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Derivative expansion and the induced Chern-Simons term at finite temperature in 2+1 dimensions

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We apply the method of derivative expansion of fermion determinants to compute the induced Chern-Simons term in (2+1)-dimensional field theories. The temperature dependence of the Chern-Simons term computed using real-time and imaginary-time methods is shown to be compatible with gauge invariance.

I. INTRODUCTION

Derivative expansion of the fermion determinant had been extensively used in recent literature[1] to study the low-energy properties of various theories. It has been used for example, to study Skyrmions,[2] to rederive the amplitudes for various anomaly-induced decays such as $\pi^0 \to 2\gamma$ (Ref. 3), to find out the temperature dependence of pion mass and decay constants,[4] to analyze the strongly interacting Higgs sector of the standard model, to study ordinary and supersymmetric nonlinear σ models, supergravity models,[5] and most recently the superstring models.[6] In the context of (1 + 1)-dimensional field theories, it has been recently employed to demonstrate the solubility of various two-dimensional models and to establish the temperature independence of the Abelian chiral anomaly.[7] In this paper, we apply this method to compute the induced Chern-Simons term at zero temperature as well as at finite temperature, for Abelian and non-Abelian gauge theories in 2 + 1 dimensions.

Recently, (2 + 1)-dimensional gauge theories[8] having a Chern-Simons term have attracted much attention in the literature.[9] In the presence of a Chern-Simons term the gauge fields are massive and in case of a non-Abelian gauge theory, gauge invariance under the large gauge transformations implies quantization of the mass parameter. To be explicit, in Euclidean space (our metric is given by $g_{11}=g_{22}=g_{33}=1$) the Chern-Simons (CS) part of the action is given by

$$S_{\rm CS}=iM\int d^3x\,\epsilon^{\mu\nu\rho}{\rm tr}(A^\mu\partial^\nu A^\rho-\tfrac{2}{3}igA^\mu A^\nu A^\rho)\ , \qquad (1)$$

where tr stands for trace over the internal indices and g for the gauge coupling constant. Under a large gauge transformation

$$A_\mu \to A'_\mu = U^{-1}A_\mu U + ig^{-1}U^{-1}\partial_\mu U$$

and

$$S_{\rm CS}\to S_{\rm CS}+2\pi i\left[\frac{4\pi M}{g^2}\right]W\ , \qquad (2)$$

where $U=e^{i\theta\cdot T}$, T^a's are the generators of the gauge transformation, and

$$W=\frac{1}{24\pi^2}\int d^3x\,\epsilon_{\mu\nu\rho}{\rm tr}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}(\partial_\rho U)U^{-1}$$

is the winding number which is an integer.[10] Hence for the path integral to be well defined,

$$4\pi M/g^2=\text{integer}\ . \qquad (3)$$

This is the origin of mass quantization.

These topologically massive gauge theories also have been considered at finite temperature[11] and for the non-Abelian gauge theories it has been argued using topological gauge invariance that the mass parameter should be temperature independent. However, for Abelian gauge theories the Chern-Simons term does not have any topological significance and hence *a priori* its temperature dependence is unknown. Furthermore, it is well known that in gauge theories with fermions, such a topological mass term is induced by their interactions,[12] and the long-distance effective gauge theory contains a CS term. In light of these results, it is interesting to see if the CS term is temperature dependent or not. We show that the induced CS term depends on temperature in such a way that the topological gauge invariance of the total action can be maintained.

The paper is organized as follows. In Sec. II we describe the derivative-expansion scheme and apply it to compute the induced Chern-Simons densities at zero temperature, for Abelian as well as non-Abelian theories. Then we proceed in Sec. III to establish their temperature dependence using techniques of finite-temperature field theory. Finally we have some concluding remarks in Sec. IV.

II. DERIVATIVE-EXPANSION SCHEME AND THE INDUCED CHERN-SIMONS TERM AT ZERO TEMPERATURE

Let us consider the case of a massive fermion field interacting with an external gauge field in three space-time dimensions. For convenience we will work in Euclidean space throughout this paper. The Euclidean action is given by

$$S=\int\bar\psi(x)(i\partial\!\!\!/+gA\!\!\!/+m)\psi(x)d^3x\ . \qquad (4)$$

The corresponding generating functional is

$$Z[A_\mu]=\int D\bar{\psi}(x)D\psi(x) \times \exp\left[-\int \bar{\psi}(x)(i\partial\!\!\!/+gA\!\!\!/+m)\psi(x)d^3x\right] . \tag{5}$$

A_μ stands for $A_\mu^a T^a$, where T^a's represent the generators of the gauge transformations in the representation of the fermions. We will choose to work with two-component Dirac spinors. The (Euclidean) Dirac algebra

$$\{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu}$$

is then satisfied by the matrices

$$\gamma_1=i\sigma_1,\quad \gamma_2=i\sigma_2,\quad \gamma_3=i\sigma_3 ,$$

where σ_i are the Pauli matrices. Furthermore, they satisfy

$$\begin{aligned}&\gamma_\mu\gamma_\nu=-\delta_{\mu\nu}-\epsilon_{\mu\nu\rho}\gamma_\rho ,\\ &\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho)=2\epsilon_{\mu\nu\rho} ,\\ &\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_\sigma)=-2(\delta_{\mu\nu}\epsilon_{\rho\lambda\sigma}+\delta_{\rho\lambda}\epsilon_{\mu\nu\sigma}\\ &\qquad -\delta_{\nu\sigma}\epsilon_{\rho\lambda\mu}+\delta_{\mu\sigma}\epsilon_{\rho\lambda\nu}) ,\end{aligned} \tag{6}$$

where we have written down the trace identities needed for future calculations.

Integration over the fermion fields yields

$$Z[A_\mu]=\mathrm{Det}(p\!\!\!/+m+gA\!\!\!/)=e^{-S_{\text{eff}}} . \tag{7}$$

Since the field A_μ is external, no Legendre transformation is required to go from the connected vacuum functional to the effective action. Here

$$\begin{aligned}S_{\text{eff}}(A_\mu)&=-\ln \mathrm{Det}(p\!\!\!/+m+gA\!\!\!/)\\ &=-\mathrm{Tr}\ln(p\!\!\!/+m+gA\!\!\!/) .\end{aligned} \tag{8}$$

Here the trace stands for the trace over Dirac matrices, trace over the internal space as well as for the integrations in momentum and coordinate spaces. This expression for the effective action cannot, in general, be evaluated and written as

$$S_{\text{eff}}=\int d^3x\, L_{\text{eff}} . \tag{9}$$

$A_\mu(x)$'s being position-dependent variables do not commute with functions of momentum and it is not at all obvious how to separate out the momentum- and space-dependent quantities, and carry out the integration in respective spaces formally indicated by the trace operation. To that end we will use the techniques of derivative expansion[1] and proceed as follows:

$$\begin{aligned}S_{\text{eff}}(A_\mu)&=-\mathrm{Tr}\ln(p\!\!\!/+m+gA\!\!\!/)\\ &=-\mathrm{Tr}\ln(p\!\!\!/+m)-\mathrm{Tr}\ln\left[1+\frac{gA\!\!\!/}{p\!\!\!/+m}\right] .\end{aligned} \tag{10}$$

Throwing away the uninteresting gauge field-independent first term, we concentrate on the Matthews-Salam determinant.[13] The new effective action can be expanded in a power series. Denoting

$$S_f(p)=\frac{p\!\!\!/-m}{p^2+m^2} ,$$

we have

$$S_{\text{eff}}(A_\mu)=\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}[S_f(p)gA\!\!\!/]^n . \tag{11}$$

We shall use the identities given below to disentangle the x and p traces. In general, for any function $\phi(x)$, we have

$$\begin{aligned}&\phi(x)p_\mu=p_\mu\phi(x)-i\partial_\mu\phi(x) ,\\ &[p^2,\phi(x)]=\Box\phi(x)+2ip_\mu\partial_\mu\phi(x) ,\end{aligned} \tag{12}$$

where $p^2=p_\mu p_\mu$ and $\Box=\partial_\mu\partial_\mu$. Using these identities and writing a heat-kernel representation for the propagator it can be proved that

$$\phi(x)\frac{1}{p^2+m^2}=\frac{1}{(p-i\partial)^2+m^2}\phi(x) , \tag{13}$$

where the operator ∂ acts on $\phi(x)$. A similar integral representation for the δ function yields

$$\phi(x)\delta(p^2+m^2)=\delta[(p-i\partial)^2+m^2]\phi(x) . \tag{14}$$

We will use these results later for our calculations. Using the above identities, each term in the expansion can be written in the form

$$\int d^3p\, F(p^2)\int d^3x\, G(A_\mu(x)) ,$$

and hence the momentum integration can be carried out yielding the effective action. Analyzing the effective action, it is clear that the contributions to the Chern-Simons term will come from terms quadratic and cubic in A_μ.

Consider the terms quadratic in A_μ. The corresponding action is given by

$$S_{\text{eff}}^{(2)}=\frac{1}{2}g^2\frac{p\!\!\!/-m}{p^2+m^2}A\!\!\!/\frac{p\!\!\!/-m}{p^2+m^2}A\!\!\!/ . \tag{15}$$

The terms having two and four γ matrices contribute to the wave-function renormalization of the gauge boson and hence are not relevant for the present purpose. Evaluation of the other two terms linear in mass yields

$$S_{\text{eff}}^{(2)\text{CS}}=-\tfrac{1}{2}g^2 m\,\text{Tr}\left[\frac{\not p}{p^2+m^2}A\frac{1}{p^2+m^2}A+\frac{1}{p^2+m^2}A\frac{\not p}{p^2+m^2}A\right]$$

$$=-\frac{mg^2}{2}\text{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\text{tr}\left[\frac{p_\mu}{p^2+m^2}A_\nu\frac{1}{p^2+m^2}A_\rho+\frac{1}{p^2+m^2}A_\mu\frac{p_\nu}{p^2+m^2}A_\rho\right]. \quad (16)$$

where Tr_D stands for trace over Dirac γ matrices. Commuting the A_μ's past the momentum operators to the right,

$$S_{\text{eff}}^{(2)\text{CS}}=-\frac{mg^2}{2}\text{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\text{tr}\left[\frac{1}{p^2+m^2}\frac{1}{(p-i\partial)^2+m^2}(p_\mu A_\nu+p_\nu A_\mu-i\partial_\nu A_\mu A_\rho)\right]. \quad (17)$$

Since the Dirac trace is antisymmetric in μ and ν, the only nonzero contribution is from the $\partial_\nu A_\mu A_\rho$ term. Carrying out the momentum integration yields

$$S_{\text{eff}}^{(2)\text{CS}}=-\frac{ig^2}{8\pi}\frac{m}{|m|}\text{tr}\int\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho d^3x\ . \quad (18)$$

We note here that a mass term in (2 + 1)-dimensional field theory violates parity[14] and hence its sign is crucial in the effective Lagrangian.

The additional contribution to the CS term coming from the terms cubic in A_μ's can be written as

$$S_{\text{eff}}^{(3)}=\tfrac{1}{3}g^3\text{Tr}\frac{\not p-m}{p^2+m^2}A\frac{\not p-m}{p^2+m^2}A\frac{\not p-m}{p^2+m^2}A\ . \quad (19)$$

Noticing that the terms having even number of γ matrices do not contribute to the CS term, we have

$$S_{\text{eff}}^{(3)\text{CS}}=-\tfrac{1}{3}mg^3\text{Tr}\left[\frac{\not p}{p^2+m^2}A\frac{\not p}{p^2+m^2}A\frac{1}{p^2+m^2}A+\frac{\not p}{p^2+m^2}A\frac{1}{p^2+m^2}A\frac{\not p}{p^2+m^2}A\right.$$
$$\left.+\frac{1}{p^2+m^2}A\frac{\not p}{p^2+m^2}A\frac{\not p}{p^2+m^2}A+m^2\frac{1}{p^2+m^2}A\frac{1}{p^2+m^2}A\frac{1}{p^2+m^2}A\right]. \quad (20)$$

Furthermore, using the identity

$$\phi(x)\frac{1}{p^2+m^2}=\frac{1}{p^2+m^2}\phi(x)+\frac{[p^2,\phi(x)]}{(p^2+m^2)^2}+\frac{[p^2,[p^2,\phi(x)]]}{(p^2+m^2)^3}+\cdots \quad (21)$$

and noting that only the first term will contribute to the CS term (all other terms have at least one derivative in them), we get

$$S_{\text{eff}}^{(3)\text{CS}}=-\frac{mg^3}{3}\left[\text{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_\sigma\text{tr}\left[\frac{p_\mu p_\rho}{(p^2+m^2)^3}A_\nu A_\lambda A_\sigma+\frac{p_\mu p_\lambda}{(p^2+m^2)^3}A_\nu A_\rho A_\sigma+\frac{p_\nu p_\lambda}{(p^2+m^2)^3}A_\mu A_\rho A_\sigma\right]\right.$$
$$\left.+m^2\text{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\text{tr}\frac{1}{(p^2+m^2)^3}A_\mu A_\nu A_\rho\right]. \quad (22)$$

The momentum integrals are all finite and hence can be evaluated easily. Keeping terms without any derivatives, we have

$$S_{\text{eff}}^{(3)\text{CS}}=-\frac{g^3}{96\pi^2}\frac{m}{|m|}\left[\text{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_\sigma\left[\delta_{\mu\rho}\text{tr}\int d^3x\ A_\nu A_\lambda A_\sigma+\delta_{\mu\lambda}\text{tr}\int d^3x\ A_\nu A_\rho A_\sigma\right.\right.$$
$$\left.\left.+\delta_{\nu\lambda}\text{tr}\int d^3x\ A_\mu A_\rho A_\sigma\right]+2\,\text{tr}\int d^3x\ \epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right]$$
$$=-\frac{g^3}{12\pi}\frac{m}{|m|}\text{tr}\int d^3x\ \epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\ . \quad (23)$$

Combining the above expression with Eq. (18) we have the induced Chern-Simons term as

$$S_{\text{eff}}^{\text{CS}}=-\frac{ig^2}{8\pi}\frac{m}{|m|}\text{tr}\int d^3x\ \epsilon_{\mu\nu\rho}(\partial_\mu A_\nu A_\rho-\tfrac{2}{3}igA_\mu A_\nu A_\rho) \quad (24)$$

in agreement with the standard results.[8] The Abelian Chern-Simons term corresponds to the first term in the above expression.

III. INDUCED CHERN-SIMONS TERM AT FINITE TEMPERATURE

We will make use of both imaginary-[15] and real-time[16] formalism to consider the finite-temperature effects.[17] In the imaginary-time method, the finite-temperature generalization is given by replacing the integration over the continuous variable p_3 by a summation over the discrete values $p_3=(n+1/2)2\pi/\beta$, where $\beta=1/kT$ and n is an integer. Considering the quadratic and cubic terms in A separately, we have for the quadratic term

$$S_{\rm eff}^{(2)\rm CS}=\frac{-mg^2}{2}\mathrm{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\sum_{n=-\infty}^{\infty}\frac{1}{\beta}\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m^2}\frac{1}{(p-i\partial)^2+m^2}\mathrm{tr}\int d^3x(-i)\partial_\nu A_\mu A_\rho\ . \tag{25}$$

Furthermore, noticing that the ∂ in the denominator of the above expression would not contribute to the CS term, Eq. (25) simplifies to

$$\begin{aligned}S_{\rm eff}^{(2)\rm CS}&=img^2\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m^2)^2}\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\nu A_\mu A_\rho\\&=\sum_{n=-\infty}^{\infty}\frac{img^2\beta}{16\pi^3}\frac{1}{(n+\frac{1}{2})^2+\left[\frac{m\beta}{2\pi}\right]^2}\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\nu A_\mu A_\rho\\&=-\frac{im}{|m|}\frac{g^2}{8\pi}\tanh\left[\frac{|m|\beta}{2}\right]\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho\ .\end{aligned} \tag{26}$$

The cubic term in A can be considered in a similar fashion and we have

$$\begin{aligned}S_{\rm eff}^{(3)\rm CS}=-\frac{mg^3}{3}\Bigg[&\mathrm{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_\sigma\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\Bigg[\int\frac{d^2p}{(2\pi)^2}\frac{p_\mu p_\rho}{(p^2+m^2)^3}\mathrm{tr}\int d^3x\ A_\nu A_\lambda A_\sigma\\&+\int\frac{d^2p}{(2\pi)^2}\frac{p_\mu p_\lambda}{(p^2+m^2)^3}\mathrm{tr}\int d^3x A_\nu A_\rho A_\sigma\\&+\int\frac{d^2p}{(2\pi)^2}\frac{p_\nu p_\lambda}{(p^2+m^2)^3}\mathrm{tr}\int d^3x\ A_\mu A_\rho A_\sigma\Bigg]\\&+m^2\mathrm{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m^2)^3}\mathrm{tr}\int d^3x\ A_\mu A_\nu A_\rho\Bigg]\end{aligned} \tag{27}$$

where we have neglected the ∂'s in the denominator because they would not contribute to the CS term. Proceeding as in the $T=0$ case we have

$$S_{\rm eff}^{(3)\rm CS}=-\frac{2mg^3}{3}\frac{1}{\beta}\sum_n\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m^2)^2}\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\ . \tag{28}$$

Following the evaluation of the Abelian term we perform the momentum integration and sum the infinite series to get

$$S_{\rm eff}^{(3)\rm CS}=-\frac{mg^3}{|m|}\frac{1}{12\pi}\tanh\left[\frac{|m|\beta}{2}\right]\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\ . \tag{29}$$

Hence, at finite temperature,

$$S_{\rm eff}^{\rm CS}=-\frac{ig^2}{8\pi}\frac{m}{|m|}\tanh\left[\frac{|m|\beta}{2}\right]\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\rho}(\partial_\mu A_\nu A_\rho-\tfrac{2}{3}igA_\mu A_\nu A_\rho)\ . \tag{30}$$

It is gratifying to see that the quadratic and cubic terms in A have the same temperature dependence essential for gauge invariance.

It is worthwhile to prove the above result using the methods of real-time formalism. In this formalism, one introduces the so-called tilde particles[16] to properly take into account the higher-loop effects and the finite-temperature propagators are obtained by a Bogoliubov transformation. For fermions

$$S(p)=U_F(|p_3|)S_0(p)U_F(|p_3|)^\dagger=S_0(p)+S_\beta(p)\ , \tag{31}$$

where $S(p)$ is the propagator at finite temperature and $S_0(p)$ is the zero-temperature propagator. Here, the unitary operator

$$U_F(\,|p_3|\,)=\frac{1}{(e^{\beta|ip_3|}+1)^{1/2}}\begin{bmatrix} e^{\beta|ip_3|/2} & 1 \\ -1 & e^{\beta|ip_3|/2}\end{bmatrix}, \tag{32}$$

and

$$S_\beta(p)=-2\pi i\delta(p^2+m^2)(\not p-m)[f(p_3)]\ . \tag{33}$$

Because of the presence of tilde particles the distribution function is a matrix and is given by

$$[f(p_3)]=\frac{1}{e^{\beta|ip_3|}+1}\begin{bmatrix} 1 & e^{\beta|ip_3|/2} \\ e^{\beta|ip_3|/2} & -1\end{bmatrix}. \tag{34}$$

It is worth mentioning that at one-loop level tilde particles completely decouple and do not contribute to the physical processes. Neglecting ∂'s from $U_F(\,|p_3|\,)$ and the denominator for the present purpose, we find the quadratic term to be

$$\begin{aligned} S_{\text{eff}}^{(2)\text{CS}}&=-\frac{mg^2}{2}\text{Tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\int\frac{d^3p}{(2\pi)^3}U_F(\,|p_3|\,)\frac{1}{p^2+m^2}U_F^\dagger(\,|p_3|\,)U_F(\,|p_3|\,)\frac{1}{p^2+m^2}U_F^\dagger(\,|p_3|\,)(-i)\int \text{tr}\,d^3x\,\partial_\nu A_\mu A_\rho\\ &=-img^2\int\frac{d^3p}{(2\pi)^3}U_F(\,|p_3|\,)\frac{1}{(p^2+m^2)^2}U_F^\dagger(\,|p_3|\,)\text{tr}\int\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho d^3x\ . \end{aligned} \tag{35}$$

To avoid terms having a product of δ functions, we used the following procedure:

$$\begin{aligned} S_{\text{eff}}^{(2)\text{CS}}&=img^2\frac{\partial}{\partial m^2}\int\frac{d^3p}{(2\pi)^3}U_F(\,|p_3|\,)\frac{1}{p^2+m^2}U_F^\dagger(\,|p_3|\,)\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho\\ &=img^2\frac{\partial}{\partial m^2}\int\frac{d^3p}{(2\pi)^3}\left[\frac{1}{p^2+m^2}-2\pi i\delta(p^2+m^2)f(p_3)\right]\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho\\ &=-img^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{(p^2+m^2)^2}\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho+\frac{mg^2}{(2\pi)^2}\frac{\partial}{\partial m^2}\int d^3p\,\delta(p^2+m^2)f(p_3)\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho\ . \end{aligned} \tag{36}$$

Since we are only computing the one-loop contribution, we have dropped the contribution of the tilde particles from the above expression.

The integral involving the δ function can be calculated in a straightforward way and we get

$$\int d^3p\,\delta(p^2+m^2)f(p_3)=2\pi i\,|m|-\frac{2\pi i}{\beta}\ln(e^{\beta|m|}+1)\ . \tag{37}$$

Substituting the above expression in Eq. (36) we have

$$\begin{aligned} S_{\text{eff}}^{(2)\text{CS}}&=\left[\frac{img^2}{|m|\,8\pi}-\frac{img^2}{4\pi|m|}\frac{e^{\beta|m|}}{e^{\beta|m|}+1}\right]\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho\\ &=\frac{img^2}{|m|\,8\pi}\left[1-\frac{2e^{\beta|m|}}{e^{\beta|m|}+1}\right]\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho=-\frac{img^2}{|m|\,8\pi}\tanh\left[\frac{\beta|m|}{2}\right]\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}\partial_\mu A_\nu A_\rho\ . \end{aligned} \tag{38}$$

A straightforward calculation similar to the above one yields for the cubic term

$$S_{\text{eff}}^{(3)\text{CS}}=-\frac{mg^3}{|m|\,12\pi}\tanh\left[\frac{\beta|m|}{2}\right]\text{tr}\int\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho d^3x\ . \tag{39}$$

Combining the quadratic and cubic terms in A,

$$S_{\text{eff}}^{\text{CS}}=-\frac{ig^2}{8\pi}\frac{m}{|m|}\tanh\left[\frac{\beta|m|}{2}\right]\text{tr}\int d^3x\,\epsilon_{\mu\nu\rho}(\partial_\mu A_\nu A_\rho-\tfrac{2}{3}igA_\mu A_\nu A_\rho)\ . \tag{40}$$

The real- and the imaginary-time methods yield the same answer as they should.

IV. CONCLUSION

Our analysis yields a temperature-dependent Chern-Simons term in both Abelian and non-Abelian gauge theories. While temperature dependence of the Abelian CS term has been noticed earlier[17] the results for the non-Abelian theory are new. The temperature dependence of the non-Abelian CS term needs some clarification. As has been noticed earlier[12] the effective action for gauge fields due to fermions is not invariant under topologically nontrivial gauge transformations. The change in the action is given by $\pm\pi\,|\,n\,|\,i$ where n is the winding number of the gauge transformation. This is true for a theory having an odd number of fermions. To restore gauge invariance one adds a parity-violating CS term to the action and the renormalized effective action is given by

$$S^R_{\text{eff}}=S_{\text{eff}}(\text{finite})\pm S_{\text{CS}} , \tag{41}$$

where S_{CS} is the induced CS action calculated earlier. Hence, although it appears that the CS term is not gauge invariant under a large gauge transformation and has developed a temperature dependence, in the context of the above discussion the total action can be made gauge invariant by choosing the minus sign in front of S_{CS}.

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