Exact solutions of the Dirac equation in spatially flat Robertson-Walker space-times

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Exact solutions of the Dirac equation (including the neutrino case, $m=0$) for three models of expanding universes are given. Gordon decomposition of the current is discussed.

I. INTRODUCTION

The behavior of relativistic particles obeying the Dirac equation in curved spaces, in particular in expanding universes, is of considerable importance in astrophysics and cosmology. Such investigations go back to Fock,[1] Tetrode,[2] Schrödinger,[3,4] and McVittie.[5] A general discussion of the $m=0$ neutrino case was given by Brill and Wheeler.[6]

Isham and Nelson[7] have solved the Dirac equation in the zero-momentum limit. They propose a quantization and obtain a mass of the order of the mass of the Universe, so the equation has a different interpretation than the electron. We are interested in the behavior of the electron and neutrino in curved spaces and shall present an exact solution for arbitrary momentum and mass.

We intend to apply the results to pair creation in expanding universes. In this connection the Dirac equation has been studied approximately by Chimento and Mollerach[8] and by Audretsch and Schäfer[9] for the radiation-dominated universe. A study of pair creation of spin-$\frac{1}{2}$ particles (massive and massless) in Robertson-Walker universes was made by Parker.[10] He showed that for massless neutrinos, as a result of conformal invariance, there was no pair creation. There have been other studies of spin-$\frac{1}{2}$ fields in such spaces.[11]

II. THE DIRAC EQUATION IN EXPANDING SPACE-TIMES

The Dirac equation in a curved space is taken to be

$$\left[i\gamma^\mu(x)\frac{\partial}{\partial x^\mu} - i\gamma^\mu(x)\Gamma_\mu(x) \right]\psi = m\psi \ . \tag{1}$$

Here $\gamma^\mu(x)$ are the curvature-dependent Dirac matrices and Γ_μ are the spin connections to be determined.

We consider the spatially flat metrics of the form

$$ds^2 = dt^2 - a^2(t)(dx_1{}^2 + dx_2{}^2 + dx_3{}^2) \ . \tag{2}$$

Thus,

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2) \ , \qquad g^{\mu\nu} = \text{diag}(1, -a^{-2}, -a^{-2}, -a^{-2}) \tag{3}$$

with $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$. For this case we obtain immediately from the defining equations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$,

$$\gamma^0(t) = \gamma_0, \quad \gamma^1(t) = -\frac{1}{a(t)}\gamma_1 \ , \qquad \gamma^2(t) = -\frac{1}{a(t)}\gamma_2, \quad \gamma^3(\tau) = -\frac{1}{a(t)}\gamma_3 \ , \tag{4}$$

where, from now on, γ_μ (without arguments and lower indices) denotes the standard (flat-space) Dirac matrices.

The spin connections $\Gamma_\mu(x)$ satisfy the equation

$$[\Gamma_\nu, \gamma^\mu(x)] = \frac{\partial \gamma^\mu(x)}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}\gamma^\rho(x) \ , \tag{5}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols for the metric (2) which we first determine to be

$$\Gamma^1_{\nu\sigma} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a\dot a & 0 & 0 \\ 0 & 0 & a\dot a & 0 \\ 0 & 0 & 0 & a\dot a \end{bmatrix}, \quad \Gamma^2_{\nu\sigma} = \begin{bmatrix} 0 & \dot a/a & 0 & 0 \\ \dot a/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma^3_{\nu\sigma} = \begin{bmatrix} 0 & 0 & \dot a/a & 0 \\ 0 & 0 & 0 & 0 \\ \dot a/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma^4_{\nu\sigma} = \begin{bmatrix} 0 & 0 & 0 & \dot a/a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dot a/a & 0 & 0 & 0 \end{bmatrix} . \tag{6}$$

Then Eq. (5) can be solved for the spin connections Γ_μ which we determine as

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{\dot a}{2}\gamma_0\gamma_1, \quad \Gamma_2 = \frac{\dot a}{2}\gamma_0\gamma_2, \quad \Gamma_3 = \frac{\dot a}{2}\gamma_0\gamma_3 \ , \tag{7}$$

so that the combination $\gamma^\mu\Gamma_\mu$ in Eq. (1) simplifies to

$$\gamma^\mu \Gamma_\mu = -\frac{3}{2}\frac{\dot a}{a}\gamma_0 \ . \tag{7'}$$

The Dirac equation (1), after multiplying it by $-i\gamma_0$, becomes

$$\left[\frac{\partial}{\partial t}+\frac{3}{2}\frac{\dot a}{a}-\frac{1}{a}\boldsymbol{\alpha}\cdot\nabla - im\gamma_0\right]\psi=0 \ . \tag{8}$$

III. THE SOLUTIONS OF THE DIRAC EQUATION

Since a is a function of t only, we can set in Eq. (2)

$$\psi(x,t)=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\begin{bmatrix} f_{\rm I}(\mathbf{k},t)\\ f_{\rm II}(\mathbf{k},t)\end{bmatrix} , \tag{9}$$

and two-component spinors obey the coupled equations

$$\begin{aligned}&\left[\partial_t+\frac{3}{2}\frac{\dot a}{a}-im\right]f_{\rm I}-\frac{i}{a}\mathbf{k}\cdot\boldsymbol{\sigma} f_{\rm II}=0 \ ,\\ &\left[\partial_t+\frac{3}{2}\frac{\dot a}{a}+im\right]f_{\rm II}-\frac{i}{a}\mathbf{k}\cdot\boldsymbol{\sigma} f_{\rm I}=0 \ .\end{aligned} \tag{10}$$

Multiplying the first equation from the left by $-ia(\mathbf{k}\cdot\boldsymbol{\sigma}/k^2)$ and inserting $f_{\rm II}$ from the second equation, we obtain, after some more algebra,

$$\left[\partial_t{}^2+\left[-\frac{1}{2}\frac{\ddot a}{a}+\frac{1}{4}\frac{\dot a^2}{a^2}-im\frac{\dot a}{a}+\frac{k^2}{a^2}\right]+m^2\right]h_{\rm I}=0 \ , \tag{11}$$

where we have set

$$h_{\rm I}(t)=f_{\rm I}(t)a^2 \ . \tag{12}$$

We shall consider three special models for expansion: (a) $a=a_0 t$, a model considered by Schrödinger,[3] (b) $a=a_0\sqrt{t}$, a model of a radiation-dominated universe, and (c) $a=e^{Ht}$, an inflationary universe. Equation (11) for these three cases becomes

$$\left[\partial_t{}^2+\left[\frac{k^2/a_0{}^2+\frac{1}{4}}{t^2}-\frac{im}{t}\right]+m^2\right]h_{\rm I}=0 \ , \tag{13a}$$

$$\left[\partial_t{}^2+\left[\frac{\frac{3}{16}}{t^2}+\frac{k^2/a_0{}^2-im/2}{t}\right]+m^2\right]h_{\rm I}=0 \ , \tag{13b}$$

$$[\partial_t{}^2+k^2e^{-2Ht}+(m-iH/2)^2]h_{\rm I}=0 \ . \tag{13c}$$

Case A. $a=a_0 t$. Changing the variable from t to $z^2=-4m^2t^2$ or $z=\mp 2imt$, we get, from Eq. (13a),

$$\left[\frac{d^2}{dz^2}-\frac{1}{4}+\frac{\pm\frac{1}{2}}{z}+\frac{\frac{1}{4}-(\pm ik/a_0)^2}{z^2}\right]h_{\rm I}=0 \tag{14}$$

and recognize the Whittaker differential equation which has *two independent* solutions:

$$h_{\rm I}^{\pm}(z)=W_{\pm 1/2,ik/a_0}(\mp 2imt) \ . \tag{15}$$

Having obtained the upper components of the wave function in (9), $f_{\rm I}^{\pm}$, using (12), the lower components $f_{\rm II}$ are obtained by

$$f_{\rm II}^{\pm}=-ia_0t\left[\partial_t+\frac{3\dot a}{2a}-im\right]\frac{\mathbf{k}\cdot\boldsymbol{\sigma}}{k^2}f_{\rm I}^{\pm} \ . \tag{16}$$

Let

$$g_{\rm I}=\frac{1}{t^2}h_{\rm I} \ , \tag{17}$$

then

$$\begin{aligned}g_{\rm II}&=-ia_0t\left[\partial_t+\frac{3}{2t}-im\right]g_{\rm I}^{\pm}\\ &=-ia_0t\left[-\frac{2}{t^3}W+(\mp)2im\frac{1}{t^2}\dot W+\frac{3}{2t^3}W-\frac{im}{t^2}W\right] .\end{aligned} \tag{18}$$

Now the Whittaker functions satisfy the identities[12]

$$\begin{aligned}\dot W_{\kappa,\mu}(z)&=(\tfrac{1}{2}+\mu-\kappa)(\tfrac{1}{2}-\mu-\kappa)\frac{1}{z}W_{\kappa-1,\mu}(z)\\ &\quad+\left[\frac{\kappa}{z}-\frac{1}{2}\right]W_{\kappa,\mu}(z) \ ,\\ \dot W_{\kappa,\mu}(z)&=-\frac{1}{z}W_{\kappa+1,\mu}(z)-\left[\frac{\kappa}{z}-\frac{1}{2}\right]W_{\kappa,\mu}(z) \ .\end{aligned} \tag{19}$$

We use the first identity for $\kappa=+\frac{1}{2}$, and the second identity for $\kappa=-\frac{1}{2}$, to show that Eq. (17) completely simplifies to

$$g_{\rm II}^{+}=-\frac{ik^2}{a_0}\frac{1}{t^2}W_{-1/2,ik/a_0}(-2imt) \ , \tag{20a}$$

$$g_{\rm II}^{-}=ia_0\frac{1}{t^2}W_{1/2,ik/a_0}(2imt) \ . \tag{20b}$$

Finally, if we denote the constant spinor components of $f_{\rm I}$ by $\binom{1}{0}$ and $\binom{0}{1}$, the four independent solutions of the form (9) are, with $k_\pm=k_1\pm ik_2$,

$$\psi_1=N_1\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{1}{a_0{}^2t^2}\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}W_{1/2,ik/a_0}(-2imt)\\ -\frac{i}{a_0}\begin{bmatrix}k_3\\k_+\end{bmatrix}W_{-1/2,ik/a_0}(-2imt)\end{bmatrix},$$

$$\psi_2=N_2\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{1}{a_0{}^2t^2}\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}W_{1/2,ik/a_0(-2imt)}\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}W_{-1/2,ik/a_0}(-2imt)\end{bmatrix}, \tag{21}$$

$$\psi_3=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{N_3}{a_0{}^2t^2}\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}W_{-1/2,ik/a_0}(+2imt)\\ \frac{ia_0}{k^2}\begin{bmatrix}k_3\\k_+\end{bmatrix}W_{1/2,ik/a_0}(+2imt)\end{bmatrix},$$

$$\psi_4=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{N_4}{a_0{}^2t^2}\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}W_{-1/2,ik/a_0}(+2imt)\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}W_{1/2,ik/a_0}(+2imt)\end{bmatrix}.$$

It is interesting to see the asymptotic form of these solutions for large times. Since $W_{k,\mu}(z)\to z^k e^{-z/2}$, for $-3\pi/2<\arg z<3\pi/2$, we obtain

$$\psi_1\simeq\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{\sqrt{-2im}}{a_0{}^2}\frac{N_1}{t^{3/2}}\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ \frac{1}{2m_0a}\begin{bmatrix}k_3\\k_+\end{bmatrix}\frac{1}{t}\end{bmatrix}e^{imt},$$

$$\psi_2\simeq\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{\sqrt{-2im}}{a_0{}^2}\frac{N_2}{t^{3/2}}\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}\end{bmatrix}e^{imt},$$

$$\psi_3\simeq\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{\sqrt{-2im}}{a_0{}^2}\frac{N_3}{t^{3/2}}\begin{bmatrix}\frac{1}{2im}\begin{bmatrix}1\\0\end{bmatrix}\frac{1}{t}\\ \frac{ia_0}{k^2}\begin{bmatrix}k_3\\k_+\end{bmatrix}\end{bmatrix}e^{-imt}, \tag{22}$$

$$\psi_4\simeq\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{\sqrt{-2im}}{a_0{}^2}\frac{N_4}{t^{3/2}}\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}\end{bmatrix}e^{-imt}.$$

We can determine the normalization constants N_i in such a way that asymptotically, i.e., in the flat-space limit, we have the usual $\delta(\mathbf{k}-\mathbf{k}')$ normalization of the electron's wave function. The norm of ψ is defined by

$$\begin{aligned}(\psi^{\mathbf{k}},\psi^{\mathbf{k}'})&=\int_t d^3x\, a^3(t)\bar{\psi}^{\mathbf{k}}\gamma^0\psi^{\mathbf{k}'}\\&=\int_t d^3x\, a_0{}^3t^3\psi_{\mathbf{k}}^\dagger\psi_{\mathbf{k}'}\\&\to\delta(\mathbf{k}-\mathbf{k}')a_0{}^3t^3\frac{\sqrt{2im}}{a_0{}^2}\frac{\sqrt{-2im}}{a_0{}^2}\frac{N^2}{t^3}.\end{aligned} \tag{23}$$

This procedure gives

$$N_1=N_2=(a_0/2m)^{1/2},\quad N_3=N_4=k/(2ma_0)^{1/2}. \tag{24}$$

Case B. $a=a_0\sqrt{t}$. Again with the new variable $z=\mp 2imt$, Eq. (13b) is transformed into a Whittaker equation

$$\left[\frac{d^2}{dz^2}+\left[-\frac{1}{4}+\frac{\pm\frac{1}{4}(1+2ik^2/a_0{}^2m)}{z}+\frac{\frac{1}{4}-(\frac{1}{4})^2}{z^2}\right]\right]h_{\mathrm{I}}=0. \tag{25}$$

The two independent solutions are

$$g_{\mathrm{I}}^{\pm}=\frac{1}{t}W_{\pm\kappa,1/4}(\mp z),\quad \kappa=\frac{1}{4}\left[1+\frac{2ik^2}{a_0{}^2m}\right],\quad z=2imt. \tag{26}$$

The lower-component spinors satisfy

$$\begin{aligned}g_{\mathrm{II}}&=-ia_0\sqrt{t}\left[\partial_t+\frac{3}{4t}-im\right]\frac{1}{t}W_{\pm\kappa,1/4}(\mp z)\\&=-ia\frac{1}{\sqrt{t}}\left[\mp 2im\dot{W}_{\pm\kappa,1/4}(\mp z)-\frac{1}{4t}W_{\pm\kappa,1/4}(\mp z)-imW_{\pm\kappa,1/4}(\mp z)\right].\end{aligned} \tag{27}$$

Again using the identities (19) we can eliminate $\dot{W}$ to obtain

$$g_{\mathrm{II}}^{+}=\frac{2k^2}{ma_0}\frac{1}{t^{3/2}}\left[W_{\kappa,1/4}(-z)-\frac{1}{8}\left[1-\frac{ik^2}{ma_0{}^2}\right]W_{\kappa-1,1/4}(-z)\right]$$

and

$$g_{\mathrm{II}}^{-}=\frac{k^2}{2ma_0}\frac{1}{t^{3/2}}\left[W_{-\kappa,1/4}(z)+\frac{2ima_0{}^2}{k^2}W_{-\kappa+1,1/4}(z)\right].$$

Hence, the four independent solutions are

$$\psi_1=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{1}{a_0{}^2t}N_1\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}W_{\kappa,1/4}(-2imt)\\ \frac{2}{ma_0}\begin{bmatrix}k_3\\k_+\end{bmatrix}\frac{1}{\sqrt{t}}\left[W_{\kappa,1/4}(-z)-\frac{1}{8}\left[1-\frac{ik^2}{ma_0{}^2}\right]W_{\kappa-1,1/4}(-z)\right]\end{bmatrix},$$

$$\psi_2=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{1}{a_0{}^2t}N_2\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}W_{\kappa,1/4}(-2imt)\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}\frac{1}{\sqrt{t}}\left[W_{\kappa,1/4}(-z)-\frac{1}{8}\left[1-\frac{ik^2}{ma_0{}^2}\right]W_{\kappa-1,1/4}(-z)\right]\end{bmatrix},$$

$$\psi_3=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{1}{a_0{}^2t}N_3\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}W_{-\kappa,1/4}(-2imt)\\ \frac{1}{2ma_0}\begin{bmatrix}k_3\\k_+\end{bmatrix}\frac{1}{\sqrt{t}}\left[W_{-\kappa,1/4}(z)+\frac{2ima_0{}^2}{k^2}W_{-\kappa+1,1/4}(z)\right]\end{bmatrix},\qquad(28)$$

$$\psi_4=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{1}{a_0{}^2t}N_4\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}W_{-\kappa,1/4}(-2imt)\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}\frac{1}{\sqrt{t}}\left[W_{-\kappa,1/4}(-z)+\frac{2ima_0{}^2}{k^2}W_{-\kappa+1,1/4}(z)\right]\end{bmatrix},$$

which behave asymptotically as

$$\psi_1\to\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{(-2im)^\kappa}{a_0{}^2}t^{\kappa-1}N_1\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ \frac{2}{ma_0}\begin{bmatrix}k_3\\k_+\end{bmatrix}\frac{1}{\sqrt{t}}\end{bmatrix}e^{imt},$$

$$\psi_2\to\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{(-2im)^\kappa}{a_0{}^2}t^{\kappa-1}N_2\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}\end{bmatrix}e^{imt},\qquad(29)$$

$$\psi_3\to\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{(-2im)^{-\kappa}}{a_0{}^2}t^{-\kappa-1}N_3\times\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ \frac{-2ma_0}{k^2}\begin{bmatrix}k_3\\k_+\end{bmatrix}\sqrt{t}\end{bmatrix}e^{-imt},$$

$$\psi_4\to\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{(-2im)^{-\kappa}}{a_0{}^2}t^{-\kappa-1}N_4\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \begin{bmatrix}k_-\\-k_3\end{bmatrix}\end{bmatrix}e^{-imt}.$$

Using these asymptotic forms we can determine the normalization constants N_i as in Eq. (23): we obtain

$$(\psi_1,\psi_1)=\delta(\mathbf{k}-\mathbf{k}')\frac{\sqrt{2m}}{a_0}e^{\pi k^2/2ma_0{}^2}N_1{}^2,$$

and so

$$N_1{}^2=N_2{}^2=\frac{a_0}{\sqrt{2m}}e^{-\pi k^2/2m}a_0{}^2,\qquad(30a)$$

$$N_3{}^2=N_4{}^2=\frac{k^2}{a_0}\frac{1}{(2m)^{3/2}}e^{-\pi k^2/2m}a_0{}^2.\qquad(30b)$$

It is interesting to note that the solutions for case B can also be given in terms of parabolic cylindrical functions because of the relation[12]

$$D_\nu(z)=2^{(\nu+1/2)/2}z^{-1/2}W_{\nu/2+1/4,\pm1/4}(z^2/2).\qquad(31)$$

For this purpose we make another change of variable in Eq. (13b), setting $t=y^2$ and scaling $h_{\rm I}\to t^{1/4}\tilde{g}_{\rm I}$ to obtain

$$(-\tfrac{1}{2}\partial_y{}^2-2m^2y^2)\tilde{g}_{\rm I}=2\left[\frac{k^2}{a_0{}^2}-im/2\right]\tilde{g}_{\rm I}.\qquad(32)$$

This is a Schrödinger equation for a unit mass moving in a potential $V=-2m^2y^2$ with "energy" $E=2(k^2/a_0{}^2-im/2)$. The solution is

$$\tilde{g}_{\rm I}^{\pm}=D_{\nu\pm}(2\sqrt{2m}\,e^{\pm i\pi/4}\sqrt{t}\,)\qquad(33)$$

with

$$\nu_+=-i\frac{k^2}{ma_0{}^2},\quad \nu_-=-1+ik^2/ma_0{}^2.$$

Case C. $a=e^{Ht}$. Here we introduce the new coordinate z by

$$z=\frac{k}{H}e^{-Ht}\,, \tag{34}$$

then Eq. (13c) becomes the Bessel equation

$$\left[z^2\frac{d^2}{dz^2}+z\frac{d}{dz}+(z^2-\nu^2)\right]h_{\mathrm{I}}=0 \tag{35}$$

with $\nu\equiv\frac{1}{2}(1+2im/H)$, and we have the solutions

$$J_{\pm\nu}(z)=J_{\pm\nu}\left[\frac{k}{H}e^{-Ht}\right]. \tag{36}$$

Proceeding as before, the four independent normalized solutions are

$$\begin{aligned}
\psi_1&=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\left[\frac{\pi k/2H}{\cos\frac{i\pi m}{H}}\right]^{1/2}\times e^{-2Ht}\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}J_\nu\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}J_{\nu-1}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix},\\
\psi_2&=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\left[\frac{\pi k/2H}{\cos\frac{i\pi m}{H}}\right]^{1/2}\times e^{-2Ht}\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}J_\nu\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_-\\-k_3\end{bmatrix}J_{\nu-1}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix},\\
\psi_3&=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\left[\frac{\pi k/2H}{\cos\frac{i\pi m}{H}}\right]^{1/2}\times e^{-2Ht}\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}J_{-\nu}\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}J_{-\nu+1}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix},\\
\psi_4&=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\left[\frac{\pi k/2H}{\cos\frac{i\pi m}{H}}\right]^{1/2}\times e^{-2Ht}\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}J_{-\nu}\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_-\\k_3\end{bmatrix}J_{-\nu+1}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix}.
\end{aligned} \tag{37}$$

IV. MASSLESS CASE, NEUTRINO SOLUTIONS

We now give for the three models, Eqs. (13), the four independent normalized solutions in the $m=0$ limit.

Case A. $a=a_0t$:

$$\psi_1=A\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ \frac{1}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}\left[1+\frac{3ia_0}{4k}\right]\end{bmatrix}\exp\left[i\frac{k}{a_0}\ln t\right],$$

$$\psi_2=A\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \frac{1}{k}\begin{bmatrix}k_-\\-k_3\end{bmatrix}\left[1+\frac{3ia_0}{4k}\right]\end{bmatrix}\exp\left[i\frac{k}{a_0}\ln t\right],$$

$$\psi_3=A\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ -\frac{1}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}\left[1-\frac{3ia_0}{4k}\right]\end{bmatrix}\exp\left[-i\frac{k}{a_0}\ln t\right],$$

$$\psi_4=A\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ -\frac{1}{k}\begin{bmatrix}k_-\\-k_3\end{bmatrix}\left[1-\frac{3ia_0}{4k}\right]\end{bmatrix}\exp\left[-i\frac{k}{a_0}\ln t\right],$$

where

$$A=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}(a_0t)^{+3/2}}\,.$$

Case B. $a=a_0\sqrt{t}$:

$$\psi_1=B\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ \frac{1}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}\end{bmatrix}\exp\left[i\frac{2k}{a_0}\sqrt{t}\right],$$

$$\psi_2=B\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \frac{1}{k}\begin{bmatrix}k_-\\-k_3\end{bmatrix}\end{bmatrix}\exp\left[i\frac{2k}{a_0}\sqrt{t}\right],$$

$$\psi_3=B\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ -\frac{1}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}\end{bmatrix}\exp\left[-i\frac{2k}{a_0}\sqrt{t}\right],$$

$$\psi_4=B\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ -\frac{1}{k}\begin{bmatrix}k_-\\-k_3\end{bmatrix}\end{bmatrix}\exp\left[-i\frac{2k}{a_0}\sqrt{t}\right],$$

where

$$B=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}a_0{}^{3/2}t^{3/4}}.$$

Case C. $a=e^{Ht}$:

$$\psi_1=C\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}J_{1/2}\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}J_{-1/2}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix},$$

$$\psi_2=C\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}J_{1/2}\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_-\\-k_3\end{bmatrix}J_{-1/2}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix},$$

$$\psi_3=C\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}J_{-1/2}\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_3\\k_+\end{bmatrix}J_{+1/2}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix},$$

$$\psi_4=C\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}J_{-1/2}\left[\frac{k}{H}e^{-Ht}\right]\\ \frac{i}{k}\begin{bmatrix}k_-\\-k_3\end{bmatrix}J_{+1/2}\left[\frac{k}{H}e^{-Ht}\right]\end{bmatrix},$$

where

$$C=\frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}e^{-2Ht}.$$

V. GORDON DECOMPOSITION OF THE CURRENT

For further interpretation of the theory it is useful to consider the Gordon decomposition of the current. Using Eq. (1) in an external field A_μ and its conjugate we can express the current as follows:

$$j^\mu=\bar{\psi}\gamma^\mu\psi=\frac{1}{2m}\bar{\psi}(i\partial_\lambda\gamma^\lambda\gamma^\mu-i\gamma^\mu\gamma^\lambda\partial_\lambda$$
$$-i[\gamma^\lambda\Gamma_\lambda,\gamma^\mu]+eA_\lambda g^{\lambda\mu})\psi.$$

This can be reexpressed as

$$j^\mu=\frac{1}{2m}(\bar{\psi}\sigma^{\lambda\mu}\psi)_{,\lambda}-\frac{i}{4m}g^{\mu\lambda}\bar{\psi}\overleftrightarrow{\partial}_\lambda\psi$$
$$-\frac{i}{4m}\bar{\psi}([\gamma^\lambda_{,\lambda},\gamma^\mu]+[\gamma^\lambda,\gamma^\mu_{,\lambda}])\psi$$
$$-\frac{i}{2m}\bar{\psi}[\gamma^\lambda\Gamma_\lambda,\gamma^\mu]\psi+\frac{e}{2m}A_\lambda g^{\lambda\mu}\bar{\psi}\psi.$$

For our metric (3) we use (7′). Furthermore, we have $\gamma^\lambda_{,\lambda}=0$, $[\gamma^\lambda,\gamma^\mu_{,\lambda}]=[\gamma_0,\gamma_\mu]\delta_{\mu j}\dot{a}/a^2$ and $\sigma^{0k}=(i/2)[\gamma^0,\ \gamma^k]=-(i/a)\gamma_0\gamma_k$, $\sigma^{jk}=(i/\ 2)[\gamma^j,\gamma^k]=(i/2a^2)[\gamma_j,\gamma_k]$. Then

$$j^\mu=\frac{1}{2m}(\bar{\psi}\sigma^{\lambda\mu}\psi)_{,\lambda}-\frac{1}{2m}g^{\mu\lambda}\bar{\psi}\left[\frac{i}{2}\overleftrightarrow{\partial}_\lambda-eA_\lambda\right]\psi$$
$$-\frac{7i}{4m}\frac{\dot{a}}{a^2}\delta_{\mu j}\bar{\psi}[\gamma_0,\gamma_\mu]\psi,$$

or, in components,

$$j_0=+\frac{i}{2ma}\nabla_k\bar{\psi}\gamma_k\gamma_0\psi-\frac{1}{2m}\bar{\psi}\left[\frac{i}{2}\overleftrightarrow{\partial}_0-eA_0\right]\psi$$

and

$$j_k=\frac{i}{2m}\partial_t\left[\frac{1}{a}\bar{\psi}\gamma_k\gamma_0\psi\right]+\frac{i}{4ma^2}\partial_j(\bar{\psi}[\gamma_j,\gamma_k]\psi)$$
$$-\frac{1}{2m}\frac{1}{a^2}\bar{\psi}(i\tfrac{1}{2}\overleftrightarrow{\partial}_k-eA_k)\psi+\frac{7i}{2m}\frac{\dot{a}}{a^2}\bar{\psi}\gamma_k\gamma_0\psi.$$

Writing $j_0=\nabla\cdot\mathbf{P}+\rho_{\text{convective}}$ and $\mathbf{j}=\partial\mathbf{P}/\partial t+\nabla\times\mathbf{M}+\mathbf{j}_{\text{convective}}+7(\dot{a}/a)\mathbf{P}$, the polarization density is given by

$$\mathbf{P}_k=\frac{i}{2ma}\bar{\psi}\gamma_k\gamma_0\psi$$

and the magnetization density by

$$M_i=\epsilon_{ijk}\frac{i}{4m}\frac{1}{a^2}\bar{\psi}[\gamma_j,\gamma_k]\psi.$$

Note the factor $1/a$ in front of $\mathbf{P}$ and the factor $1/a^2$ in front of $\mathbf{M}$ and $\mathbf{j}_{\text{convective}}$, also the new term $7(\dot{a}/a)\mathbf{P}$ in the current $\mathbf{j}$ as compared to the flat-space case. Thus, these physical quantities are all time dependent. The current $j^\mu=\bar{\psi}\gamma^\mu\psi$ is conserved because $\gamma^\mu_{,\mu}=0$ in our metric and, using Eq. (1) and its conjugate,

$$\bar{\psi}\left[-i\gamma^\mu\frac{\overleftarrow{\partial}}{\partial x_\mu}-\Gamma_\mu\gamma^\mu\right]=m\bar{\psi}.$$

Note that $\gamma^0\gamma^{\mu\dagger}\gamma^0=\gamma^\mu$, but $\gamma^0\Gamma^\dagger_\mu\gamma_0=-\Gamma_\mu$. Hence, the scalar product can be defined by the integral over a spacelike surface

$$(\psi_1,\psi_2)=\int_\Sigma\sqrt{-g}\,\bar{\psi}_1\gamma^\mu\psi_2 d\sigma_\mu,$$

which we have used to normalize our solutions.

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