Connection of some two-dimensional bosonic and fermionic models to scalar curvature

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It is found that the equations of motion of a variety of two-dimensional bosonic and fermionic models are a consequence of the constancy of the scalar curvature.

Models of field theory are very useful laboratories for understanding the dynamics of fields in a relatively simple manner. But there are very many such models and it will be very useful to find a common origin for them. The bosonic models, Liouville, sine-Gordon, and nonlinear σ models in two dimensions, were previously obtained from a common Lagrangian.[1] Here we attempt to obtain the equations of motion of both bosonic and fermionic models when the scalar curvature is a constant. In order to minimize the complexity of these various models, we limit ourselves to two dimensions. Let g_{AB} ($A,B=0,1$) be the metric on a two-dimensional manifold, where the indices 0 and 1 refer to time and space, respectively. We parametrize the metric as

$$g_{AB}=\begin{bmatrix} P & Q \\ Q & R \end{bmatrix}, \quad g=PR-Q^2 , \tag{1}$$

where P, Q, and R are functions of x and t. The Riemann curvature tensor is given by

$$R_{ABCD}=\frac{1}{2}\left[\frac{\partial^2 g_{AD}}{\partial x^B \partial x^C}+\frac{\partial^2 g_{BC}}{\partial x^A \partial x^D}-\frac{\partial^2 g_{BD}}{\partial x^A \partial x^C}-\frac{\partial^2 g_{AC}}{\partial x^B \partial x^D}\right]+(\Gamma_{FAD}\Gamma^F_{BC}-\Gamma_{FAC}\Gamma^F_{BD}) , \tag{2}$$

where Γ is the affine connection given by

$$\Gamma^A_{BC}=\tfrac{1}{2}g^{AE}\left[\frac{\partial g_{EB}}{\partial x^C}+\frac{\partial g_{EC}}{\partial x^B}-\frac{\partial g_{BC}}{\partial x^E}\right] . \tag{3}$$

In two dimensions there is only one independent component of the Riemann tensor that must be proportional to the scalar curvature r. So we have[2]

$$r=\frac{2}{g}R_{0101} . \tag{4}$$

The scalar curvature r can then be written after some algebraic manipulations as

$$r=-\frac{1}{g^{1/2}}\left[\left[\frac{P_x-Q_t}{g^{1/2}}\right]_x+\left[\frac{R_t-Q_x}{g^{1/2}}\right]_t\right]+\frac{1}{2g^2}\begin{vmatrix} P & Q & R \\ P_x & Q_x & R_x \\ P_t & Q_t & R_t \end{vmatrix} , \tag{5}$$

where the last term is a determinant and the subscripts denote partial derivatives. In the models discussed in this paper, the determinant term does not contribute.

We first obtain the well-known Liouville and sine-Gordon models. We note that for

$$g_{AB}=\begin{bmatrix} e^\rho & 0 \\ 0 & e^\rho \end{bmatrix} \tag{6}$$

the Liouville equation

$$\tfrac{1}{2}(\rho_{xx}+\rho_{tt})=e^\rho \tag{7}$$

results from Eqs. (5) and (6) for $r=-2$. For the metric

$$g_{AB}=\begin{bmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{bmatrix} , \tag{8}$$

one obtains, from Eqs. (5) and (8) when $r=-2$,

$$\alpha_{tx}=\sin\alpha , \tag{9}$$

and from

$$g_{AB}=\begin{bmatrix} -\mu^2\cos^2\frac{u}{2} & 0 \\ 0 & -\mu^2\sin^2\frac{u}{2} \end{bmatrix} , \tag{10}$$

$$u_{tt}-u_{xx}+\mu^2\sin u=0 , \tag{11}$$

results from Eqs. (5) and (10) when $r=-2$.

The constant value of r can be adjusted. Crampin, Pirani, and Robinson[3] obtained Eq. (9) with $r=0$ and Dolan[4] obtained Eqs. (6) and (9) with $r=-2$. In the case of two-dimensional gravity take

$$g_{AB}=\begin{bmatrix} 1 & 0 \\ 0 & e^2 \end{bmatrix} , \tag{12}$$

where $e\equiv e^1_1$ is a component of a zweibein. We obtain, from Eqs. (5) and (12),

$$\ddot{e}/e=-r/2 \ . \tag{13}$$

This is the equation of motion in the synchronous gauge.[5] When we compare Eq. (13) with the corresponding equation $\ddot{e}/e=-2\lambda$, we note that λ is proportional to r.

Let us next consider another bosonic model that leads to a one-dimensional Schrödinger equation. Take the metric

$$g_{AB}=\begin{bmatrix} \frac{m^2}{2}(1+u^2) & \frac{m^2}{2}(1-u^2) \\ \frac{m^2}{2}(1-u^2) & \frac{m^2}{2}(1+u^2) \end{bmatrix} , \tag{14}$$

in which case we obtain, from Eqs. (5) and (14),

$$u_{xx}+2u_{xt}+u_{tt}+m^2ru=0 \ . \tag{15}$$

Rotate the axis as follows:

$$x'=x\cos\theta+t\sin\theta \ ,$$
$$t'=-x\sin\theta+t\cos\theta \ . \tag{16}$$

In order to remove the u_{xt} term, we take $\sin\theta=\cos\theta=1/\sqrt{2}$ and obtain, from Eqs. (15) and (16),

$$\frac{d^2u}{dx^2}+m^2ru=0 \ . \tag{17}$$

For constant r, this is the one-dimensional Schrödinger equation with a constant potential. But if we allow a functional form for r, then Eq. (17) describes a general one-dimensional Schrödinger equation. It should be noted that we started in two dimensions, but ended up with a dynamical equation in one dimension. This phenomena might be more general and may be applicable to higher dimensions. It should also be observed that the potential and curvature become related in the Schrödinger interpretation of Eq. (17).

So far, the previous models were all bosonic and, hence, the equations of motion involved second derivatives with respect to space and time. We now want to apply the above methods to the fermionic models. These involve equations of motion with only the first derivatives. From Eq. (5) we expect to obtain derivatives of these equations. In some special case it might be possible to obtain the equations themselves after one integration. We will show that such is the case with the fermionic sector of the massless Schwinger model whose Lagrangian density is given by

$$L(x)=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(x)[i\partial\!\!\!/-e\gamma^\mu A_\mu(x)]\psi(x) \ , \tag{18}$$

where $F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)$ and $A_\mu(x)$ is the vector potential. In two dimensions we take

$$\gamma^0=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^1=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \psi=\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} ; \tag{19}$$

and the equations of motion for the fermionic fields in the axial gauge[6] $[A_1(x,t)=0;\ A_0(x,t)=A(x,t)]$ are

$$\psi_{1t}-\psi_{1x}=-ieA\psi_1 \ , \tag{20a}$$

$$\psi_{2t}+\psi_{2x}=-ieA\psi_2 \ . \tag{20b}$$

In order to obtain Eq. (20b), consider the metric

$$g_{AB}=\begin{bmatrix} P & P \\ P & -P \end{bmatrix} . \tag{21}$$

We find, from Eqs. (5) and (18) with $r=0$,

$$\left[\frac{-P_t+P_x}{P}\right]_x=\left[\frac{P_t+P_x}{P}\right]_t . \tag{22}$$

Equation (22) can be written for $P=\psi_2$ as

$$\psi_{2t}-\psi_{2x}=-C(t)\psi_2 \ , \tag{23a}$$

$$\psi_{2t}+\psi_{2x}=D(x)\psi_2 \ , \tag{23b}$$

where $C(t)$ and $D(x)$, respectively, are functions of t and x only. If we identify $D(x)=-ieA(x)$, then Eq. (23b) becomes identical to Eq. (20b) in the static case.

Now we want to show that there are several fermionic models whose equations of motion can be combined to give the same equation involving scalar currents. The models we have looked at are the following.

(i) Free fermions. The Lagrangian density is

$$L(x)=\bar{\psi}(i\partial\!\!\!/-m)\psi \ , \tag{24}$$

and the equations of motion are

$$\psi_{1t}-\psi_{1x}=-im\psi_2 \ , \tag{25a}$$

$$\psi_{2t}+\psi_{2x}=-im\psi_1 \ . \tag{25b}$$

(ii) Massive Schwinger model. The Lagrangian density is

$$L(x)=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\partial\!\!\!/-m-eA\!\!\!/)\psi \ , \tag{26a}$$

and the equations of motion are

$$\psi_{1t}-\psi_{1x}=-i[m\psi_2+e(A_0-A_1)\psi_1] \ , \tag{26b}$$

$$\psi_{2t}+\psi_{2x}=-i[m\psi_1+e(A_0+A_1)\psi_2] \ . \tag{26c}$$

(iii) Massive Thirring model. The Lagrangian density is

$$L(x)=\bar{\psi}(i\partial\!\!\!/-m)\psi-\frac{\lambda}{2}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) \ , \tag{27a}$$

and the equations of motion are

$$\psi_{1t}-\psi_{1x}=-i(2\lambda\psi_1S_2+m\psi_2) \ , \tag{27b}$$

$$\psi_{2t}+\psi_{2x}=-i(2\lambda S_1\psi_2+m\psi_1) \ , \tag{27c}$$

where

$$S_1=\psi_1^*\psi_1, \quad S_2=\psi_2^*\psi_2 \ . \tag{28}$$

(iv) Massive Thirring Schwinger model. The Lagrangian density is

$$L(x)=\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/-m)\psi-\frac{\lambda}{2}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{29a}$$

and the equations of motion are

$$\psi_{1t}-\psi_{1x}=-i[2\lambda S_2\psi_1+m\psi_2+e(A_0-A_1)\psi_1]\ , \tag{29b}$$

$$\psi_{2t}+\psi_{2x}=-i[2\lambda S_1\psi_2+m\psi_1+e(A_0+A_1)\psi_2]\ . \tag{29c}$$

(v) Gross-Neveu model (single flavor). The Lagrangian density is

$$L(x)=\bar{\psi}(i\partial\!\!\!/-m)\psi-\frac{\lambda}{2}(\bar{\psi}\psi)(\bar{\psi}\psi)\ , \tag{30a}$$

and the equations of motion are

$$\psi_{1t}-\psi_{1x}=-i[m+\lambda(\psi_1^*\psi_2+\psi_2^*\psi_1)]\psi_2\ , \tag{30b}$$

$$\psi_{2t}+\psi_{2x}=-i[m+\lambda(\psi_1^*\psi_2+\psi_2^*\psi_1)]\psi_1\ . \tag{30c}$$

All these models give the same scalar current equation which is derived here for the massive Thirring model.

Take the equation of motion for ψ_1 and its complex conjugate,

$$\psi_{1t}-\psi_{1x}=-i(2\lambda S_2\psi_1+m\psi_2)\ , \tag{31a}$$

$$\psi_{1t}^*-\psi_{1x}^*=i(2\lambda\psi_1^*S_2+m\psi_2^*)\ . \tag{31b}$$

Multiply the top equation by ψ_1^* from the left, the bottom equation by ψ_1 from the right, add both terms, and make use of the anticommutation relations of ψ_1 and ψ_2. Then we obtain

$$S_{1t}-S_{1x}=-im(\psi_1^*\psi_2-\psi_2^*\psi_1)\ . \tag{32}$$

Similar operations with ψ_2 equations give

$$S_{2t}+S_{2x}=im(\psi_1^*\psi_2-\psi_2^*\psi_1)\ . \tag{33}$$

Comparison with Eq. (32) immediately gives

$$-S_{1t}+S_{1x}=S_{2t}+S_{2x}\ . \tag{34}$$

All the models considered above lead to this equation. Now consider generally the metric with $P=0$,

$$g_{AB}=\begin{bmatrix}0 & Q\\ Q & R\end{bmatrix}\ . \tag{35}$$

Equations (5) and (35) with $r=0$ give

$$Q_t(2Q_x-R_t)-Q(2Q_x-R_t)_t=0\ . \tag{36}$$

This equation is integrated to give

$$2Q_x-R_t=E(x)Q\ , \tag{37}$$

where $E(x)$ is an integration function independent of t. When $E(x)=0$ we obtain

$$2Q_x-R_t=0\ . \tag{38}$$

Specifically, for the metric

$$g_{AB}=\begin{bmatrix}0 & \frac{1}{2}(S_1-S_2)\\ \frac{1}{2}(S_1-S_2) & S_1+S_2\end{bmatrix}\ , \tag{39}$$

Eq. (38) yields precisely Eq. (34). If we define

$$j^\mu=\bar{\psi}\gamma^\mu\psi\ , \tag{40}$$

then Eq. (34) is nothing but

$$\partial_\mu j^\mu=0\ , \tag{41}$$

which is the statement of current conservation. What has been shown here is that Eq. (41) can be related to the vanishing of the scalar curvature in a certain two-dimensional manifold.

It can be shown that a general expression consistent with $\partial_\mu j^\mu=0$ is given by

$$\psi_{1t}-\psi_{1x}+i(f_1\psi_1+f_3\psi_2)=0\ ,$$

$$\psi_{2t}+\psi_{2x}+i(f_2\psi_2+f_3\psi_1)=0\ .$$

When we assign different values to the functions f_1, f_2, and f_3, the equations of motion for different models can be recovered. Unlike bosonic models, the equations of motion for fermionic models cannot be directly obtained from the metric.

Recently it has been pointed out that many equations in physics have a geometrical integrability origin.[7] It is expected that the present approach will provide an alternative method for gaining a deeper understanding of these nonlinear equations.

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