A fluid of multidimensional objects

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This paper contains a variational formulation of the theory of a perfect fluid composed of multidimensional objects in space-time of arbitrary dimension; a nonstandard variational principle is based on the discussion of the Noether identity related to covariance of the Lagrangian under the action of diffeomorphisms of space-time. The initial-value problem and the boundary conditions are considered.

I. INTRODUCTION

The perfect fluid of point particles turns out to have an interesting generalization, proposed and elaborated by Stachel.[1-3] This is the perfect fluid composed of multidimensional objects. In four-dimensional space-time, the dimension of the objects can be 0 (point particles), 1 (strings), or 2 (membranes). In view of an already existing application[4] to the Kaluza-Klein cosmology, the generic case of $(k-1)$-dimensional objects in m-dimensional space-time is considered.

The principal aim of this paper is to formulate a variational principle for such a fluid. This is achieved in Secs. II–IV. In Sec. II, I discuss a nonstandard variational principle useful in the theory of fluids. The starting point of the discussion is the Noether identity related to the covariance of the Lagrangian under diffeomorphisms of space-time. To formulate the variational principle, I use the method of differential forms;[5-7] the method seems to play here a more essential role than for other variational principles described in its framework. In Sec. III the variational principle is specialized to the standard case of the perfect fluid of point particles, $k=1$. In Sec. IV it is generalized to arbitrary k.

The remaining part of the paper is devoted to two special problems in the theory of fluids composed of multidimensional objects. In Sec. V, I show that the fluid equations of motion are evolutionary at least in some important special cases. In Sec. VI the boundary conditions are examined.

II. THE DIFFEOMORPHIC VARIATIONAL PRINCIPLE

We consider a dynamical system which consists of a scalar $(m-k)$-form N (a basic fluid variable called *flux*) and a number of l-forms ϕ_A of arbitrary tensorial type (auxiliary fields). The Lagrangian m-form is

$$L=L(N,dN,\phi_A,\tilde{D}\phi_A,\theta^i) ,$$

where $\tilde{D}$ is the covariant exterior derivative determined by the connection 1-forms ω^i_j, and θ^i are orthonormal frame 1-forms; the connection is assumed to be metric, $\omega_{ij}+\omega_{ji}=0$. The independent variation of L with respect to N, ϕ_A, θ^i, and ω^i_j,

$$\delta L=-H\wedge\delta N+\delta\phi_A\wedge L^A-\delta\theta^i\wedge t_i+\tfrac{1}{2}\delta\omega^i_j\wedge s_i^j + \text{an exact form} ,$$

serves as a definition of the k-form H (later called the *enthalpy* k-form), the $(m-l)$-forms L^A, the energy-momentum $(m-1)$-forms t_i, and the spin $(m-1)$-forms s_i^j.

We assume that the Lagrangian L is Lorentz invariant and diffeomorphic covariant. The first assumption will not be discussed in detail. We mention only that, if ϕ_A are scalars (or if they are not present), this assumption implies that the spin $(m-1)$-forms vanish, $s_{ij}=0$, and the energy-momentum tensor is symmetric: $\theta_i\wedge t_j=\theta_j\wedge t_i$.

The second assumption is

$$L(h^*N,d(h^*N),h^*\phi_A,h^*D\phi_A,h^*\theta^i) =h^*L(N,dN,\phi_A,D\phi_A,\theta^i) , \quad (2.1)$$

for any diffeomorphism h (h^* denotes pullback of the forms). Taken infinitesimally, this leads to the following Noether identity:[8-11]

$$(Z\lrcorner\theta^i)Dt_i-(Z\lrcorner\Theta^i)\wedge t_i+\tfrac{1}{2}(Z\lrcorner\Omega^i_j)\wedge s_i^j +(-1)^k dH\wedge(Z\lrcorner N)-H\wedge(Z\lrcorner dN) +(-1)^l(Z\lrcorner\phi_A)\wedge DL^A +(Z\lrcorner D\phi_A)\wedge L^A=0 \quad (2.2)$$

for an arbitrary vector field Z. Above, $\Theta^i=\frac{1}{2}Q^i{}_{jh}\theta^j\wedge\theta^h$ and $\Omega^i_j=\frac{1}{2}R^i{}_{jgh}\theta^g\wedge\theta^h$ denote the torsion and curvature two-forms, respectively. The second Noether identity implied by (2.1) (obtained from the boundary terms) will be useful only in the case when ϕ_A are not present and L does not depend on dN:

$$H\wedge(Z\lrcorner N)+(Z\lrcorner N)+(Z\lrcorner\theta^i)\wedge t_i+Z\lrcorner L=0 . \quad (2.3)$$

We look for a stationary point (N,ϕ_A) of the action integral $\int_\Omega L$ among the family of trial fields (N',ϕ'_A) over Ω such that $N'=h^*N$ for a diffeomorphism h, on the boundary $\phi'_A \mid_{\partial\Omega} = \phi_A \mid_{\partial\Omega}$ and $h \mid_{\partial\Omega} = id_{\partial\Omega}$. In other words, the variations of ϕ_A are arbitrary (restricted by the boundary constraints only), whereas the variations of N are induced by diffeomorphisms of space-time. Infinitesimally,

$$\delta N = \underset{Z}{\mathfrak{L}} N = Z \lrcorner\, dN + d(Z \lrcorner\, N) ,$$

and the vector field Z vanishes on the boundary, $Z \mid_{\partial\Omega} = 0$. Therefore, under such variation of N,

$$\delta L = (-1)^k dH \wedge (Z \lrcorner\, N) - H \wedge (Z \lrcorner\, dN) + d\left[(Z \lrcorner\, N) \wedge \frac{\partial L}{\partial N} + (Z \lrcorner\, dN) \wedge \frac{\partial L}{\partial dN} \right] .$$

This leads to the following equations of motion:

$$(-1)^k dH \wedge (Z \lrcorner\, N) - H \wedge (Z \lrcorner\, dN) = 0 , \tag{2.4}$$

and they should hold for any vector field Z. The left-hand side of (2.4) is the contribution from the field N to the identity (2.2). Therefore, if equations resulting from variations with respect to other fields are satisfied ($L^A = 0$), then Eq. (2.4) is equivalent to the energy-momentum-conservation equation

$$\tilde{D} t_i = Q^j{}_i \wedge t_j - \tfrac{1}{2} R^{jh}{}_i \wedge s_{jh} ,$$

where $Q^j{}_i = Q^j{}_{ih} \theta^h$ and $R^j{}_{hi} = R^j{}_{hig} \theta^g$ are torsion 1-forms and curvature 1-forms, respectively. If the auxiliary fields are scalars (or if they are not present) then the above equation can be written as

$$D t_i = 0 ,$$

where D denotes the Levi-Civita covariant derivative.

The name "fluid variable" associated with N is motivated by the above equivalence. Notice that we had to assume that N is a scalar in order to obtain covariant equations. The degree of the form N is, however, arbitrary.

I emphasize that the description of a fluid given here is Eulerian—the Lagrange coordinates are not used. The method of variation, however, is close to the variation of world lines present in the Lagrangian description of a fluid.

In the discussion below we shall use the $(m-l)$-forms

$$\eta_{i_1 \cdots i_l} = \frac{1}{(m-l)!} \eta_{i_1 \cdots i_m} \theta^{i_{l+1}} \wedge \cdots \wedge \theta^{i_m} ,$$

where $\eta_{i_1 \cdots i_m}$ is completely skew symmetric and determined, with respect to orthonormal frames by the condition $\eta_{1 \cdots m} = 1$. For instance, the energy-momentum tensor t_i^j is defined by the equality $t_i = \eta_j t_i^j$ and the Lagrangian function $\mathcal{L}$ is related to L by $L = \mathcal{L}\eta$.

III. THE PERFECT FLUID OF POINT PARTICLES

The form N will have now the degree $m-1$ and will represent the flux of particles (baryons). We assume also that N is spacelike.

The most simple and physically relevant case is that of the *perfect fluid at zero temperature.* In this case N is the only dynamical variable:

$$L = L(N, \theta^i) . \tag{3.1}$$

Write $N = N^i \eta_i$ and $N^i = n u^i$ where u^i is the normalized velocity, $u^i u_i = 1$. The scalar Lagrangian (3.1) should have the form

$$L = \mathcal{L}(n)\eta . \tag{3.2}$$

Using the identity (2.3) it is easy to show that the energy-momentum tensor of the fluid has the form

$$t_i^j = (\epsilon + p) u^j u_i - p \delta_i^j , \tag{3.3}$$

where $\epsilon = -\mathcal{L}$ is the energy density in the rest frame and p is given by the thermodynamical relation

$$-p\delta(1/n) = \delta(\epsilon/n) ; \tag{3.4}$$

therefore it is interpreted as pressure. The 1-form $H = H_i \theta^i$ is given by

$$H_i = \frac{\epsilon + p}{n} u_i , \tag{3.5}$$

which allows one to call it the enthalpy 1-form.

Equation (2.4) takes on the form

$$(\mathrm{div} N) H_i - N^j (\nabla_i H_j - \nabla_j H_i) = 0 , \tag{3.6}$$

where ∇_i denotes the Levi-Civita covariant derivative. If $\epsilon + p \neq 0$, this equation *implies* the continuity equation

$$dN = (\mathrm{div} N)\eta = 0 . \tag{3.7}$$

The continuity equation plays a special role from the point of view of the variational principle under consideration. Since $d(h^* N) \equiv h^* dN$, Eq. (3.7) is a necessary condition for the existence of a stationary point of the action integral in a given family of trial forms N'. This variational principle is the only one known to the author which allows one to derive the continuity equation rather than to postulate it; simultaneously it explains why the derivation of this equation in other approaches is not possible. The case $\epsilon + p = 0$ is of minor interest since it implies $d\epsilon/dn = 0$; i.e., L corresponds to the cosmological term.

The Lagrangian (3.1) admits a straightforward generalization to *perfect fluids at arbitrary temperature.* In this case, the fluid Lagrangian L depends on two dynamical variables: the flux N and the specific entropy S (i.e., entropy per particle in the rest frame). We require that the specific entropy be constant along world lines of the fluid:

$$dS \wedge N = 0 . \tag{3.8}$$

The Lagrangian which takes into account the constraint (3.8) then has the form

$$L = -\epsilon(n,S)\eta + \lambda dS \wedge N . \tag{3.9}$$

The variation of the Lagrangian (3.9) with respect to the Lagrange multiplier λ leads to the conservation law (3.8). The variation, with respect to the specific entropy

S, gives the equation

$$nT\eta+d(\lambda N)=0\ , \tag{3.10}$$

where the temperature T is determined by the thermodynamic identity

$$\delta(\epsilon/n)=-p\delta(1/n)+T\delta S\ .$$

The second term in (3.9) does not contribute to the energy-momentum tensor (3.3). It is an interesting exercise to transform directly Eq. (3.6) with the "enthalpy" vector given now by

$$H_i=\frac{\epsilon+p}{n}u_i-\lambda\nabla_i S\ , \tag{3.11}$$

into the standard form

$$(\epsilon+p)u^j\nabla_j u_i=(\delta_i^j-u^j u_i)\nabla_j p\ . \tag{3.12}$$

Note that (3.10) allows one to determine λ if it is given on an initial hypersurface transversal to u^i.

Note also that the standard variation with respect to N would lead to vanishing of H_i given by (3.11). Although the equations of motion (3.6) or (3.12) would be satisfied in this case [the continuity equation (3.7) would have to be imposed], the form of the velocity vector would exclude rotational motions of the fluid. This is the so-called Lin difficulty present in variational derivations of equations of motion of relativistic and nonrelativistic fluids which use the Eulerian description.[12,13] Our method of variation gives an interesting insight into the origin of this difficulty.

IV. THE PERFECT FLUID OF $(k-1)$-DIMENSIONAL OBJECTS IN m-DIMENSIONAL SPACE-TIME

Our aim now is to generalize the notion of the perfect fluid of particles, $k=1$, to an arbitrary k. The basic idea is sketched already in Sec. II. However, if N is an arbitrary $(m-k)$-form and $k\neq 1$, $k\neq m-1$ then the number of component equations contained in (2.4) is less than the number of components of N. Worse, in the important case $m=4$, $k=2$ the equations have a nonevolutionary character.

Inspired by a paper by Stachel,[2] I shall apply the variational principle of Sec. II to a theory of k-dimensional foliation, in m-dimensional space-time. Such foliation is a continuum of k-dimensional surfaces or, physically, k-dimensional world sheets, each sheet representing the set of events belonging to the history of a $(k-1)$-dimensional object.

The foliation will be tangent to the distribution V of vector fields Z given by the equation

$$V=\{Z:Z\,\lrcorner\,N=0\}\ . \tag{4.1}$$

There are two conditions which N must satisfy in order to determine a k-dimensional foliation. The first condition is algebraic; N has to be simple:

$$N=\omega_{k+1}\wedge\cdots\wedge\omega_m\ , \tag{4.2}$$

where ω_i are linearly independent 1-forms. We assume also that N is spacelike in order that the distribution (4.1) be timelike. The second condition is differential; it is given by the Frobenius theorem

$$dN=N\wedge\omega \tag{4.3}$$

for a certain 1-form ω.

We consider now the Lagrangian

$$L=L_0(N,\theta^i)+\lambda_1\wedge(N-\omega_{k+1}\wedge\cdots\wedge\omega_m)+\lambda_2\wedge(dN-N\wedge\omega)\ . \tag{4.4}$$

The variation, with respect to the Lagrange multipliers λ_1 and λ_2, gives the constraints (4.2) and (4.3), respectively. The variation with respect to ω's gives certain restrictions on the Lagrange multipliers λ_1 and λ_2. If these restrictions and the constraints (4.2) and (4.3) are satisfied then the k-forms $H_1=-\lambda_1$ and $H_2=(-1)^{k+1}d\lambda_2-\lambda_2\wedge\omega$ satisfy identically the basic equation (2.4). To prove this statement, observe that the second and the third terms in (4.4) (denoted L_1 and L_2, respectively) do not contain the frames θ^i. Thus the energy-momentum $(m-1)$-forms corresponding to the "Lagrangians" L_1 and L_2 will vanish and the statement follows from the considerations of Sec. II. Therefore, the equation of motion (2.4) has to be satisfied for H which originate from the variation of L_0 itself:

$$\delta L_0=-H\wedge\delta N-\delta\theta^i\wedge t_i\ .$$

If we now take Z which satisfies (4.1) and use the constraint (4.3), Eq. (2.4) gives

$$(Z\,\lrcorner\,\omega)N\wedge H=0\ .$$

Under the assumption

$$N\wedge H\neq 0\ , \tag{4.5}$$

this leads to $Z\,\lrcorner\,\omega=0$. This implies that ω is a linear combination of the 1-forms ω_i; in turn from (4.2) and (4.3) we obtain

$$dN=0\ . \tag{4.6}$$

Equation (4.5) is once more a necessary condition for the existence of a stationary point of this variational principle. It enables us to interpret physically the $(m-k)$-form N. The integral

$$\int_{\Sigma_{m-k}}N\ ,$$

represents the number of particles $(k=1)$, strings $(k=2),\ldots,$ membranes $(k=m-1)$ crossing an $(m-k)$-dimensional surface Σ_{m-k} in space-time. If Σ_{m-k} is the boundary of an $(m-k+1)$-dimensional surface, $\Sigma_{m-k}=\partial\Sigma_{m-k+1}$, then the number of objects coming into Σ_{m-k+1} is equal to the number of objects going out of Σ_{m-k+1}:

$$\int_{\partial\Sigma_{m-k+1}}N=0\ .$$

We calculate the energy-momentum tensor. According to the identity (1.3), we have

$$Z^j t_j=-H\wedge(Z\,\lrcorner\,N)-Z\,\lrcorner\,L_0\ .$$

Write $L_0 = \mathcal{L}\eta$; write also

$$N = \frac{1}{k!} N^{i_1 \cdots i_k} \eta_{i_1 \cdots i_k}$$

and

$$H = \frac{1}{k!} H_{i_1 \cdots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} .$$

Then, simple algebra leads to

$$t_j^i = -\frac{k+1}{k!} N^{l_1 \cdots l_k} H_{[l_1 \cdots l_k} \delta_{j]}^i - \Lambda \delta_j^i , \tag{4.7}$$

or, equivalently

$$t_j^i = \frac{1}{(k-1)!} N^{l_1 \cdots l_{k-1} i} H_{l_1 \cdots l_{k-1} j} - \left[\frac{1}{k!} N^{l_1 \cdots l_k} H_{l_1 \cdots l_k} + \mathcal{L} \right] \delta_j^i . \tag{4.8}$$

Define now the normalized k-velocity $u^{i_1 \cdots i_k}$ and the scalar n by the relations

$$N^{i_1 \cdots i_k} = n u^{i_1 \cdots i_k} ,$$

where $n > 0$ and

$$\frac{1}{k!} u^{i_1 \cdots i_k} u_{i_1 \cdots i_k} = (-1)^{k-1} .$$

The sign on the right-hand side above expresses the fact that $u^{i_1 \cdots i_k}$ is timelike. The scalar n represents the number of (particles), strings, . . . , membranes crossing the unit area of an $(m-k)$-dimensional surface in the rest frame (t = const) perpendicular to these objects. We shall call it concentration.

Because of condition (4.2), n is the only scalar which can be built out of N. Therefore, we have

$$L_0(N, \theta^i) = \mathcal{L}(n)\eta ,$$

and varying N with θ^i fixed, we get

$$H = (-1)^{km} \frac{1}{k!} \frac{h}{n} u_{i_1 \cdots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} ,$$

where h is given by

$$h = -n \frac{d\mathcal{L}}{dn} . \tag{4.9}$$

The energy-momentum tensor (4.8) can now be rewritten as

$$t_j^i = h \frac{1}{(k-1)!} u^{l_1 \cdots l_{k-1} i} u_{j l_1 \cdots l_{k-1}} - (\mathcal{L} + h) \delta_j^i . \tag{4.10}$$

The velocity k-form

$$u = \frac{1}{k!} u_{i_1 \cdots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} ,$$

which is simple because of (4.2), after a convenient choice of the orthonormal frame (θ^i), can be written as

$$u = \theta^0 \wedge \cdots \wedge \theta^{k-1} . \tag{4.11}$$

The tensor

$$q_j^i = \frac{1}{(k-1)!} u^{l_1 \cdots l_{k-1} i} u_{j l_1 \cdots l_{k-1}}$$

has the form

$$q_j^i = \theta^{0i} \theta^0{}_j - \theta^{1i} \theta^1{}_j - \cdots - \theta^{k-1 i} \theta^{k-1}{}_j ;$$

i.e., it projects onto the distribution V given by (4.1). From (4.10) we get

$$t^{ij} \theta^0{}_i \, \theta^0{}_j = -\mathcal{L} ;$$

therefore,

$$\epsilon = -\mathcal{L}$$

is still the energy density in the rest frame of the fluid determined by the 1-form θ^0. In contrast with the case of particles, the rest frame is determined nonuniquely; there exists a $(k-1)$-dimensional family of rest frames. Introducing pressure $p = h - \epsilon$ [it is equivalent to (3.4) owing to (4.9)], we have

$$t_j^i = (\epsilon + p) q_j^i - p \delta_j^i , \tag{4.12}$$

in analogy with (3.3).

The energy density is non-negative in any frame if $\epsilon \geq 0$ and $\epsilon + p \geq 0$. The flux of energy is timelike or null in any frame if $p \leq \epsilon$. So the energetic conditions are

$$-\epsilon \leq p \leq \epsilon \geq 0 , \tag{4.13}$$

and they do not depend on the dimensions k and m.

The equation of motion (2.4), because of the continuity equation (4.5), reduces to

$$(Z \lrcorner N) \wedge dH = 0 .$$

In the component notation it reads

$$N^{i_1 \cdots i_k} \nabla_{[j} H_{i_1 \cdots i_k]} = 0 ,$$

or

$$N^{i_1 \cdots i_k} [\nabla_j H_{i_1 \cdots i_k} + (-1)^k k \nabla_{i_1} H_{i_2 \cdots i_k j}] = 0 . \tag{4.14}$$

Making use of the thermodynamical relation (3.4), we can transform (4.14) into the form

$$(\epsilon + p) u^{i_1 \cdots i_k} \nabla_{i_1} u_{i_2 \cdots i_k j} = (k-1)! (\delta_j^i - q_j^i) \nabla_i p , \tag{4.15}$$

analogous to (3.12).

We can add to the flux N some additional dynamical variables to describe the fluid, in particular, the specific entropy S in analogy with the case of particles. The condition (3.8) now means that the specific entropy is constant along world sheets of the fluid. As in the case of particles, the constraint (3.8) does not affect the algebraic form of the energy-momentum tensor (4.12), but pressure and energy density will depend now on two parameters: n and S.

V. ARE FLUID EQUATIONS EVOLUTIONARY?

To get some insight into the Cauchy problem for the equations of motion of a fluid composed of multidimensional objects, we consider some special cases. We restrict our considerations to fluids at zero temperature and in flat space-time.

An important role in our discussion will play the thermodynamic coefficient of proportionality c defined by the formula

$$H_{i_1 \cdots i_k} = (-1)^{km} c N_{i_1 \cdots i_k} .$$

The alternative definitions of this coefficient are $c = (\epsilon + p)/n^2 = (d\epsilon/dn)/n$.

Although the subject is known, in order to enable comparisons, we shall first consider the case of ordinary fluids, $k=1$. In this case the fluid equations of motion (4.5) and (2.4), in the component notation, have the form

$$\nabla_i N^i = 0, \quad N^j(\nabla_i H_j - \nabla_j H_i) = 0 .$$

In Cartesian coordinates $(x^i) = (t, x^\alpha)$, the above equations read (the overdot denotes time differentiation):

$$\dot{N}^0 + \partial_\alpha N^\alpha = 0 , \tag{5.1}$$

$$N^\alpha(\dot{H}_\alpha - \partial_\alpha H_0) = 0 , \tag{5.2}$$

$$N^0(\partial_\alpha H_0 - \dot{H}_\alpha) + N^\beta(\partial_\alpha H_\beta - \partial_\beta H_\alpha) = 0 . \tag{5.3}$$

We notice that multiplying (5.3) by N^α, we obtain (5.2) since $N^0 \geq n$ is necessarily different from zero. Therefore, the system of equations can be reduced to (5.1) and (5.3).

Assume now that N^i and $\partial_\alpha N^i$ are given on an initial hypersurface $t = \text{const}$. Then the time derivatives $\dot{N}^0$ and $\dot{H}_\alpha$ are determined by these Cauchy data owing to (5.1) and (5.3), respectively. The question is whether $\dot{H}^\alpha$ is also determined.

Since we have

$$n^2 = N^i N_i ,$$

the time derivative of H^α is given by

$$(-1)^m \dot{H}^\alpha = c\dot{N}^\alpha + \frac{1}{n}\frac{dc}{dn} N^\alpha (N_0 \dot{N}^0 + N_\beta \dot{N}^\beta) ;$$

therefore, we have to solve for $\dot{N}^\beta$ the equations

$$\left[c\delta^\alpha_\beta + \frac{1}{n}\frac{dc}{dn} N^\alpha N_\beta \right] \dot{N}^\beta = \text{Cauchy data} . \tag{5.4}$$

Their solution is unique, if the corresponding determinant does not vanish:

$$c^{m-2}\left[c + \frac{1}{n}\frac{dc}{dn} N^\alpha N_\alpha \right] \neq 0 . \tag{5.5}$$

Since $c \geq 0$ because of the energetic conditions (4.13), in order to satisfy this requirement for arbitrary $N^\alpha N_\alpha \leq 0$, it is necessary and sufficient that the conditions

$$c > 0 \quad \text{and} \quad \frac{dc}{dn} \leq 0 \tag{5.6}$$

be satisfied.

For the equation of state given by a power law,

$$\epsilon = \mu n^{\nu - 1}, \quad \mu = \text{const} > 0 , \quad \nu = \text{const}$$

(alternatively $p = \nu\epsilon$), conditions (5.6) give $-\epsilon < p \leq \epsilon$. Since the energetic conditions already imply $-\epsilon \leq p \leq \epsilon$, conditions (5.6) are weak indeed; they exclude only the case of the cosmological constant, $p \neq -\epsilon$.

As the second case, we consider a fluid composed of membranes, $k = m-1$. In this case, it is convenient to use the duals $*N^i$ and $*H_i$ of the flux $(m-1)$-vector and the enthalpy $(m-1)$-covector, respectively, in order to write the equations of motion

$$dN = 0, \quad dH = 0 ,$$

in component form:

$$\partial_i * N_j - \partial_j * N_i = 0 ,$$

$$\partial_i * H^i = 0 .$$

In the Cartesian coordinates, the above equations read

$$*\dot{N}_\alpha - \partial_\alpha * N_0 = 0 , \tag{5.7}$$

$$\partial_\alpha * N_\beta - \partial_\beta * N_\alpha = 0 , \tag{5.8}$$

$$*\dot{H}^0 + \partial_\alpha * H^\alpha - = 0 . \tag{5.9}$$

Equation (5.8) plays the role of a constraint on the initial data $*N^i$ and $\partial_\alpha * N^i$. This constraint is conserved in time; i.e., its time derivative vanishes due to (5.7).

Similar to the case of the fluid composed of point particles, the time derivatives $*\dot{N}_\alpha$ and $*\dot{H}^0$ are determined by the initial data [due to (5.7) and (5.9), respectively] and we inquire whether $*\dot{N}_0$ is also determined. Since in the case of membranes we have

$$n^2 = - * N^i * N_i ,$$

the time derivative of $*H^0$ is

$$*\dot{H}^0 = c * \dot{N}^0 - \frac{1}{n}\frac{dc}{dn} * N^0 (* N_0 * \dot{N}^0 + * N_\alpha * \dot{N}^\alpha) ,$$

and we have to solve for $*\dot{N}^0$ the equation

$$\left[c - \frac{1}{n}\frac{dc}{dn} (*N^0)^2 \right] * \dot{N}^0 = \text{Cauchy data} . \tag{5.10}$$

The solution is unique for arbitrary $(*N^0)^2 \geq 0$, if the conditions (5.6) hold. The initial-value problem leads therefore to the same restrictions as in the case of the fluid of point particles.

As a third case, we consider a fluid composed of strings $(k=2)$. For this case, the condition of simplicity (4.2) becomes essential. This condition complicates the considerations; to simplify them we consider the case of four-dimensional space-time only.

The component form of the equations of motion for the fluid composed of strings is

$$\nabla_j N^{ij} = 0 ,$$

$$N^{jk}(\nabla_k H_{ij} + \nabla_j H_{ki} + \nabla_i H_{jk}) = 0 .$$

In four dimensions, it is convenient to use the traditional vector notation. Let $\mathbf{N} = (N^{\alpha 0})$ and $\mathbf{M} = (\frac{1}{2}\epsilon^{\alpha\beta\gamma} N_{\beta\gamma})$ be the three-dimensional vectors; then the above equations have the form

$$\dot{\mathbf{N}}+\nabla\times\mathbf{M}=0\ , \quad (5.11)$$

$$\nabla\cdot\mathbf{N}=0\ , \quad (5.12)$$

$$\mathbf{N}\times(c\mathbf{M})^{\cdot}-\mathbf{N}\times[\nabla\times(c\mathbf{N})]+\mathbf{M}[\nabla\cdot(c\mathbf{M})]=0\ . \quad (5.13)$$

$$\mathbf{M}\cdot(c\mathbf{M})^{\cdot}-M\cdot[\nabla\times(c\mathbf{N})]=0\ . \quad (5.14)$$

Moreover, the simplicity condition (4.2) gives the restriction

$$\mathbf{N}\cdot\mathbf{M}=0\ . \quad (5.15)$$

Let us note that multiplying (5.13) by $\mathbf{M}\times$, we get (5.14) multiplied by the vector $\mathbf{N}$; since $\mathbf{N}\neq\mathbf{0}$ [see (5.17) below], it means that (5.14) can be dropped.

The Cauchy data are now $\mathbf{N}$, $\mathbf{M}$, $\partial_\alpha\mathbf{N}$, and $\partial_\alpha\mathbf{M}$ subject to the constraints (5.12) and (5.15). Note that (5.12) is conserved in time due to (5.11). Because of (5.11), $\dot{\mathbf{N}}$ is determined by the initial data. The component of $\dot{\mathbf{M}}$ parallel to $\mathbf{N}$ can be expressed by the initial data, if we consider the equation

$$\mathbf{N}\cdot\dot{\mathbf{M}}+\dot{\mathbf{N}}\cdot\mathbf{M}=0\ , \quad (5.16)$$

obtained by differentiation of (5.15). [Notice that (5.15) is equivalent to (5.16) under the condition that (5.15) is imposed only as a constraint on the initial data.] The component of $(c\mathbf{M})^{\cdot}$ orthogonal to $\mathbf{N}$ is expressed, due to (5.13), by the initial data. The question is still whether the component $\dot{\mathbf{M}}_\perp$ of $\dot{\mathbf{M}}$ orthogonal to $\mathbf{N}$ can be expressed by the Cauchy data.

Since now we have

$$n^2=\mathbf{N}^2-\mathbf{M}^2\ , \quad (5.17)$$

we have to solve for the time derivative of the orthogonal component $\dot{\mathbf{M}}_\perp$, the equation

$$\mathbf{N}\times\left[c\dot{\mathbf{M}}-\frac{1}{n}\frac{dc}{dn}\mathbf{M}(\mathbf{M}\cdot\dot{\mathbf{M}})\right]=\text{Cauchy data}\ ,$$

analogous to (5.4) and (5.10). An alternative form of this equation is

$$c\dot{\mathbf{M}}_\perp-\frac{1}{n}\frac{dc}{dn}\mathbf{M}_\perp(\mathbf{M}_\perp\cdot\dot{\mathbf{M}}_\perp)=\text{Cauchy data}\ .$$

The corresponding 2×2 determinant should be different from zero:

$$c\left[c-\frac{1}{n}\frac{dc}{dn}\mathbf{M}_\perp\cdot\mathbf{M}_\perp\right]\neq0\ . \quad (5.18)$$

Since $\mathbf{M}_\perp\cdot\mathbf{M}_\perp\geq0$ is otherwise arbitrary, the condition (5.18) leads to exactly the same restrictions as in the cases of point particles and membranes.

The above discussion shows that from the point of view of the initial-value problem there is no essential difference between fluids of point particles, strings, and membranes. It is expected that in the case of arbitrary k and m the initial-value problem is well posed under conditions (5.6).

VI. THE JUNCTION CONDITIONS

This section is an extension of an Appendix in Ref. 2. We consider a hypersurface Σ given locally by the equation $\varphi=0$ and such that for $\varphi<0$ we have a fluid and for $\varphi>0$ we have a vacuum. Across the hypersurface Σ the matter quantities can be discontinuous. We require, however, that the basic equations of motion $Dt_i=0$ and $dN=0$ be satisfied in the sense of distributions. These differentials in the sense of distributions are[14]

$$dN=d\varphi\wedge N\,|_\Sigma\delta(\varphi)+\text{a regular part}\ ,$$

$$Dt_i=d\varphi\wedge t_i\,|_\Sigma\delta(\varphi)+\text{a regular part}\ ,$$

and the above requirement leads to the junction conditions:

$$d\varphi\wedge N\,|_\Sigma=0\ , \quad (6.1)$$

$$d\varphi\wedge t_i\,|_\Sigma=0\ . \quad (6.2)$$

In the equations below we shall omit the symbol $|_\Sigma$. The component form of the junction conditions is

$$N^{i_1\cdots i_k}\nabla_{i_k}\varphi=0\ , \quad (6.3)$$

$$t_j^i\nabla_i\varphi\equiv(\epsilon+p)q_j^i\nabla_i\varphi-p\nabla_i\varphi=0\ . \quad (6.4)$$

Assuming that N does not vanish on the boundary, the condition (6.1) or (6.3) implies that the gradient $\nabla\varphi$ is orthogonal to the distribution V given by (4.1). In turn, it implies that world sheets tangent to the distribution V are contained in the hypersurface Σ. Thus the hypersurface Σ is built out of the $(m-k-1)$-parameter family of world sheets. It means that $q_j^i\nabla_i\varphi=0$. Therefore, because of (6.4), pressure p vanishes on the boundary $p=0$. This case is standard.

On the boundary, the flux N can change its character from spacelike to null. In such a situation, condition (4.5) is not satisfied and the continuity equation does not follow. Although its archetypal form (4.3) should hold, it may not imply (6.1) since the form ω may itself contain $\delta(\varphi)$. Equation (6.2) remains as a single boundary condition.

We represent the limiting value of $*N$ as $\varphi\to-0$ in the form

$$*N=\kappa\wedge\theta^1\wedge\cdots\wedge\theta^{k-1}\ ,$$

where all 1-forms in the above decomposition are orthogonal, κ is null, and $\theta^1,\ldots,\theta^{k-1}$ are spacelike with square equal to -1. We require that the energy-momentum tensor be finite as $\varphi\to-0$. This amounts to two conditions,

$$\frac{\epsilon+p}{n^2}\to c=\text{finite}\quad\text{and}\quad p\to\text{finite}\ ,$$

and gives the following boundary form of the energy-momentum tensor:

$$t_j^i=c\kappa^i\kappa_j-p\delta_j^i\ .$$

The boundary condition (6.1) is now

$$c\kappa_j(\kappa_i\nabla^i\varphi)-p\nabla_j\varphi=0\ .$$

The case $c=0$ is trivial since then $p=0$ and the energy-momentum tensor vanishes on the boundary. If $c\neq0$, the above equation implies that pressure vanishes on the boundary, $p=0$, as in the standard case and, moreover, we obtain the equation

$$\kappa_i \nabla^i \varphi = 0 \ . \tag{6.5}$$

Remembering that the equation

$$T(\Sigma) = \{ Z : Z^i \nabla_i \varphi = 0 \}$$

determines the distribution of vector spaces tangent to the boundary Σ, the following three types of solutions of Eq. (6.5) are possible.

(1) The vector $\nabla^i \varphi$ is proportional to κ^i. It follows that $V \subset T(\Sigma)$. Therefore, as in the standard case, it implies that world sheets on the boundary are contained in Σ. In contrast with the standard case both Σ and V (on the boundary) are null. It means that the objects on the boundary move with the speed of light and the boundary of the fluid itself moves with the speed of light.

(2) The vector $\nabla \varphi$ is spacelike and orthogonal to V, $\nabla \varphi \perp V$. This case cannot happen for the membranes, $k \neq m-1$. In this case still $V \subset T(\Sigma)$ so the world sheets are contained in the boundary Σ. The hypersurface Σ is timelike and V on the boundary is null. It means that the objects on the boundary move with the speed of light, whereas the speed of the boundary itself is less. The boundary might be static.

(3) $\nabla \varphi = (\nabla \varphi)_V + (\nabla \varphi)_\perp$, where $(\nabla \varphi)_V$ is spacelike and belongs to V, whereas $(\nabla \varphi)_\perp$ belongs to the orthogonal complement of V in the space orthogonal to the null vector κ^i and may vanish. This case cannot happen for particles $k \neq 1$. In this case the vector $(\nabla \varphi)_V$ is not tangent to Σ. Therefore, $\dim[V \cap T(\Sigma)] = k-1$ and the $(k-1)$-dimensional objects can have a $(k-2)$-dimensional boundary moving on the timelife hypersurface Σ with the speed of light. This type of behavior is well known in the special case of open strings. For objects with a boundary this case has to happen necessarily.

These three cases exhaust all possibilities.

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