu_x \delta^D(x-y) . \tag{2.4}$$

The commutation relations for the operators are

$$[\Pi_\mu, X_\nu] = [P_\mu, X_\nu] = \frac{1}{i}\delta_{\mu\nu} \tag{2.5}$$

and

$$[\Pi_\mu, Q(x)] = [X_\mu, Q(X)] = 0 . \tag{2.6}$$

The covariant derivatives defined by

$$\mathcal{D}_\mu O(X) = i[\Pi_\mu, O(X)] \tag{2.7}$$

and

$$F_{\mu\nu}(X) = \frac{1}{i}[\Pi_\mu, \Pi_\nu] \tag{2.8}$$

are functions of X only. Because of the short-distance singularity of the field theory, it is not possible to carry out a straightforward derivative expansion of the expression in Eq. (2.1). Previously and recently proposed schemes are too complicated and the algebras are too tedious for the evaluation of the four-derivative terms. Recently observing that Eq. (2.1) is invariant under the momentum translation, $\Pi_\mu \to \Pi_\mu + p_\mu$, we proposed that the average over the arbitrary internal momentum can be implemented at the very beginning stage without disturbing the full character of the operators and the trace structure. Thereby it provides a natural regularization function[18]

$$\text{Tr}Q\ln[\Pi^2 + U(X)] = \frac{1}{\delta^D(0)}\int \frac{d^Dp}{(2\pi)^D}\text{Tr}Q\ln[(\Pi_\mu + p_\mu)^2 + U(X)] , \tag{2.9}$$

where

$$\delta^D(0) = \int \frac{d^Dp}{(2\pi)^D} = \frac{V_p}{(2\pi)^D} \tag{2.10}$$

is exactly the infinite factor required for the regularization to work. The expression in Eq. (2.9) is the generating functional for the n-point vertex functions. p_μ is the loop momentum while Π_μ carries the momentum of the external fields.

In addition to satisfying the regularization function, the introduction of the momentum integration without disturbing the full trace operation offers the needed freedom for manipulations, such as cyclic permutations of the operators and integrations by parts, in order to bring Eq. (2.9) into the trace of a function of the covariant derivatives in the covariant derivative expansion. The procedure for the covariant derivative expansion becomes exceedingly simple.

(1) Expand Eq. (2.9) in a power series of Π_μ.

(2) Average over the momentum angular dependence so that the integrand becomes a function of p^2 only.

(3) Perform integration by parts in the momentum integration when necessary in order to manipulate the expression into a form such that the power of the momentum in each term matches exactly the number of factors of Π. Equation (2.9) now becomes

$$\mathrm{Tr}Q\ln(\Pi^2+U)=\frac{1}{\delta^D(0)}\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}Q\left[\ln(p^2+U)+\frac{2}{D}p^2(\Delta^2\Pi^2-\Delta\Pi^\mu\Delta\Pi_\mu)\right.$$
$$-\frac{4}{D(D+2)}p^4[2\Delta^3\Pi^2\Delta\Pi^2+\Delta^2\Pi^2\Delta^2\Pi^2-2\Delta^2\Pi^\mu\Delta\Pi_\mu\Delta\Pi^2$$
$$-2\Delta\Pi^\mu\Delta^2\Pi_\mu\Delta\Pi^2-2\Delta\Pi^\mu\Delta\Pi_\mu\Delta^2\Pi^2$$
$$\left.+2(\Delta\Pi^\mu\Delta\Pi_\mu)^2+(\Delta\Pi^\mu\Delta\Pi^\nu)^2]\right] . \tag{2.11}$$

(4) Use the freedom of making cyclic permutation of the operators within the trace operation to regroup each term in the power-series expansion so that the dependence on the operators Π_μ only occurs through the commutators of Π_μ, or equivalently the covariant derivatives, and no other dependence on any isolated operator Π_μ. Therefore the expansion becomes the expansion of covariant derivatives which is a function of the operators X_μ only. For example,

$$\mathrm{Tr}Q(\Delta^2\Pi^2-\Delta\Pi^\mu\Delta\Pi_\mu)=-\tfrac{1}{2}\mathrm{Tr}Q[\Pi^\mu,\Delta]^2=\tfrac{1}{2}\mathrm{Tr}Q(\mathcal{D}^\mu\Delta)^2 . \tag{2.12}$$

Since Q commutes with Π_μ, it is clear that Q becomes a bystander throughout the entire algebraic manipulation and would not alter the form of the final result of Ref. 18.

(5) Perform the trace operation by using the diagonal basis of X:

$$\mathrm{Tr}f(X)=\int d^Dx\langle y\,|f(X)|\,x\rangle\,|_{y=x}=\delta^D(0)\int d^Dx\,f(x) . \tag{2.13}$$

It is now clear that the infinite factor $\delta^D(0)$ from the trace operation exactly cancels out the same factor in Eq. (2.9), as the regularization is expected to work. The final expression for the covariant derivative expansion is

$$\mathrm{Tr}Q(X)\ln[-\Pi^2+U(X)]=i\int d^Dx\int\frac{d^Dp_E}{(2\pi)^D}\mathrm{tr}Q(x)\left[\ln[p_E{}^2+U(x)]-\frac{1}{D}p_E{}^2(\mathcal{D}_\mu\Delta)^2\right.$$
$$-\frac{2}{D(D+2)}p_E{}^2\{2[\Delta(\mathcal{D}^2\Delta)]^2+[(\mathcal{D}^\mu\Delta)(\mathcal{D}^\nu\Delta)]^2$$
$$-2[(\mathcal{D}^\mu\Delta)(\mathcal{D}_\mu\Delta)]^2-F^{\mu\nu}\Delta^2F_{\mu\nu}\Delta^2$$
$$\left.-4iF^{\mu\nu}\Delta(\mathcal{D}_\mu\Delta)\Delta(\mathcal{D}_\nu\Delta)\}\right] , \tag{2.14}$$

where $\Delta(x)=1/[p_E{}^2+U(x)]$. The trace tr is for internal symmetry and spin spaces only. The space-time coordinate has been rotated back to Minkowski space while the momentum p_E is kept in Euclidean space. As can be seen clearly the dependence on the background fields appears only through the propagator function. Only the method of Ref. 18 leads naturally to this compact expression with the conventional momentum-space and Feynman-diagram interpretation.

In $D=4$ dimensions, the first term gives the well-known contribution to the effective potential and is quadratically divergent.[11,20] The other logarithmically divergent term is the $O(F_{\mu\nu}{}^2)$ term. All other terms in the derivative expansion are finite.

If U is not a multiple of the unit matrix, then U has more than one distinct eigenvalue $U_a(x)$. It does not commute with $\mathcal{D}_\mu U$, $F_{\mu\nu}$, or other higher covariant derivatives. Therefore, with the exception of the first term in Eq. (2.14), it will not be possible to combine all factors of Δ into a single power to perform the momentum integration in a trivial manner. It is necessary to project out the eigenmodes $\Delta=\sum_a\Delta_a\mathcal{P}_a$ and then collect various factors of Δ_a for the momentum integration.

For the purpose of isolating the divergent contribution one can safely ignore the noncommutativity of U with the covariant derivatives since the commutator $[\Delta,F_{\mu\nu}]=\Delta[F_{\mu\nu},U]\Delta$ contains an extra convergent factor. Therefore, the momentum integration can be performed easily. The logarithmically divergent term is given by

$$\frac{i}{(4\pi)^2}\frac{1}{\epsilon}\mathrm{tr}(U^2-\tfrac{1}{6}F_{\mu\nu}{}^2) , \tag{2.15}$$

which is identical to the well-known result of 't Hooft.[21]

III. THE SO(N) LINEAR σ MODEL

The SO(N) linear-σ-model Lagrangian is

$$\mathcal{L}(\Phi)=\tfrac{1}{2}(\partial_\mu\Phi)^2-V(\Phi^2)$$

with (3.1)

$$V(\mathbf{\Phi}^2)=\tfrac{1}{2}m^2\mathbf{\Phi}^2+\frac{1}{4!}\lambda(\mathbf{\Phi}^2)^2-\boldsymbol{\epsilon}\cdot\mathbf{\Phi}\ ,$$

where the last term is the symmetry-breaking term. The effective action is given by[19]

$$\int d^4x\,\mathcal{L}_{\rm eff}(\mathbf{\Phi})=\int d^4x\,\mathcal{L}(\mathbf{\Phi})+\frac{i}{2}\mathrm{Tr}\ln[-P^2+U(X)]\ . \qquad (3.2)$$

The matrix elements of U are most conveniently expressed in terms of the SO(N) spherical variables $\sigma=|\mathbf{\Phi}|=(\mathbf{\Phi}^2)^{1/2}$ and $\boldsymbol{\phi}=\mathbf{\Phi}/|\mathbf{\Phi}|$ such that $\boldsymbol{\phi}^2=1$:

$$\langle x\,|\,U_{ij}\,|\,y\rangle=\frac{\delta^2}{\delta\Phi_i(x)\Phi_j(y)}\int d^4z\;V(\mathbf{\Phi}^2)=[u_L(\sigma)\mathcal{P}^L_{ij}+u_T(\sigma)\mathcal{P}^T_{ij}]\delta^D(x-y)\ , \qquad (3.3)$$

where $\mathcal{P}^L_{ij}=\phi_i\phi_j$ and $\mathcal{P}^T_{ij}=\delta_{ij}-\phi_i\phi_j$ are the longitudinal and the transverse projection operators with multiplicities $n_L=\mathrm{Tr}\mathcal{P}^L=1$ and $n_T=\mathrm{Tr}\mathcal{P}^T=N-1$, respectively. The corresponding eigenvalues of U are

$$u_L(\sigma)=m^2+\frac{\lambda}{2}\sigma^2\quad\text{and}\quad u_T(\sigma)=m^2+\frac{\lambda}{6}\sigma^2\ . \qquad (3.4)$$

Similarly we can express the propagation function

$$\Delta=\Delta_L(\sigma)\mathcal{P}^L+\Delta_T(\sigma)\mathcal{P}^T$$

with (3.5)

$$\Delta_L(\sigma)=\frac{1}{p^2+u_L(\sigma)}\quad\text{and}\quad\Delta_T(\sigma)=\frac{1}{p^2+u_T(\sigma)}\ .$$

The effective Lagrangian can be obtained by substituting these explicit forms into Eq. (2.14) with the gauge fields set equal to zero and combining the result with Eq. (3.2):

$$\mathcal{L}_{\rm eff}=\mathcal{L}(\Phi)+\mathcal{L}_{\rm ct}(\Phi)+\mathcal{L}_S(u_L)+(N-1)\mathcal{L}_S(u_T)+\mathcal{L}_M\ , \qquad (3.6)$$

where $\mathcal{L}(\Phi)$ is the Lagrangian in Eq. (3.1) and can be expressed in terms of the new variables σ and $\boldsymbol{\phi}$:

$$\mathcal{L}(\Phi)=\tfrac{1}{2}\sigma^2(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{2}m^2\sigma^2-\frac{1}{4!}\lambda\sigma^4+\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}\sigma\ . \qquad (3.7)$$

$\mathcal{L}_{\rm ct}$ is the counterterm contribution:

$$\mathcal{L}_{\rm ct}=\tfrac{1}{2}\delta Z[\sigma^2(\partial\boldsymbol{\phi})^2+(\partial\sigma)^2]-\tfrac{1}{2}\delta Z_m m^2\sigma^2-\frac{1}{4!}\delta Z_\lambda\lambda\sigma^4+\delta Z_\epsilon\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}\sigma\ . \qquad (3.8)$$

$\mathcal{L}_S(u)$ is the one-loop correction for the neutral scalar field as calculated in Refs. 11 and 13. In four dimensions it is given by

$$\mathcal{L}_S(u)=\frac{1}{4(4\pi)^2}\left[u^2\left[\frac{2}{\epsilon}-\ln\frac{u}{\mu^2}\right]+\tfrac{1}{6}u^{-1}(\partial u)^2+\tfrac{1}{60}u^{-2}(\partial^2u)^2-\tfrac{1}{45}u^{-3}(\partial u)^2(\partial^2u)+\tfrac{1}{120}u^{-4}(\partial u)^4\right]\ . \qquad (3.9)$$

$\mathcal{L}_M$ represents the contribution from two distinct eigenvalues u_L and u_T propagating in the same loop and is finite in four dimensions:

$$\begin{aligned}\mathcal{L}_M=(\tfrac{1}{3}\lambda)^2\int\frac{d^Dp_E}{(2\pi)^D}\Bigg\{&\frac{1}{D}p_E{}^2\Delta_L{}^2\Delta_T{}^2\sigma^4(\partial\boldsymbol{\phi})^2\\&+\frac{2}{D(D+2)}p_E{}^4\Bigg[\Delta_L{}^2\Delta_T{}^2(3\Delta_L{}^2-\Delta_T{}^2)\sigma^4(\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi})^2+4\Delta_L{}^2\Delta_T{}^4\sigma^4(\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi})^2\\&+2\Delta_T{}^3\Delta_L{}^3\sigma^4(\partial^2\boldsymbol{\phi})^2+2\Delta_L\Delta_T(\Delta_T-3\Delta_L)(\Delta_T{}^3-3\Delta_L{}^3)\partial_\alpha\sigma^2\partial_\beta\sigma^2(\partial^\alpha\boldsymbol{\phi}\cdot\partial^\beta\boldsymbol{\phi})\\&-4\Delta_L{}^2\Delta_T{}^2(\Delta_T{}^2-3\Delta_L{}^2)\sigma^2\partial_\mu\sigma^2(\partial^\mu\boldsymbol{\phi}\cdot\partial^2\boldsymbol{\phi})+2\Delta_L\Delta_T(\Delta_T{}^4-3\Delta_L{}^4)\sigma^2\partial^2\sigma^2(\partial\boldsymbol{\phi})^2\\&-2\frac{\lambda}{3}\Delta_L\Delta_T(\Delta_T{}^5-9\Delta_L{}^5)\sigma^2(\partial\sigma^2)^2(\partial\boldsymbol{\phi})^2\\&-\frac{1}{2}\left[\frac{\lambda}{3}\right]^2\Delta_L{}^2\Delta_T{}^2(9\Delta_L{}^4-3\Delta_L{}^2\Delta_T{}^2+\Delta_T{}^4)\sigma^4(\partial\sigma^2)^2(\partial\boldsymbol{\phi})^2\Bigg]\Bigg\}\ .\end{aligned} \qquad (3.10)$$

We have used dimensional regularization with

$$\frac{2}{\epsilon}=\frac{2}{4-D}+\frac{3}{2}-\gamma-\ln\frac{\mu^2}{4\pi} \qquad (3.11)$$

and $\gamma=0.5772$ is Euler's constant.

Up to this point this background field calculation has been performed without any prior assumption on the symmetry structure of the vacuum or any restriction on the ranges of the parameters m and λ.

IV. THE SPONTANEOUS-SYMMETRY-BREAKING $m_\sigma \to \infty$ LIMIT

When the condition for the spontaneous symmetry breaking is satisfied, $\langle \sigma \rangle \neq 0$, Eq. (3.4) gives the masses for the two orthogonal particle states:

$$m_\sigma{}^2 = m_L{}^2 = m^2 + \frac{\lambda}{2}\langle \sigma \rangle^2 \; ,$$
$$m_\pi{}^2 = m_T{}^2 = m^2 + \frac{\lambda}{6}\langle \sigma \rangle^2 \; . \tag{4.1}$$

The parameters m^2 and λ can be determined by

$$\lambda = \frac{3}{\langle \sigma \rangle^2}(m_\sigma{}^2 - m_\pi{}^2) \; ,$$
$$m^2 = -\tfrac{1}{2}(m_\sigma{}^2 - 3m_\pi{}^2) \; . \tag{4.2}$$

The $m_\sigma \to \infty$ limit corresponds to $\lambda \to \infty$ and $m^2 \to -\infty$ such that $f = (-6m^2/\lambda)^{1/2} \to \langle \sigma \rangle$ is kept finite. In this limit one would like to eliminate the σ field completely. To one-loop order it is sufficient to use the field equation derived from the Lagrangian Eq. (3.7):

$$\sigma(\partial\boldsymbol{\phi})^2\sigma - m^2\sigma - \tfrac{1}{6}\lambda\sigma^3 + \boldsymbol{\epsilon}\cdot\boldsymbol{\phi} = 0 \; . \tag{4.3}$$

We choose the mass scaling factor in Eq. (3.8) to be

$$M^2 = -2m^2 \to \infty \; . \tag{4.4}$$

$\sigma(x)$ can be solved iteratively using the equation

$$\sigma^2 = f^2\left[1 + \frac{2}{M^2}u_T\right] , \tag{4.5}$$

where

$$u_T = \left[(\partial\boldsymbol{\phi})^2 + \frac{1}{\sigma}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi} - \frac{1}{\sigma}\partial^2\sigma\right]$$
$$\to U_T\left[1 - \frac{1}{M^2 f}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}\right] , \tag{4.6}$$

we have defined

$$f^2 = \frac{3M^2}{\lambda} \; , \tag{4.7}$$

and

$$U_T = \left[(\partial\boldsymbol{\phi})^2 + \frac{1}{f}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}\right] . \tag{4.8}$$

Since, from Eqs. (3.4) and (4.4),

$$u_L = M^2 + 3u_T \; , \tag{4.9}$$

it is clear that u_T is of the order 1 while u_L is of the order M^2. In the large-M limit, the tree Lagrangian Eq. (3.7) becomes the nonlinear σ model Lagrangian:

$$\mathcal{L} \to \frac{f^2}{2}(\partial\boldsymbol{\phi})^2 + \boldsymbol{\epsilon}\cdot\boldsymbol{\phi} \; . \tag{4.10}$$

For the one-loop contribution

$$\mathcal{L}_S(u_L) \to \frac{1}{4(4\pi)^2}\left[\frac{2}{\bar{\epsilon}_L}(6M^2u_T + 9u_T{}^2) - 3M^2u_T - \frac{27}{2}u_T{}^2\right], \quad (N-1)\mathcal{L}_S(u_T) \to (N-1)\mathcal{L}_S(u_T) \; , \tag{4.11}$$

$$\mathcal{L}_M \to \frac{1}{4(4\pi)^2}\left[\left[2u_T\ln\frac{u_T}{M^2} + M^2 + 4u_T\right](\partial\boldsymbol{\phi})^2 + \tfrac{1}{3}(\partial^2\boldsymbol{\phi})^2 + \frac{1}{3}\left[\ln\frac{u_T}{M^2} + \frac{10}{3}\right][(\partial\boldsymbol{\phi})^2]^2 - \frac{4}{3}\left[\ln\frac{u_T}{M^2} + \frac{17}{6}\right](\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi})^2 \right.$$
$$\left. + \tfrac{1}{3}u_T{}^{-1}(\partial^2 u_T)(\partial\boldsymbol{\phi})^2 - \tfrac{1}{6}u_T{}^{-2}(\partial u_T)^2(\partial\boldsymbol{\phi})^2\right] . \tag{4.12}$$

The effective Lagrangian is divergent as $2/\bar{\epsilon}_L = 2/\bar{\epsilon} - \ln\mu^2/M^2$ or as $M \to \infty$. The divergent parts (excluding $\ln M^2$)

$$\mathcal{L}_D = \frac{1}{4(4\pi)^2}\left[3\left[\frac{4}{\bar{\epsilon}_L} - 1\right]M^2u_T + \frac{2}{\bar{\epsilon}_L}(N+8)u_T{}^2 + M^2(\partial\boldsymbol{\phi})^2\right] \tag{4.13}$$

can be removed completely by a suitably chosen counterterm Eq. (3.8).

V. COUNTERTERMS AND INFINITE RENORMALIZATION

Since we do not know exactly how the unknown coefficients of Eq. (3.8), the δZ's depend on M^2, we have to take special care in taking the large-M limit. We rewrite Eq. (3.8) by first eliminating the $\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}$ term through the use of Eq. (4.3) and then use Eq. (4.5) to express σ^2 in terms of u_T:

$$\mathcal{L}_{\text{ct}} = \tfrac{1}{2}(\delta Z - 2\delta Z_\epsilon)\left[f^2\left[1 + \frac{2}{M^2}u_T\right](\partial\boldsymbol{\phi})^2 + \frac{f^2}{M^4}\frac{1}{\left[1 + \frac{2}{M^2}u_T\right]}(\partial u_T)^2\right] + \tfrac{1}{2}(\delta Z_m - \delta Z_\lambda + 2\delta Z_\epsilon)f^2u_T$$
$$- \tfrac{1}{2}(\delta Z_\lambda - 4\delta Z_\epsilon)\frac{f^2}{M^2}u_T{}^2 \; . \tag{5.1}$$

Since every divergent term in Eq. (4.13) is contained in Eq. (5.1), they can be absorbed by rewriting the parameters

δZ's such that

$$(\delta Z-2\delta Z_\epsilon)+\frac{2}{(4\pi)^2}\frac{M^2}{f^2}=C_f,\quad (\delta Z_m-\delta Z_\lambda+2\delta Z_\epsilon)+\frac{3}{2(4\pi)^2}\frac{M^2}{f^2}\left[\frac{4}{\bar\epsilon_L}-1\right]=2C_\epsilon\ ,$$
$$(\delta Z_\lambda-4\delta Z_\epsilon)\frac{f^2}{M^2}-\frac{1}{(4\pi)^2}(N+8)\frac{1}{\bar\epsilon_L}=\frac{1}{2(4\pi)^2}C\ . \tag{5.2}$$

We assume that the physical parameters, such as f_π and m_π should be independent of M^2. We will show later in Eqs. (5.9) and (5.10) that the parameters C_f and C_ϵ can grow at most like $\ln M^2$. Therefore if there is no conspiracy, the counterterm parameters δZ's must increase at most like M^2 and C approaches a constant at large M^2. Among the four linearly independent parameters only three linearly independent combinations are actually necessary to render finite renormalization. We are free to choose $\delta Z_m=0$ so that there will not be any ambiguity whether the parameter $M=-2m\rightarrow\infty$ should be the renormalized or the unrenormalized one. However such a choice should have no consequence.

The Eqs. (4.13) and (5.1) combine to yield

$$\mathcal{L}_D+\mathcal{L}_{\rm ct}=\tfrac{1}{2}C_f\left[1+\frac{2}{M^2}u_T\right]f^2(\partial\boldsymbol{\phi})^2+C_\epsilon f^2u_T-\frac{1}{4(4\pi)^2}[2u_T(\partial\boldsymbol{\phi})^2+Cu_T{}^2]\ . \tag{5.3}$$

After removing all $O(M^2)$ terms we can now safely take the large-M limit and replace u_T by U_T which has been defined by Eq. (4.8). We obtain the effective Lagrangian

$$\begin{aligned}\mathcal{L}^L_{\rm eff}=&\tfrac{1}{2}f^2(1+C_f-2C_\epsilon)(\partial\boldsymbol{\phi})^2+(1+C_\epsilon)U_T+\mathcal{L}_I\\&+\frac{1}{4(4\pi)^2}\left[-(C+\tfrac{25}{2})U_T{}^2+\tfrac{2}{9}[5(\partial\boldsymbol{\phi})^4-17(\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi})^2]-\frac{2}{f}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}U_T+\frac{1}{3}(\partial^2\boldsymbol{\phi})^2\right.\\&\left.-\left[(N-3)U_T{}^2-\frac{1}{3}(\partial\boldsymbol{\phi})^4+\frac{4}{3}(\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi})^2+\frac{2}{f}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}U_T\right]\ln\frac{U_T}{M^2}\right]\ ,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\mathcal{L}_I=&\frac{1}{4(4\pi)^2}\{\tfrac{1}{6}[2U_T{}^{-1}(\partial^2U_T)-U_T{}^2(\partial U_T)^2](\partial\boldsymbol{\phi})^2\\&+(N-1)[\tfrac{1}{6}U_T{}^{-1}(\partial U_T)^2+\tfrac{1}{60}U_T{}^{-2}(\partial^2U_T)^2-\tfrac{1}{45}U_T{}^{-3}(\partial U_T)^2(\partial^2U_T)+\tfrac{1}{120}U_T{}^{-4}(\partial U_T)^4]\}\ .\end{aligned} \tag{5.5}$$

It can readily be seen that the effective Lagrangian is manifestly invariant except the explicit symmetry breaking due to the $\boldsymbol{\epsilon}$. There is no other noninvariant term of the type suggested in Refs. 5, 8, and 9. Except for the noninvariant term, one can recover the answer of Ref. 9 from our result of Eq. (5.4) (Ref. 22).

The pion fields may be defined as the component of the $\boldsymbol{\phi}$ orthogonal to $\boldsymbol{\epsilon}$:

$$\boldsymbol{\pi}=f\left[\boldsymbol{\phi}-\frac{\boldsymbol{\epsilon}(\boldsymbol{\epsilon}\cdot\boldsymbol{\phi})}{\epsilon^2}\right] \tag{5.6}$$

and

$$\boldsymbol{\pi}\cdot\boldsymbol{\epsilon}=0,\quad \sigma=\frac{\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}}{\epsilon}=\left[1-\frac{\pi^2}{f^2}\right]^{1/2}\ . \tag{5.7}$$

A natural procedure for renormalization is to choose that counterterm such that the physical parameters

$$f=f_\pi \tag{5.8a}$$

and

$$\epsilon=f_\pi m_\pi{}^2 \tag{5.8b}$$

in the tree Lagrangian acquire no higher-order corrections:

$$C_f=\frac{1}{4(4\pi)^2}\frac{m_\pi{}^2}{f_\pi{}^2}\left[N+2C+24-2(N-2)\ln\frac{M^2}{m_\pi{}^2}\right]\ , \tag{5.9}$$

$$C_\epsilon=\frac{2}{4(4\pi)^2}\frac{m_\pi{}^2}{f_\pi{}^2}\left[(N-1)\left[1-\ln\frac{M^2}{m_\pi{}^2}\right]+\left[C+\frac{27}{2}\right]\right]\ . \tag{5.10}$$

The effective Lagrangian in Eq. (5.4) is a partial but not complete derivative expansion, since $U_T=(1/f)\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}+(\partial\boldsymbol{\phi})^2$ contains a derivative term and can be further expanded. However $(1/f)\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}$ is a small symmetry-breaking term and vanishes in the symmetry limit. Furthermore the effective Lagrangian is nonanalytic at $U_T=0$. The expansions in powers of $(\partial\boldsymbol{\phi})^2$ and of $(1/f)\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}$ are incompatible. Therefore the limits $(\partial\boldsymbol{\phi})^2\rightarrow0$ and $(1/f)\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}\rightarrow0$ are noninterchangeable.

If the expansion in powers of $(\partial\boldsymbol{\phi})^2$ is performed first, the part of the contribution from U_T must be in the combination of $(\partial\phi)^2/(1/f)\epsilon\cdot\phi$. Thus the expansion is only valid for large symmetry breaking and it is not possible to recover the symmetry limit. Disregarding this limitation can lead to misleading conclusions. For example, if one removes the term $\frac{1}{6}U_T^{-1}(\partial U_T)^2$ from $\mathcal{L}_I$, then $\mathcal{L}_I$ is homogeneous in U_T with degree zero. The leading terms of the expansion, which can be obtained with setting $(\partial\phi)^2=0$ or $U_T=(1/f)\epsilon\cdot\phi$, would therefore be completely independent of the symmetry-breaking parameter ϵ. This can be illustrated by one particular term:

$$U_T^{-4}(\partial U_T)^4 \to (\epsilon\cdot\phi)^{-4}(\partial\epsilon\cdot\phi)^4 = \left[1-\frac{\pi^2}{f^2}\right]^{-2}\left[\partial\left[1-\frac{\pi^2}{f^2}\right]^{1/2}\right]^4 . \tag{5.11}$$

It appears as if there would be noninvariant terms surviving after the limit $\epsilon\to 0$. However the expansion is only valid for large ϵ and $\epsilon\to 0$ limit is not allowed. In the reconstruction of the effective Lagrangian from Feynman diagrams, it is not possible to keep track of the symmetry-breaking effects and to identify the source of the nonanalyticity. These terms would appear as anomalous noninvariant terms but in fact they are induced by the external symmetry-breaking term and the nonanalyticity in U_T.

If instead one expands the effective Lagrangian in powers of $\epsilon\cdot\phi$ first, one immediately displays the infrared problem explicitly. While the expansion does not contain infinite divergent parts, the infrared nonanalyticity manifests as an unconventional form of the effective Lagrangian containing terms such as

$$(\partial\phi)^4\ln(\partial\phi)^2,\quad \left[\frac{1}{(\partial\phi)^2}\right]^4[\partial(\partial\phi)^2]^4,\ \ldots$$

which cannot be subjected to the usual treatment of perturbation calculation. Conversely one would not be able to find such an effective Lagrangian from the reconstruction from the Feynman diagrams.

VI. THE NONLINEAR σ MODEL

The SO(N) nonlinear σ model is defined by the Lagrangian

$$\mathcal{L}_{\rm NL}(\Phi)=\tfrac{1}{2}f^2(\partial\Phi)^2+f\epsilon\cdot\Phi , \tag{6.1}$$

with the constraint $\Phi^2=1$. In this section we shall introduce a new and intuitive method to calculate covariantly the effective Lagrangian for field theory with constraint.

In the background-field calculation the standard procedure is to expand the Lagrangian Eq. (6.1) in a Taylor series around the classical background field ϕ: $\Phi=\phi+\omega$. Since the classical background field ϕ must also satisfy the constraint $\phi^2=1$, the quantum displacement ω must be constrained by

$$\omega^2+2\omega\cdot\phi=0 . \tag{6.2}$$

In the functional path-integration formulation the quantum displacement ω is to be integrated over the N-dimensional sphere with unit radius. ω^2 is not necessarily small and cannot be ignored. It is convenient to decompose ω into the longitudinal and transverse components by the projection operators $\mathcal{P}_L=\phi\phi$ and $\mathcal{P}_T=1-\phi\phi$:

$$\begin{aligned}\omega_L&=\mathcal{P}_L\omega=\phi(\phi\cdot\omega)=\phi\omega_L ,\\ \omega_T&=\mathcal{P}_T\omega=\omega-\phi(\phi\cdot\omega) .\end{aligned} \tag{6.3}$$

Dyadic notation will be used in this section. Equation (6.2) becomes

$$\omega_L{}^2+\omega_T{}^2+2\omega_L=0 . \tag{6.4}$$

The constraint Eq. (6.4) can be used to eliminated ω_L:

$$\omega_L=-1+\sqrt{1-\omega_T{}^2}\to -\tfrac{1}{2}\omega_T{}^2+O(\omega_T{}^4) . \tag{6.5}$$

In terms of the independent ω_T the Taylor expansion of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm NL}(\phi+\omega)&=\mathcal{L}_{\rm NL}(\phi)-f^2\omega\cdot\left[\partial^2\phi-\frac{1}{f}\epsilon\right]-\tfrac{1}{2}f^2\omega\cdot\partial^2\omega\\ &=\mathcal{L}_{\rm NL}-f^2\omega_T\cdot\left[\partial^2\phi-\frac{1}{f}\epsilon\right]-\tfrac{1}{2}f^2\omega_T\cdot\left[\partial^2+(\partial\phi)^2+\frac{1}{f}\epsilon\cdot\phi\right]\omega_T+O(\omega_T{}^4) .\end{aligned} \tag{6.6}$$

The requirement of the vanishing of the linear term gives the field equation for ϕ:

$$\mathcal{P}_T\left[\partial^2\phi-\frac{1}{f}\epsilon\right]=\partial^2\phi+\phi(\partial\phi)^2-\frac{1}{f}\epsilon+\frac{1}{f}\phi(\epsilon\cdot\phi)=0 . \tag{6.7}$$

In the presence of the external source $\mathbf{J}(x)$, the right-hand side of the equation is modified to be $\mathcal{P}_T\mathbf{J}(x)$ and the equation can be used to express ϕ as a function of $\mathbf{J}$. The introduction of the external source $\mathbf{J}(x)$ allows the classical background field to be arbitrary. For any choice of ϕ, one can always find a proper $\mathbf{J}(x)$ so that the classical field ϕ is at the stationary point.

It is essential that one put in the proper projection operator $\mathcal{P}_T$ associated with every factor of ω_T to project out the independent components. A more proper way is to introduce a set of $N-1$ orthonormal basis vectors $\hat{\mathbf{e}}_i$, $i=1,\ldots,N-1$ for the transverse space with the conditions that

$$\hat{\mathbf{e}}_i\cdot\phi=0 , \tag{6.8}$$

$$\hat{\boldsymbol{\epsilon}}_i\cdot\hat{\boldsymbol{\epsilon}}_j=\delta_{ij}\ , \tag{6.9}$$

$$\sum_i \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i=\mathcal{P}_T\ . \tag{6.10}$$

We can then define the $N-1$ orthogonal transverse vectors

$$\omega_i=\hat{\mathbf{e}}_i\cdot\boldsymbol{\omega}_T\ . \tag{6.11}$$

The Green's function in the presence of the background is given by

$$G_{ij}{}^{-1}(x-y)=\hat{\mathbf{e}}_i\cdot[\partial^2+(\partial\boldsymbol{\phi})^2+f\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}]\hat{\mathbf{e}}_j\delta(x-y)\ . \tag{6.12}$$

To one-loop order, the effective action can be obtained by performing the Gaussian integration:

$$e^{iI_{\rm eff}}=\int D[\omega_i]\exp\left[i\int d^4x\,\mathcal{L}_{\rm NL}(\boldsymbol{\phi}+\boldsymbol{\omega})\right]$$
$$=\exp\left[i\int d^4x\,\mathcal{L}_{\rm NL}(\boldsymbol{\phi})\right]\det[\bar{G}^{-1}(P,X)]^{-1/2}\ , \tag{6.13}$$

where

$$\bar{G}_{ij}{}^{-1}(P,X)=\hat{\mathbf{e}}_i(X)\cdot\left[P^2+(\partial\boldsymbol{\phi})^2(X)+\frac{1}{f}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}(X)\right]\hat{\mathbf{e}}_j(X)\ . \tag{6.14}$$

The operators P and X are defined in Sec. I. The one-loop corrected effective action is then given by

$$\int d^4x\,\mathcal{L}_{\rm eff}^{\rm NL}(\boldsymbol{\phi})=\int d^4x\,\mathcal{L}_{\rm NL}(\boldsymbol{\phi})+\frac{i}{2}{\rm Tr}\ln\bar{G}^{-1}(P,X)\ . \tag{6.15}$$

The calculation of the derivative expansion can be greatly simplified and the unwanted dependence on the $\hat{\mathbf{e}}_i$ can be completely eliminated by the following observation.

We defined a vector gauge field

$$\underline{v}_\mu=\frac{1}{i}\partial_\mu\boldsymbol{\phi}\boldsymbol{\phi}-\boldsymbol{\phi}\frac{1}{i}\partial_\mu\boldsymbol{\phi} \tag{6.16}$$

and the corresponding generalized momentum operator

$$\Pi_\mu=P_\mu-\underline{v}_\mu\ . \tag{6.17}$$

It follows that the field strength has a simple form

$$\underline{F}_{\mu\nu}=\frac{1}{i}[\Pi_\mu,\Pi_\nu]=i(\partial_\mu\boldsymbol{\phi}\partial_\nu\boldsymbol{\phi}-\partial_\nu\boldsymbol{\phi}\partial_\mu\boldsymbol{\phi})\ . \tag{6.18}$$

Note that $\underline{v}_\mu$ is not the pure gauge field. The pure gauge field is in fact equal to $2\underline{v}_\mu$ and the corresponding field strength is equal to zero. The unique advantage of our choice of $\underline{v}_\mu$ is that Π_μ so defined commutes with the projection operators $\mathcal{P}_T$ and $\mathcal{P}_L$,

$$[\Pi_\mu,\mathcal{P}_T]=[\Pi_\mu,\mathcal{P}_L]=0\ , \tag{6.19}$$

and $\underline{F}_{\mu\nu}$ is purely transverse:

$$\mathcal{P}_L\underline{F}_{\mu\nu}=\underline{F}_{\mu\nu}\mathcal{P}_L=0\ ,$$
$$\mathcal{P}_T\underline{F}_{\mu\nu}=\underline{F}_{\mu\nu}\mathcal{P}_T=\underline{F}_{\mu\nu}\ . \tag{6.20}$$

By comparing the two expressions

$$\mathcal{P}_TP^2\mathcal{P}_T=\mathcal{P}_T(P^2+[P^2,\mathcal{P}_T])\mathcal{P}_T$$
$$=\mathcal{P}_T\left[P^2+2\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}-\frac{2}{i}\partial_\mu\boldsymbol{\phi}\boldsymbol{\phi}P^\mu\right]\mathcal{P}_T \tag{6.21}$$

and

$$\mathcal{P}_T\Pi^2\mathcal{P}_T=\mathcal{P}_T\left[P^2+\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}-2\frac{1}{i}\partial_\mu\boldsymbol{\phi}\boldsymbol{\phi}P^\mu\right]\mathcal{P}_T\ , \tag{6.22}$$

we deduce the identity

$$\mathcal{P}_TP^2\mathcal{P}_T=\mathcal{P}_T(\Pi^2+\partial\boldsymbol{\phi}\partial\boldsymbol{\phi})\ . \tag{6.23}$$

Since $[\mathcal{P}_T,\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}]=0$ the extra $\mathcal{P}_T$ on the right-hand side of the equation is omitted. It follows from Eq. (6.8) that

$$\hat{\mathbf{e}}_i=\mathcal{P}_T\hat{\mathbf{e}}_i\ . \tag{6.24}$$

Therefore we can write

$$\bar{G}_{ij}{}^{-1}(P,X)=\hat{\mathbf{e}}_i(X)\cdot G^{-1}(\Pi,X)\cdot\hat{\mathbf{e}}_j(X)\ , \tag{6.25}$$

with

$$G^{-1}(\Pi,X)=\Pi^2+\frac{1}{f}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}(X)+(\partial\boldsymbol{\phi})^2(X)$$
$$-\partial\boldsymbol{\phi}(X)\partial\boldsymbol{\phi}(X)\ . \tag{6.26}$$

Since $[G^{-1},\mathcal{P}_T]=0$, we can diagonalize G^{-1} and $\mathcal{P}_T$ simultaneously. The trace operation in Eq. (6.13) can then be evaluated:

$${\rm Tr}\ln\bar{G}^{-1}(P,X)=\sum_i\hat{\mathbf{e}}_i\cdot\ln G^{-1}(\Pi,X)\cdot\hat{\mathbf{e}}_i$$
$$={\rm Tr}\mathcal{P}_T\ln G^{-1}(\Pi,X)\ . \tag{6.27}$$

Therefore we have succeeded in eliminating the entire dependence on the orthonormal basis $\hat{\mathbf{e}}_i$. The resulting trace has the same form as that of the Eq. (2.14) with the identifications

$$Q(X)=\mathcal{P}_T=\underline{1}-\boldsymbol{\phi}(X)\boldsymbol{\phi}(X)\ ,$$
$$\Pi_\mu=P_\mu-\underline{v}_\mu(X)=P_\mu-\frac{1}{i}(\partial_\mu\boldsymbol{\phi}\boldsymbol{\phi}-\boldsymbol{\phi}\partial_\mu\boldsymbol{\phi})\ , \tag{6.28}$$
$$U(X)=\frac{1}{f}\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}(X)+(\partial\boldsymbol{\phi})^2(X)-\partial\boldsymbol{\phi}(X)\partial\boldsymbol{\phi}(X)\ .$$

The momentum integration of Eq. (2.14) can be more easily evaluated if we further expand the expression in the powers of $\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}$ according to $U(x)=U_T(x)-\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}$ with $U_T(x)=(\partial\boldsymbol{\phi})^2+(1/f)\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}$ as defined in Eq. (4.8). Using the identities $\mathcal{P}_T\partial_\mu\boldsymbol{\phi}=\partial_\mu\boldsymbol{\phi}$ and ${\rm tr}\mathcal{P}_T=N-1$ and the integral formula

$$\int\frac{d^Dp_E}{(2\pi)^D}\frac{p_E{}^{2s}}{(p_E{}^2+U)^n}=(4\pi)^{-D/2}U^{s-n+D/2}\frac{\Gamma\left[\frac{D}{2}+s\right]\Gamma\left[n-s-\frac{D}{2}\right]}{\Gamma\left[\frac{D}{2}\right]\Gamma(n)}\ , \tag{6.29}$$

we obtain the nonlinear σ model effective Lagrangian up to the four-derivative terms:

$$\mathcal{L}_{\text{eff}}^{\text{NL}} = \tfrac{1}{2}f^2(1+C_f^N)(\partial\phi)^2 + (1+C_\epsilon^N)f\epsilon\cdot\phi + \mathcal{L}_I + \frac{1}{4(4\pi)^2}\left[U_T{}^2 - \frac{1}{f}\epsilon\cdot\phi U_T - 2(\partial_\mu\phi\cdot\partial_\nu\phi)^2 + \tfrac{1}{2}(\partial\phi)^4 - \left[(N-3)U_T{}^2 - \frac{1}{3}(\partial\phi)^4 + \frac{4}{3}(\partial_\mu\phi\cdot\partial_\nu\phi)^2 + \frac{2}{f}\epsilon\cdot\phi U_T \right]\left[\ln\frac{U_T}{\mu^2} - \frac{2}{\bar\epsilon_N} \right] \right], \tag{6.30}$$

where $2/\bar\epsilon_N = 2/\bar\epsilon$ and $\mathcal{L}_I$ is the same as defined in Eq. (5.5).

We have also carried out the corresponding calculation using the nonmanifestly covariant method of Ref. 10 and obtained the same result. It is interesting that one can actually identify the corresponding expressions and contributions such as those in Eq. (6.28) between the two methods. The details of that calculation will be published elsewhere.

VII. COMPARISON BETWEEN THE LINEAR AND THE NONLINEAR σ MODELS

Now we are ready to make a direct comparison of the results between the $M \to \infty$ effective Lagrangian of the linear σ model, Eq. (5.4), and that of the nonlinear σ model, Eq. (6.30). They are in many ways very similar and yet there exist few terms which are distinctly different. The $\frac{1}{3}(\partial^2\phi)^2$ term in Eq. (5.4) cannot be generated from the nonlinear σ model. Since the parameters occur in different ways in both cases, fair comparison requires a careful analysis.

(1) The basic nonlinear σ model Lagrangian with the counterterms and $\mathcal{L}_I$, the first three terms in Eqs. (5.4) and (6.30) are the same. The coefficients of $\ln U_T$ are also identical. Therefore both effective Lagrangians contain the same infrared structure and divergence.

(2) The ultraviolet divergent terms of the linear σ model, the $\ln M^2$ term in Eq. (5.4), can be matched by the nonlinear σ model, Eq. (6.30), by identifying[5]

$$\ln\mu^2 + \frac{2}{\bar\epsilon_N} = \ln M^2 - K , \tag{7.1}$$

where K is a finite constant independent of M.

The difference between the two effective Lagrangians becomes

$$\mathcal{L}_{\text{eff}}^{\text{L}} - \mathcal{L}_{\text{eff}}^{\text{N}} = \frac{1}{4(4\pi)^2}\left[-(C+\tfrac{27}{2})U_T{}^2 - \frac{1}{f}\epsilon\cdot\phi U_T + \tfrac{11}{18}(\partial\phi)^4 - \tfrac{16}{9}(\partial_\mu\phi\cdot\partial_\nu\phi)^2 + \tfrac{1}{3}(\partial^2\phi)^2 + K\left[(N-3)U_T{}^2 - \tfrac{1}{3}(\partial\phi)^4 + \tfrac{4}{3}(\partial_\mu\phi\cdot\partial_\nu\phi)^2 + \frac{2}{f}\epsilon\cdot\phi U_T \right] \right]. \tag{7.2}$$

Equation (7.2) remains finite as $M^2 \to \infty$ or $\epsilon \to 0$.

(3) One can further eliminate the growing N dependence by defining

$$C + \tfrac{27}{2} = K(N-3) - A . \tag{7.3}$$

Equation (7.2) then becomes

$$\mathcal{L}_{\text{eff}}^{\text{L}} - \mathcal{L}_{\text{eff}}^{\text{N}} = \frac{1}{4(4\pi)^2}\left[-\frac{1}{f}\epsilon\cdot\phi U_T + \tfrac{1}{3}(\partial^2\phi)^2 + \tfrac{11}{18}(\partial\phi)^4 - \tfrac{16}{9}(\partial_\mu\phi\cdot\partial_\nu\phi)^2 + AU_T{}^2 + K\left[-\tfrac{1}{3}(\partial\phi)^4 + \tfrac{4}{3}(\partial_\mu\phi\cdot\partial_\nu\phi)^2 + \frac{2}{f}\epsilon\cdot\phi U_T \right] \right]. \tag{7.4}$$

The parameters A and K are strictly finite constants independent of N and M^2.

(4) Apart from the obvious term $\frac{1}{3}(\partial^2\phi)^2$ which cannot be eliminated by any choice of parameters A and K, there are four linearly independent terms but only two free parameters A and K. Therefore at most only two of the four linear independent terms can be eliminated. For example, one can eliminate the two invariant terms at the symmetry limit $\epsilon \to 0$ by fixing

$$K = \tfrac{4}{3}$$

and

$$A = -\tfrac{1}{6} .$$

Then, there is a net difference between the effective Lagrangian of the linear σ model at $m_\sigma \to \infty$ and the effective Lagrangian of the nonlinear σ model,

$$\mathcal{L}_{\text{eff}}^{\text{L}} - \mathcal{L}_{\text{eff}}^{\text{N}} = \frac{1}{4(4\pi)^2}\left[\frac{1}{3}(\partial^2\phi)^2 + \frac{4}{3f}\epsilon\cdot\phi(\partial\phi)^2 + \frac{3}{2f^2}(\epsilon\cdot\phi)^2 \right] \tag{7.5}$$

which is finite at $M^2 \to \infty$, $\epsilon \to 0$, and $N \to \infty$. In short the nonlinear σ model gives correctly the divergent parts of the effective Lagrangian of the linear σ model at $m_\sigma \to \infty$ up to the four-derivative terms in one-loop perturbation calculation.

(5) The unique term $[1/12(4\pi)^2](\partial^2 \boldsymbol{\phi})^2$ generates the meson-loop contribution to the pion vector form factor $[1/6(4\pi)^2]q^2/f_\pi^2$, for the linear σ model, while the nonlinear σ model has no meson-loop contribution to the pion vector form factor. In the SU(2)$\times$SU(2) or SO(4) σ model with the Yukawa coupling with quarks, the counterpart of the quark-loop contribution is $[1/(4\pi)^2](N_c/3)(\partial^2 \boldsymbol{\phi})^2$, where N_c is the number of colors.[23] The quark-loop contribution to the pion vector form factor is $[2/(4\pi)^2](N_c/3)(q^2/f_\pi^2)$.

(6) We have demonstrated by explicit calculation of the effective Lagrangians of the σ model that the two processes, (1) taking $m_\sigma \to \infty$ and (2) calculating the quantum corrections, do not commute. It is natural to ask what has been lost by taking the $m_\sigma \to \infty$ limit first to obtain the nonlinear σ model and why.

Fundamentally these two processes commute if and only if the decoupling theorem is applicable, which is definitely not the case for the linear σ model with spontaneous symmetry breaking. Consequently the loop contributions in which the heavy σ propagator forms part of or the entire loop cannot be absorbed completely by the renormalization of the physical constants and remain finite in the $m_\sigma \to \infty$ limit. By taking the $m_\sigma \to \infty$ limit first, one has essentially thrown out these contributions completely. From the dimension of the operator $(\partial^2 \boldsymbol{\phi})^2$, it is clear that this term is a natural and unique candidate for the decoupling violation.

From a more naive point of view, the $M^2 = -2m^2 \to \infty$ limit of the linear σ model Lagrangian Eq. (3.1) yields the nonlinear σ model Lagrangian Eq. (6.1) plus $O(1/M^2)$ terms such as $-(f^2/M^2)(\partial \boldsymbol{\phi})^4$. The one-loop contributions of these terms are linearly divergent. If one uses a reasonable cutoff M^2, the infinite factor cancels out to yield finite result, which is the origin of the $(\partial^2 \boldsymbol{\phi})^2$ term.

VIII. CONCLUSION

Is the nonlinear σ model the $m_\sigma \to \infty$ limit of the linear σ model? We have proposed to investigate this question by comparing the one-loop effective action expansions for both cases with the SO(N) symmetry group.

For this purpose we have introduced completely new background-field methods for the linear model and the nonlinear model. The calculation is manifestly covariant at all stages and it is not required to introduce formally the pion fields. The reduction to the pion fields is optional, only for the purpose of applications and comparisons, in contrast with the earlier works where the asymmetry has been introduced at the beginning stage in the definition of the pion fields. It is difficult to see how any noninvariant term can appear and therefore no noninvariant term is present in either model.

We have verified the substitution rule of Appelquist and Bernard[5] $1/\epsilon_N \to \ln M$ for the nonlinear σ model to reproduce the logarithmically divergent term of the linear σ model and have generalized it to include the symmetry-breaking term for arbitrary N. For $N=4$ and neglecting the symmetry-breaking effect, our results are in agreement except the noninvariant terms.

In addition to the divergent terms our calculation also includes all finite terms and the symmetry-breaking effects. The latter is important for the infrared problem. Our results show that both models exhibit the identical infrared behavior.

The leading large-N contribution can also be matched by appropriately chosen renormalization constants. Therefore the only differences between the two effective Lagrangians are the nonleading finite terms. Nonetheless the differences are real. They may be interpreted as the consequence of a violation of the decoupling. Whether the significance of these differences carries beyond the perturbation calculation is not very clear. However there may be some subtle difficulties not anticipated in the construction of the effective action for the nonlinear σ model due to the constraint, such that Eq. (6.30) may not be valid even to one-loop order Lagrangian.[24] The investigation of this aspect is presently in progress.

For the time being, the linear σ model and the nonlinear σ model are the only viable models to provide the spontaneous symmetry breaking urgently needed to understand any unified picture in particle physics. Even with the limited scope of applicability, our new method and analysis have taken a new small step toward the understanding of this fundamental problem.

ACKNOWLEDGMENTS

I would like to thank Professor J. Goldstone, Professor R. Jackiw, and members of the Center for Theoretical Physics at MIT for their hospitality during his visit at the Center. I am indebted to Dr. C. M. Fraser of Imperial College for checking my calculation and correcting a minor error. This work was supported in part through funds provided by the U.S. Department of Energy under Contracts Nos. DE-AC02-76ER03069 and DE-AS05-77ER05490.

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