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Stephenson-Kilmister-Yang theory of gravity and its dynamics

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We investigate the Stephenson-Kilmister-Yang (SKY) gravitational Lagrangian in the framework of SO(3,1) gauge theories. It is proved that by an appropriate choice of dynamical variables this theory can be cast in a Hamiltonian form. Thirty-six dynamical variables $X^{k}{}_{(\alpha)(\beta)}, Y^{k(\alpha)(\beta)}$ resemble the electric field and the magnetic induction in Yang-Mills theories of internal symmetries. Their evolution is governed by a Maxwell-type system of differential equations. There are 16 constraints for the initial values of the dynamical variables, of which 10 are first class and 6 are second class. The full gauge group of the theory is parametrized by 13 "functions" on spacetime and is essentially larger than the 10-parameter full gauge group of generic SO(3,1) theories of gravity. Additional 3-parameter gauge transformations in the set of field variables are generated by a nonstandard action of Lorentz boosts. There are 3 gauge variables related to these transformations. Only 10 of the 13 gauge transformations act independently in the set of dynamical variables. Therefore the theory has $36-(16+10)=10$ independent degrees of freedom in the phase space. It is also shown that the SKY gravity naturally couples to matter Yang-Mills fields maintaining all of its features. A brief discussion of a conceivable coupling with vector matter fields is presented.

I. INTRODUCTION

There are at least two reasons why quadratic Lagrangians became a field of profound interest in gravitation.

(i) First, the development of gauge theories of internal symmetries in elementary-particle physics inspired research in gauge formulations of the Einstein theory.[1] This inevitably led to the Sciama-Kibble-Trautman gauge approach to gravity and to the replacement of standard Riemann-Einstein spacetimes with Riemann-Cartan geometries.[2-4]

Treating tetrad and connection coefficients as independent variational potentials we take torsion and curvature as corresponding field strengths and then the gravitational Lagrangian L is an invariant density constructed from tetrad and connection coefficients as well as from their first partial derivatives. The field equations follow from the Einstein-Palatini variational principle with the tetrad components $e^{(\alpha)}{}_{\mu}$ and the connection coefficients $\Gamma_{\mu}{}^{(\alpha)(\beta)}$ as independent variational potentials. Therefore we have two subsets of field equations:

$$(E1)^{\lambda}{}_{(\alpha)}=\delta L/\delta e^{(\alpha)}{}_{\lambda}=0\ , \tag{1.1a}$$

$$(E2)^{\lambda}{}_{(\alpha)(\beta)}=\delta L/\delta \Gamma_{\lambda}{}^{(\alpha)(\beta)}=0\ . \tag{1.1b}$$

Such a picture is standard in contemporary SO(3,1) or SL(2,C) gauge formulations of gravity (cf. papers by Trautman,[5] Hehl, von der Heyde, and co-workers,[6,7] Tseytlin,[8] Ivanenko and Sandanashvily,[9] Ne'eman,[10] Szczyrba,[11] Antonowicz and Szczyrba,[12] Blagoević and Nikolić,[13,14] and Grensing and Grensing[15]) and the most natural gravitational Lagrangian for such a theory is constructed from the scalar curvature R and invariants quadratic in torsion and curvature. Several such quadratic Lagrangians were proposed and examined in the literature (Yang,[16] Fairchild and co-workers,[17,18] Hehl, von der Heyde and co-workers,[6] Nieh and Rauch[19] Neville,[20] Sezgin and van Nieuwenhuizen,[21] Hayashi and Shirafuji,[22] Miyamoto, Nakano, Ohtani, and Tamura,[23] Fukui and co-workers,[24] and Schweitzer[25]).

Some authors investigated classical solutions of such theories (spherical and axially symmetric, plane waves): Baekler-Yasskin,[26] Hehl and co-workers,[27,28] Lenzen,[29] Baekler,[30] McCrea,[31] Benn, Derelli, and Tucker,[32] Zhang,[33] Mielke,[34] Chen, Chern, Hsu, and Yeung,[35] Cannale, De Ritis, and Tarantino,[36] and Müller and Schmidt.[37] Others discussed the linear approximations of particular theories and their particle spectra: Sezgin and van Nieuwenhuizen,[21] Hayashi and co-workers.[22-25] Problems related to the Birkhoff theorem were investigated profoundly by Rauch and Nieh[38] in a general case, and by Riegert[39] in conformal gravity. Strominger[40] proved the positive-energy theorem for a special quadratic Lagrangian.

(ii) Second, essential difficulties in the quantization of Einstein's gravity have stimulated interest in higher-order gravitational Lagrangians built from invariants of the curvature tensor. In the 1960s DeWitt[41] suggested that quadratic terms in the gravitational Lagrangian may cure the divergence problems. Detailed calculations on the renormalizability of higher-order Lagrangians in gravity were performed by Stelle,[42] Julve and Tonin,[43] Salam and Stradthdee,[44] Tomboulis,[45] Hasslacher and Mottola,[46] Fradkin and Tseytlin,[47] Kaku,[48] Boulware, Horowitz and Strominger,[49] and Kawasaki and co-workers.[50] The most recent results in that direction and a comprehensive bibliography can be found in Refs. 51–56.

Let us observe an essential difference between classical (i) and quantum (ii) approaches to quadratic gravitational Lagrangians. In the classical SO(3,1) gauge picture the tetrad and connection coefficients are treated as independent variational potentials and torsion appears naturally in the theory. In the papers devoted to quantum aspects of the problem the Einstein-Hilbert picture prevails. The connection coefficients are computed from tetrads (from the metric) by means of the Levi-Civita formula. Spacetime is always Riemannian and torsion is absent.

To be precise there are, however, some papers devoted to the quantum aspects of the gravitational theories with torsion[20,21] but Riemann-Cartan spacetimes are not very popular in quantum gravity yet.

On the classical level the field equations derived from the Einstein-Hilbert (EH) and Einstein-Palatini (EP) variational principles are not equivalent (only for the Einstein Lagrangian in vacuum do they lead to the same field equations). Let us take as an example the Stephenson-Kilmister-Yang (SKY) Lagrangian

$$L=\tfrac{1}{4}\sqrt{-g}\,R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta}{}^{\mu\nu}\ . \tag{1.2}$$

In the Einstein-Hilbert picture spacetime is Riemannian and we get the following system of fourth-order equations for the components of a metric:

$$\tfrac{1}{2}[\nabla_\beta\nabla_\alpha(R^{\nu\beta\alpha\mu}+R^{\mu\beta\alpha\nu})+\tfrac{1}{4}g^{\mu\nu}R^{\alpha\beta}{}_{\epsilon\tau}R_{\alpha\beta}{}^{\epsilon\tau}-R^{\alpha\beta\mu}{}_{\tau}R_{\alpha\beta}{}^{\nu\tau}]=0\ . \tag{1.3}$$

In general, the Einstein-Palatini approach gives rise to spacetimes with torsion. If we consider, however, a special case of Riemannian spacetimes satisfying the EP variational equations for the SKY Lagrangian then we get the system

$$\nabla_\alpha R^{\nu\beta\alpha\mu}=0\ , \tag{1.4a}$$

$$\tfrac{1}{4}g^{\mu\nu}R^{\alpha\beta}{}_{\epsilon\tau}R_{\alpha\beta}{}^{\epsilon\tau}-R^{\alpha\beta\mu}{}_{\tau}R_{\alpha\beta}{}^{\nu\tau}=0\ . \tag{1.4b}$$

It is easy to see that on the level of Riemannian spacetimes the EP SKY equations (1.4) have fewer solutions than the EH SKY equations (1.3). On the other hand, the SKY equations in a Riemann-Cartan spacetime have additional solutions with torsion and we may conclude that the sets of solutions for EP and EH SKY equations are completely different.

First-order theories derived from the Einstein-Palatini variational principle can be cast in a Hamiltonian form based on the natural symplectic (Poisson) structure in the set of all conceivable configurations of their fields.[11,12] For second-order theories a general symplectic-Hamiltonian formulation has not been presented in the literature yet. Efforts of several authors have not given rise to any essential progress in that field.[57–59] The formula for the symplectic two-form seems to be particularly elusive. Recently, however, the present author has found a new approach to the problem that enabled him to define a symplectic two-form for any gravitational theory with second- (and higher-) order Lagrangians, find appropriate dynamical variables for the theory in question, as well as write the dynamical and constraint equations in terms of these variables. Moreover, a complete canonical classification of Lagrangians quadratic in the curvature tensor has been accomplished.[60]

The results of that paper correspond to Boulware's analysis of quadratic Lagrangians in gravity.[54] Taking into account these two papers we may state that both the EP and EH variational principles give rise to infinite-dimensional Hamiltonian systems. Thus, both approaches are acceptable from the physical point of view, especially if quantizations of those systems are considered.[61]

The SO(3,1) theories, however, form a much richer class and their dynamics reveals many interesting features, which has not been observed for systems of the EH field equations. Without any doubt peculiarities of the SO(3,1) dynamics have their counterparts in the structure of supergravity theories, which are considered to be much more viable than SO(3,1) models.

In the present paper we study the SKY system in Riemann-Cartan spacetime. In spite of a strong degeneracy of the SKY Lagrangian our analysis leads to a reasonable and consistent dynamical picture. These results question arguments given by some authors[25] against SKY gravity, which in our opinion are based on an incomplete analysis of its structure.

In view of present results we may expect to have an ample class of SO(3,1) theories with a consistent dynamics. Some of them share certain characteristics of SKY gravity[62] and others are different.[63]

Now we outline the results of the present paper.

For a fixed slicing of spacetime in a family of three-dimensional surfaces we take the following quantities as the dynamical variables:

$$X^k{}_{(\alpha)(\beta)}=-\hat{\varepsilon}\,\hat{R}_{(\alpha)(\beta)0}{}^k\ , \tag{1.5a}$$

$$Y^{k(\alpha)(\beta)}=\tfrac{1}{2}\epsilon^{kuv}\hat{R}^{(\alpha)(\beta)}{}_{uv}\ . \tag{1.5b}$$

[The caret over the symbol of a geometric object denotes its components in a special coordinate system compatible with the slicing (cf. Appendix A).] We may write Eqs. (1.4a) in Maxwell form:

$${}^\dagger\hat{D}_0X^k{}_{(\alpha)(\beta)}=\epsilon^{kuv}({}^\dagger\hat{D}_u+\partial_u\ln N)(Y^r{}_{(\alpha)(\beta)}\bar{\mathscr{g}}_{rv})\ , \tag{1.6}$$

$${}^\dagger\hat{D}_kX^k{}_{(\alpha)(\beta)}=0\ . \tag{1.7}$$

Here $\bar{\mathscr{g}}_{ij}=(\bar{g})^{-1/2}\bar{g}_{ij}$ is the three-metric density of weight -1 on slices and the covariant 3-derivatives ${}^\dagger\hat{D}_0$ and ${}^\dagger\hat{D}_k$ are defined in Appendix A.

The relations between the connection coefficients and the curvature tensor give rise to the second pair of the Maxwell equations:

$${}^\dagger\hat{D}_0Y^{k(\alpha)(\beta)}=-\epsilon^{kuv}({}^\dagger\hat{D}_u+\partial_u\ln N)(X^{r(\alpha)(\beta)}\bar{\mathscr{g}}_{rv})\ , \tag{1.8}$$

$${}^\dagger\hat{D}_kY^{k(\alpha)(\beta)}=0\ . \tag{1.9}$$

The system (1.6)–(1.9) resembles the Maxwell form of the Yang-Mills equations with $X^k{}_{(\alpha)(\beta)}$ corresponding to the electric field $E^k{}_A$ and $Y^{k(\alpha)(\beta)}$ to the magnetic induction B^{kA}.

Equations (1.4b) can be formulated as constraints for the dynamical variables

$$-\epsilon_{pij}X^i{}_{(\alpha)(\beta)}Y^{j(\alpha)(\beta)}=0\ , \tag{1.10}$$

$$X^k{}_{(\alpha)(\beta)}X^{s(\alpha)(\beta)}+Y^k{}_{(\alpha)(\beta)}Y^{s(\alpha)(\beta)}=0\ . \tag{1.11}$$

Therefore we have $18+18=36$ dynamical equations (1.6) and (1.8) for 36 dynamical variables (1.5) as well as $6+9=15$ constraints (1.7), (1.10), and (1.11) for the dynamical variables. [Equations (1.9) follow from the Bianchi identities and therefore they do not bring any new information.] As we show in Sec. V this system of constraints is incomplete. There exists a sixteenth constraint providing the time conservation of (1.11).

Such a formulation of the dynamics is very elegant. Only the role of the metric density $\bar{\mathcal{g}}_{ij}$ seems to be unclear. We prove in Sec. V that this quantity can be determined algebraically in terms of the dynamical variables and their spatial derivatives up to a conformal (scale) factor τ. The function τ satisfies a first-order partial differential equation on slices, which can be solved by the method of characteristics. Therefore the metric density $\bar{\mathcal{g}}_{ij}$ on slices can be treated as a quantity depending functionally on the dynamical variables and on the lapse N (N enters the above-mentioned partial differential equation for τ). Now the constraints are consistent with the dynamics and the formal initial-value problem is well posed.

In order to determine the number of independent degrees of freedom of the theory in question we have to know the full group of its gauge transformations. SO(3,1) theories are always invariant with respect to local Lorentz rotations and the action of the diffeomorphism group of spacetime. Surprisingly, the gauge group for SKY gravity is essentially larger. We have the following set of gauge transformations preserving the set of solutions: (i) Diff$M$ transformations with 4 gauge variables N and N^k; (ii) standard SO(3) rotations with 3 gauge variables $\hat{\Gamma}_0{}^{(a)(b)}$; (iii) standard boost transformations with 3 gauge variables $n^{(a)}$; (iv) nonstandard SO(3) rotations with 3 gauge variables expressed by three of nine triad components $\hat{e}^{(a)}{}_k$; (v) nonstandard boost transformations with 3 gauge variables $\hat{\Gamma}_0{}^{(0)(a)}$. The detailed analysis shows that in the space of the dynamical variables $X^k{}_{(\alpha)(\beta)}, Y^{k(\alpha)(\beta)}$ only the transformations (i), (ii), and (v) act independently. These transformations correspond to 10 gauge variables N, N^k, $\hat{\Gamma}_0{}^{(\alpha)(\beta)}$, which appear in the dynamical field equations.

Eventually, we have 36 dynamical variables subject to 16 constraints and 10 gauge transformations. That means, we have $36-(16+10)=10$ independent degrees of freedom in the phase space (five degrees of freedom in the configuration space).

The fact that SKY gravity is a direct gravitational counterpart of Yang-Mills theories of internal symmetries is evidently seen when we couple these two fields. Such a system is naturally consistent for the energy-momentum tensor of the Yang-Mills fields is symmetric and traceless. Coupling SKY gravity to other matter fields causes some problems but even those can be overcome. We discuss these questions in Sec. VI.

The present paper is the first of a planned series devoted to the dynamics of quadratic Lagrangians in spacetimes with torsion. It is based on ideas and methods developed in our previous papers devoted to the dynamics of SO(3,1), SL(2,C), and GL(4,R) gauge theories of gravity.[11,12,64–66] The analysis of general gravitational Lagrangians performed in Refs. 11, 12, and 65 indicates that for particular Lagrangians the problem of dynamical variables, their evolution, constraints, gauge transformations, and independent degrees of freedom should be studied for each case separately. Even in the Einstein-Cartan theory with tensor or spinor matter fields the situation is essentially different for different matter Lagrangians.[64,12,66]

The analysis of the SKY theory confirms this point of view. Here, new gauge transformations appear of which the nonstandard action of boosts is the most interesting. This gauge invariance induces three new gauge variables $\hat{\Gamma}_0{}^{(0)(a)}$. [In the Einstein-Cartan (EC) theory these quantities are determined as functions of the dynamical variables and the translational gauge variable N.] Also the role of the three-metric $\bar{g}_{ij}$ on slices is different in both theories. In the EC theory $\bar{g}_{ij}$ are dynamical quantities. In SKY gravity, on the contrary, they are not dynamical and can be determined by means of the dynamical and gauge variables.

We may expect that for other gravitational theories the dynamical picture has its own features and such an analysis for various classes of gravitational Lagrangians will be the subject of our subsequent publications.[62] Especially interesting is the case of Hehl–von der Heyde-type Lagrangians with dynamical torsion whose Hamiltonian analysis has recently been accomplished.[63]

The Hamiltonian dynamics of quadratic Lagrangians in gravity was also discussed by Blagoević and Nikolić.[13,14] These authors separated subclasses of Lagrangians quadratic in torsion and curvature that have the same types of primary constraints. Unfortunately, they did not present a complete set of constraints restricting the discussion to the secondary ones. Our analysis of the SKY theory, where a nontrivial tertiary constraint appears, shows that higher-order constraints are essential in the dynamical picture.

Finally, we would like to say a few words about the history of the Lagrangian (1.2). The essential paper that led to further investigation in this direction was that by Yang.[16] Observing that the field equations in gauge field theories of internal symmetries can be treated as conditions for sourceless fields

$$\partial_\lambda f^{I\lambda\mu}=0 \ ,$$

where

$$f^{I\lambda\mu}=\partial^\lambda A^{I\mu}-\partial^\mu A^{I\lambda}+c^I{}_{LM}A^{L\lambda}A^{M\mu} \ ,$$

Yang derived Eqs. (1.4a) as the conditions for a sourceless GL(4,R) gauge field represented by the Levi-Civita connection of a metric on spacetime. Yang's paper inspired a series of publications devoted to the system (1.4a). Pavelle[67,68] observed that this system had already been proposed by Kilmister and Newman[69] in the 1960s. Later Hehl with collaborators[6] added one more name as they discovered Stephenson's[70] contribution to the problem (see also Higgs[70]). Pavelle,[67,68] Ni,[71] and Thompson[72] proved that all vacuum solutions of Einstein's equations satisfy (1.4a). Ni gave a more profound analysis of the problem pointing out that all vacuum solutions of the Einstein theory with a cosmological constant as well as all solutions of Nordström's theory satisfy (1.4a). Moreover, other non-Einsteinian solutions for (1.4a) were found.[67,68,71–74]

These papers proved that the system (1.4a) does not satisfy the Birkhoff theorem. In particular, the two-parameter family of solutions discussed in Ref. 73 contains the Schwarzschild metric and all metrics of this family except the Schwarzschild one have "naked" singularities as $r=0$. The system (1.4a) has a lot of unphysical solutions even for the stationary SO(3)-symmetric case. The situation changes if we consider the complete system (1.4a), (1.4b). Baekler, Hehl, and Mielke[27] found all torsionless spherically symmetric solutions for the SKY system (1.4). They are the Schwarzschild–de Sitter, Nariai and Ni metrics (see also Refs. 26 and 75).

Therefore the complete SKY system (1.4) has a much more reasonable set of solutions than the original one (1.4a) and can be taken as the basis for a physical theory of a model character. On the other hand, such a reasonable reduction of the set of solutions may be considered as a strong argument in favor of the Einstein-Palatini variational principle in gravity.

Notation. Throughout the paper greek indices run from 0 to 3, latin indices run from 1 through 3. D_λ denotes the SO(3,1)-covariant derivative and strictly corresponds to $\mathcal{D}_\lambda$ in Refs. 64, 65, and 12. $\hat{D}_0$ and $\hat{D}_k$ are SO(3)-covariant derivatives on the surfaces of the slicing. In Refs. 64, 65, and 12 they were denoted by capital script characters. The dagger-covariant derivatives ${}^\dagger\hat{D}_0$ and ${}^\dagger\hat{D}_k$ are defined in Appendix A. The reader should take into account the difference between $\hat{D}_\lambda$ and ${}^\dagger\hat{D}_\lambda$ operators. The symbols $\mathscr{g}_{ij}$ and $\mathscr{h}_{ij}$ are used to denote tensor densities of weight -1 on three-dimensional slices. In contrast to Refs. 64, 65, and 12, in the present paper tensor densities of weight $+1$ are denoted by roman characters and not by script ones.

II. THE COVARIANT FORMULATION OF THE SKY THEORY OF GRAVITY

We start with the following gravitational Lagrangian:

$$L=\tfrac{1}{4}eR^{(\alpha)(\beta)}{}_{\mu\nu}R_{(\alpha)(\beta)}{}^{\mu\nu}\ . \tag{2.1}$$

Its Euler-Lagrange equations read

$$(E1)^\lambda{}_{(\alpha)}=\delta L/\delta e^{(\alpha)}{}_\lambda=V^\lambda{}_{(\alpha)}=0\ , \tag{2.2a}$$

$$(E2)^\lambda{}_{(\alpha)(\beta)}=\delta L/\delta\Gamma_\lambda{}^{(\alpha)(\beta)} =-D_\mu(eR_{(\alpha)(\beta)}{}^{\mu\lambda})=0\ . \tag{2.2b}$$

Here

$$V^\lambda{}_{(\alpha)}=\tfrac{1}{4}ee^\lambda{}_{(\alpha)}R^{(\omega)(\tau)}{}_{\mu\nu}R_{(\omega)(\tau)}{}^{\mu\nu}-eR_{(\omega)(\tau)}{}^{\lambda\nu}R^{(\omega)(\tau)}{}_{\mu\nu}e^\mu{}_{(\alpha)} \tag{2.3}$$

is the canonical energy-momentum tensor of the gravitational field. This quantity is symmetric and traceless.

A very important question is whether the system of variational equations (2.2) has any solution. Recently, some interesting results in that direction have been obtained by Baekler and co-workers,[27–30] McCrea,[31] Benn, Derelli, and Tucker,[32] and others.[32–36] In particular, the following duality principle was proposed in Refs. 27 and 75. If we define the involutive double-dual operation for the curvature tensor

$${}^*R^{*(\alpha)(\beta)}{}_{\mu\nu}=\tfrac{1}{4}\epsilon^{(\alpha)(\beta)(\rho)(\sigma)}eR_{(\rho)(\sigma)}{}^{\tau\omega}\epsilon_{\mu\nu\tau\omega} \tag{2.4}$$

then the field equations (2.2b) can be rewritten as

$$-\tfrac{1}{4}\epsilon_{(\alpha)(\beta)(\rho)(\sigma)}D_\mu{}^*R^{*(\sigma)(\rho)}{}_{\tau\omega}\epsilon^{\mu\nu\tau\omega}=0\ . \tag{2.2b$'$}$$

On the other hand, the energy-momentum equations (2.2a) read

$$(E1)^\mu{}_\nu=-e({}^+R_{(\alpha)(\beta)\nu\omega}{}^-R^{(\alpha)(\beta)\mu\omega} +{}^-R_{(\alpha)(\beta)\nu\omega}{}^+R^{(\alpha)(\beta)\mu\omega})=0\ , \tag{2.2a$'$}$$

where

$${}^\pm R^{(\alpha)(\beta)}{}_{\nu\omega}=\tfrac{1}{2}(R^{(\alpha)(\beta)}{}_{\nu\omega}\pm{}^*R^{*(\alpha)(\beta)}{}_{\nu\omega})\ ,$$

are the self- and anti-self-dual parts of the Riemann tensor, respectively. We see that for self- and anti-self-dual curvature tensors the field equations (2.2a′) are satisfied. Moreover, in this case Eqs. (2.2b′) are satisfied by virtue of the Bianchi identities for the Riemann tensor. We conclude that the vacuum SKY equations have at least two subclasses of solutions consisting of self- and anti-self-dual connections. In a special case of vanishing torsion we have, from (2.4),

$${}^*R^{*\alpha\beta}{}_{\mu\nu}=R^{\alpha\beta}{}_{\mu\nu}+\tfrac{1}{2}(\delta^\alpha{}_\mu\delta^\beta{}_\nu-\delta^\alpha{}_\nu\delta^\beta{}_\mu)R -(\delta^\alpha{}_\mu R_\nu{}^\beta-\delta^\alpha{}_\nu R_\mu{}^\beta+\delta^\beta{}_\nu R_\mu{}^\alpha-\delta^\beta{}_\mu R_\nu{}^\alpha)\ . \tag{2.4$'$}$$

Therefore for Riemannian spacetimes the self-dual geometries satisfy the equations

$$R_\mu{}^\alpha-\tfrac{1}{4}\delta^\alpha{}_\mu R=0\ . \tag{2.5}$$

By virtue of the contracted Bianchi identities (2.5) are equivalent to

$$R_\mu{}^\alpha=\text{const}\times\delta^\alpha{}_\mu\quad\text{(Einstein spaces)}\ . \tag{2.5$'$}$$

Similarly, for the anti-self-dual Riemannian case we get

$$R=0,\quad C^{\alpha\beta\mu\nu}=0\quad\text{(Nordström spaces)}\ ,$$

where $C^{\cdots}$ is the Weyl tensor.

We conclude that the vacuum SKY equations have at least two series of solutions: Riemann-Einstein and Riemann-Nordström spaces. For spherically symmetric geometries the converse result is also true.[27] The solutions of the vacuum SKY equations belong to self- or anti-self-dual geometries. Therefore the corresponding metrics are Schwarzschild–de Sitter or Nariai in the first class and the Ni metric in the latter.

III. THE FIELD EQUATIONS IN THE (3+1)-FORM

The gravitational momenta for the SKY Lagrangian (2.1) read

$$P^{\lambda\tau}{}_{(\alpha)(\beta)}=\partial L/\partial(\partial_\lambda\Gamma_\tau{}^{(\alpha)(\beta)})=eR_{(\alpha)(\beta)}{}^{\lambda\tau}\ , \tag{3.1a}$$

$$U^{\lambda\tau}{}_{(\alpha)}=\partial L/\partial(\partial_\lambda e^{(\alpha)}{}_\tau)=0\ . \tag{3.1b}$$

In the (3 + 1) formulation we have[11,64]

$$\hat{P}^{0k}{}_{(\alpha)(\beta)}=-\hat{e}\,\hat{R}_{(\alpha)(\beta)0s}\bar{g}^{sk}\ ,\qquad \hat{P}^{pq}{}_{(\alpha)(\beta)}=\hat{e}\,\hat{R}_{(\alpha)(\beta)mn}\bar{g}^{mp}\bar{g}^{nq}\ . \tag{3.2}$$

Let us introduce the very convenient notation

$$X^{k}{}_{(\alpha)(\beta)}=\hat{P}^{0k}{}_{(\alpha)(\beta)}\ , \tag{3.3a}$$

$$Y^{k(\alpha)(\beta)}=\tfrac{1}{2}\epsilon^{kuv}\hat{R}^{(\alpha)(\beta)}{}_{uv}\ , \tag{3.3b}$$

$$\hat{R}^{(\alpha)(\beta)}{}_{uv}=\epsilon_{uvk}Y^{k(\alpha)(\beta)}\ . \tag{3.3b$'$}$$

We show that by means of these variables the field equations can be written in an elegant Maxwell-type form. The quantities $X^{k}{}_{(\alpha)(\beta)}$ and $Y^{k(\alpha)(\beta)}$ may be considered as counterparts of the electric field $E^{k}{}_{A}$ and the magnetic induction B^{kA} in Yang-Mills theories.[76]

The equations $(\hat{E}1)^{0}{}_{p}=0$ read

$$-\epsilon_{pij}X^{i}{}_{(\alpha)(\beta)}Y^{j(\alpha)(\beta)}=0\ . \tag{3.4}$$

If we define the symmetric tensor density of weight 2,

$$C^{ks}=\hat{e}[(\hat{E}1)^{k}{}_{(a)}\hat{e}^{(a)s}+\bar{g}^{ks}(\hat{E}1)^{0}{}_{(0)}]\ , \tag{3.5}$$

then 6 equations

$$C^{ks}=0 \tag{3.6}$$

are equivalent to 7 equations

$$(\hat{E}1)^{0}{}_{(0)}=0,\quad (\hat{E}1)^{k}{}_{(a)}=0\ . \tag{3.7}$$

This fact is due to the vanishing of the trace of the gravitational energy-momentum tensor (2.3).

Taking into account relations (B2) and (3.3b$'$) we write (3.6) in the explicit form

$$C^{ks}=X^{k}{}_{(\alpha)(\beta)}X^{s(\alpha)(\beta)}+Y^{k(\alpha)(\beta)}Y^{s}{}_{(\alpha)(\beta)}$$
$$=0\ . \tag{3.6$'$}$$

Because of the symmetry of the gravitational energy-momentum tensor the equations $(\hat{E}1)^{k}{}_{(0)}=0$ are equivalent to (3.4). The equations $(\hat{E}2)^{0}{}_{(0)(b)}=0$ read

$$^{\dagger}\hat{D}_{p}X^{p}{}_{(0)(b)}=0\ . \tag{3.8a}$$

The equations $(\hat{E}2)^{0}{}_{(a)(b)}=0$ read

$$^{\dagger}\hat{D}_{p}X^{p}{}_{(a)(b)}=0\ . \tag{3.8b}$$

From the equations $(\hat{E}2)^{k}{}_{(\alpha)(\beta)}=0$ we get the first part of the dynamical "Maxwell equations"

$$^{\dagger}\hat{D}_{0}X^{k}{}_{(\alpha)(\beta)}=\epsilon^{kuv}(^{\dagger}\hat{D}_{u}+\partial_{u}\ln N)(Y^{t}{}_{(\alpha)(\beta)}\bar{\mathcal{g}}_{tv})\ . \tag{3.9}$$

The definition (3.3) and the Bianchi identities for the Riemann tensor (A24) give rise to the second part of the dynamical "Maxwell equations":

$$^{\dagger}\hat{D}_{0}Y^{k(\alpha)(\beta)}=-\epsilon^{kuv}(^{\dagger}\hat{D}_{u}+\partial_{u}\ln N)(X^{t(\alpha)(\beta)}\bar{\mathcal{g}}_{tv})\ . \tag{3.10}$$

The corresponding equations for potentials read

$$^{\dagger}\hat{D}_{0}\hat{\Gamma}_{k}{}^{(\alpha)(\beta)}=-X^{t(\alpha)(\beta)}\bar{\mathcal{g}}_{tk}\ . \tag{3.10$'$}$$

From the Bianchi identities we get the relations

$$^{\dagger}\hat{D}_{k}Y^{k(\alpha)(\beta)}=0\ , \tag{3.11}$$

which are the counterparts of (3.8).

The covariant dagger derivatives $^{\dagger}\hat{D}_{0}$ and $^{\dagger}\hat{D}_{k}$ are defined in Appendix A. We also note the following relations resulting from (A20):

$$Y^{k(\alpha)(\beta)}=\tfrac{1}{4}\epsilon^{kuv}(^{\dagger}\hat{D}_{u}\hat{\Gamma}_{v}{}^{(\alpha)(\beta)}-{}^{\dagger}\hat{D}_{v}\hat{\Gamma}_{u}{}^{(\alpha)(\beta)})\ . \tag{3.12}$$

Now we discuss the symplectic dynamics of the system (3.4), (3.6), and (3.8)–(3.10). It is known[11,12] that for a general SO(3,1) gauge theory of gravity the gravitational symplectic variables are

$$\hat{U}^{0k}{}_{(a)},\hat{e}^{(a)}{}_{k},\hat{P}^{0k}{}_{(\alpha)(\beta)},\hat{\Gamma}_{k}{}^{(\alpha)(\beta)},M_{(a)},n^{(a)}\ . \tag{3.13}$$

The quantities $n^{(a)}$ are the direction coefficients of the tetrad field $e^{(\alpha)}{}_{\tau}$ with respect to the vector field $\mathcal{N}$ normal to the slicing of spacetime. Their conjugate momenta $M_{(a)}$ are given by linear combinations of $(\hat{E}2)^{0}{}_{(0)(b)}$, the symplectic variables $\hat{U}^{0k}{}_{(a)},\hat{e}^{(a)}{}_{k},\hat{P}^{0k}{}_{(\alpha)(\beta)},\hat{\Gamma}_{k}{}^{(\alpha)(\beta)}$ as well as their spatial derivatives. The detailed formulas are presented in Appendix B.

For the SKY Lagrangian the general approach presented in Refs. 11 and 12 should be simplified and modified. In this case we have primary (kinematical) constraints (3.1b) that lead to the relations

$$\hat{U}^{0k}{}_{(a)}=0\ . \tag{3.14}$$

The symplectic form on the initial surface σ reads

$$\Omega(X_1,X_2)=\int_{\sigma}[\delta_1\hat{P}^{0k}{}_{(\alpha)(\beta)}\wedge\delta_2\hat{\Gamma}_{k}{}^{(\alpha)(\beta)}+\delta_1M_{(a)}\wedge\delta_2n^{(a)}]dx^1\wedge dx^2\wedge dx^3\ . \tag{3.15}$$

Here $\delta_i\hat{P}^{0k}{}_{(\alpha)(\beta)}$, $\delta_i\hat{\Gamma}_{k}{}^{(\alpha)(\beta)}$, $\delta_iM_{(a)}$, $\delta_in^{(a)}$, $i=1,2$ represent vectors X_i tangent to the space of geometric configuration of the system. The symbol $\wedge$ denotes antisymmetrization with respect to the subscripts 1 and 2.

The symplectic analysis of the kinematical structure of the Yang theory shows the following. (i) Equations (3.4) and (3.8b) are symplectic constraints; they are called the spatial translational Hamiltonian constraints and SO(3)-rotational Hamiltonian constraints, respectively, and are typical for any SO(3,1) gauge theory of gravity; it is important that the left-hand sides of these constraints are functions of the dynamical symplectic variables

$$X^{k}{}_{(\alpha)(\beta)}\ \text{and}\ \hat{\Gamma}_{k}{}^{(\alpha)(\beta)} \tag{3.16}$$

and they do not depend on $M_{(a)}$. (ii) For general SO(3,1) gauge theory the left-hand sides of Eqs. (3.8a) depend on the dynamical symplectic variables $\hat{U}^{0k}{}_{(a)}$, $\hat{e}^{(a)}{}_{k}$, $\hat{P}^{0k}{}_{(\alpha)(\beta)}$, $\hat{\Gamma}_{k}{}^{(\alpha)(\beta)}$, their spatial derivatives as well as on the momenta $M_{(a)}$. For the SKY Lagrangian, however, the left-hand sides of constraints (3.8a) are functions of the dynamical symplectic variables (3.16) (no dependence on $M_{(a)}$). We show later that these three additional constraints on the SKY theory are related to additional three-parameter gauge transformations and to three additional gauge variables $\hat{\Gamma}_{0}{}^{(0)(c)}$.

Now we discuss the dynamics of the theory.

(1) It follows from (3.8) and (B1) that the dynamics of the quantities $M_{(a)}$ is trivial. That is,

$$M_{(a)}=0\ . \tag{3.17}$$

The symplectic constraints (3.17) give rise to a degeneracy of the symplectic two-form Ω on the space of solutions and the corresponding gauge subspace is generated by the action of standard boost transformations in the space of the symplectic variables (3.13). We do not discuss this problem in detail because it is typical for any SO(3,1) gauge theory of gravity and has been explained profoundly in Refs. 11 and 12.

(2) In the space of dynamically admissible initial data on a surface σ the constraints (3.17) hold and therefore the momenta $M_{(a)}$ as well as their conjugate positions $n^{(a)}$ are eliminated from the symplectic two-form Ω. The reduced system of symplectic variables is (3.16).

(3) Equations (3.4), (3.6), and (3.8) are constraints for the reduced symplectic variables; Eqs. (3.9) and (3.10) govern their dynamics.

(4) Because Eqs. (3.4) and (3.8b) are Hamiltonian constraints they are preserved in the process of evolution by virtue of the contracted Bianchi identities.[11,12] Moreover, the time-maintenance conditions for (3.8a) are $(\hat{E}1)^k{}_{(0)}=0$, which are equivalent to (3.4).

(5) The time-maintenance conditions for the constraints (3.6)

$$\hat{D}_0 C^{ks}=0 \tag{3.18}$$

together with the dynamical equations (3.9) and (3.10) as well as with constraints (3.4) give rise to the following system of algebraic equations for the three-metric density $\bar{\mathscr{g}}_{ij}$:

$$A^{kspq}\,\bar{\mathscr{g}}_{pq}=0 \ . \tag{3.19}$$

Here

$$A^{kspq}=\epsilon^{kzq}W_z{}^{ps}+\epsilon^{kzp}W_z{}^{qs}+\epsilon^{szq}W_z{}^{pk}+\epsilon^{szp}W_z{}^{qk} \tag{3.20}$$

and

$$W_z{}^{ij}=\tfrac{1}{2}({}^{\dagger}\hat{D}_z Y^{i(\alpha)(\beta)}X^j{}_{(\alpha)(\beta)}-Y^{j(\alpha)(\beta)}\,{}^{\dagger}\hat{D}_z X^i{}_{(\alpha)(\beta)}) \ . \tag{3.21}$$

Now we discuss the algebraic equations (3.19). First of all, we observe that the tensor density (of weight 2) $W_z{}^{ij}$ has the properties

$$W_z{}^{ij}=W_z{}^{ji}, \quad W_z{}^{zj}=0 \ . \tag{3.22}$$

These properties follow from (3.4) and (3.8).

Therefore the tensor density (of weight 3) A^{kspq} has the following properties:

$$\begin{aligned} &A^{kspq}=-A^{pqks} \ , \\ &A^{kspq}=A^{skpq}=A^{ksqp} \ . \end{aligned} \tag{3.23}$$

Moreover, it follows from (3.4) that $W_z{}^{ij}$ do not depend on the connection coefficients $\bar{\gamma}_z{}^p{}_q$. Therefore we may write

$$W_z{}^{ij}=\tfrac{1}{2}({}^{\dagger}_{\xi}\hat{D}_z Y^{i(\alpha)(\beta)}X^j{}_{(\alpha)(\beta)}-Y^{j(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_z X^i{}_{(\alpha)(\beta)}) \ , \tag{3.21$'$}$$

where $\xi_z{}^p{}_q$ is a fixed (auxiliary) symmetric 3-connection on slices. That is to say, $W_z{}^{ij}$ and A^{kspq} are functions of the reduced symplectic variables (3.16) and their spatial derivatives.

The relations (3.19) can be written in the operator form

$$A(\mathscr{g})=0 \ , \tag{3.24}$$

where the 6×6 matrix $[M^{\{ab\}\{cd\}}]$ of A is given by (C9). In order to have nontrivial solutions for $\bar{\mathscr{g}}_{ij}$ we must assume that

$$\det[M^{\{ab\}\{cd\}}]_{6\times 6}=0 \ . \tag{3.25}$$

By virtue of formulas (C11) and (C14) this condition is equivalent to

$$\epsilon_{qjr}W_z{}^{pq}W_p{}^{ij}W_i{}^{zr}=0 \ . \tag{3.26}$$

This equation presents the sixteenth symplectic constraint.

We have assumed that the determinant of the matrix $[M^{\{ab\}\{cd\}}]$ vanishes. Now we assume that the rank of that matrix is equal to 5. Of course, it can happen that for some values of reduced symplectic variables this rank is less than 5. We eliminate such degenerate cases. On the other hand, the situation when the rank is equal to 5 is generic. In such a case Eqs. (3.19) determine $\bar{\mathscr{g}}_{ij}$ up to a scalar factor. Let $\hbar_{ij}$ be a solution of (3.19); then $\bar{\mathscr{g}}_{ij}=\tau\hbar_{ij}$ is also a solution, where $\tau=\tau(x^0,x^k)$ is a one-parameter family of functions on slices (a function on spacetime).

We have the following situation. The time-maintenance conditions for the constraints (3.6) give rise to one symplectic constraint (3.26) and they determine five components of the density $\bar{\mathscr{g}}_{ij}$.

We may pose the question: why do six time-maintenance conditions for (3.6) determine only five components of $\bar{\mathscr{g}}_{ij}$? To answer this question we observe that because of the relation

$$\tfrac{1}{2}C^{ks}\bar{g}_{ks}=\hat{e}(\hat{E}1)^0{}_{(0)} \tag{3.27}$$

the constraints (3.6) implicitly contain the tenth Hamiltonian constraint

$$(\hat{E}1)^0{}_{(0)}=0 \ . \tag{3.28}$$

That, in turn, preserves in time by virtue of the contracted Bianchi identities. Hence, only five of six time-maintenance conditions for the system (3.6) are independent.

Now we are led to ask whether the symplectic constraint (3.26) is preserved in time. We have the time-maintenance condition

$$\epsilon_{qjr}\hat{D}_0 W_z{}^{pq}W_p{}^{ij}W_i{}^{zr}=0 \ . \tag{3.29}$$

Taking into account the dynamical equations (3.9), (3.10), and the constraints (3.4) we get

$$\begin{aligned}\hat{D}_0 W_z{}^{pq}=(1/4N)\{[&-\epsilon^{quv}\,{}^{\dagger}\hat{D}_z\,{}^{\dagger}\hat{D}_u(X^{t(\alpha)(\beta)}N\bar{\mathscr{g}}_{tv})X^p{}_{(\alpha)(\beta)}+\epsilon^{puv}\,{}^{\dagger}\hat{D}_u(X^{t(\alpha)(\beta)}N\bar{\mathscr{g}}_{tv})\,{}^{\dagger}\hat{D}_z X^q{}_{(\alpha)(\beta)}\\ &-\epsilon^{quv}\,{}^{\dagger}\hat{D}_z\,{}^{\dagger}\hat{D}_u(Y^t{}_{(\alpha)(\beta)}N\bar{\mathscr{g}}_{tv})Y^{p(\alpha)(\beta)}+\epsilon^{quv}\,{}^{\dagger}\hat{D}_u(Y^t{}_{(\alpha)(\beta)}N\bar{\mathscr{g}}_{tv})\,{}^{\dagger}\hat{D}_z Y^{p(\alpha)(\beta)}\\ &-X^{t(\alpha)}{}_{(\tau)}Y^{q(\tau)(\beta)}X^p{}_{(\alpha)(\beta)}N\bar{\mathscr{g}}_{tz}-X^{t(\sigma)}{}_{(\alpha)}X^q{}_{(\sigma)(\beta)}Y^{p(\alpha)(\beta)}N\bar{\mathscr{g}}_{tz}]+(p\leftrightarrow q)\} \ .\end{aligned} \tag{3.30}$$

Remark: We have to remember that the lapse N is a time density of weight 1 on slices. That is, if we perform a change of local coordinates (consistent with the slicing) $x^{0'}=x^{0'}(x^0)$, $x^{k'}=x^{k'}(x^0,x^k)$ then $N'=(\partial x^{0'}/\partial x^0)^{-1}N$. This fact, however, has no influence on the definition of spatial covariant derivatives of N for the transition function $\partial x^{0'}/\partial x^0$ is independent of x^s. Let $\hbar_{ij}$ be a fixed solution of (3.19). Then these quantities are functions of the reduced symplectic variables and their x^k derivatives. We assume that $\hbar_{ij}$ is a *positive-definite* tensor density (of weight -1) on the initial surface σ. Let $\xi_k{}^p{}_q$ be the Riemannian connection of the tensor field

$$h_{ij}=\hbar_{ij}(\det[\hbar_{mn}])^{-1} .$$

If

$$\bar{\mathfrak{g}}_{ij}=\tau\hbar_{ij} , \tag{3.31}$$

then

$$\begin{aligned} &\bar{g}_{ij}=\tau^{-2}h_{ij}, \quad \sqrt{\bar{g}}=\tau^{-3}\sqrt{h} , \\ &\bar{g}^{ij}=\tau^2 h^{ij} , \\ &c_k{}^p{}_q=\bar{\gamma}_k{}^p{}_q-\xi_k{}^p{}_q \\ &\qquad=-\tau^{-1}(\partial_k\tau\delta^p{}_q+\partial_q\tau\delta^p{}_k-\partial_u\tau h_{kq}h^{pu}) , \end{aligned} \tag{3.32}$$

where $\bar{\gamma}_k{}^p{}_q$ is the Riemannian connection of $\bar{g}_{ij}$.

If we substitute (3.30) into (3.29) making use of (3.31), (3.32), and of the constraints (3.4), (3.6), and (3.8), then we obtain the following first-order linear differential equation for the function $N\tau$ on the initial surface σ:

$$\begin{aligned} &\tfrac{1}{2}\partial_z(N\tau)\{[-{}^{\dagger}_{\xi}\hat{D}_j X^{t(\alpha)(\beta)}X^p{}_{(\alpha)(\beta)}\hbar_{tr}(W_p{}^{ij}W_i{}^{zr}+W_p{}^{iz}W_i{}^{jr})+2\,{}^{\dagger}_{\xi}\hat{D}_r X^{t(\alpha)(\beta)}X^p{}_{(\alpha)(\beta)}\hbar_{tj}W_p{}^{ij}W_i{}^{zr} \\ &\qquad +X^{t(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_j X^p{}_{(\alpha)(\beta)}\hbar_{tr}W_p{}^{iz}W_i{}^{jr}-X^{t(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_r X^p{}_{(\alpha)(\beta)}\hbar_{tj}W_p{}^{ij}W_i{}^{rz}-X^{t(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_q X^z{}_{(\alpha)(\beta)}\hbar_{tj}W_p{}^{ij}W_i{}^{qp}] \\ &\qquad +(\text{analogous term with } Y^{\cdots})\} \\ &\qquad +\tfrac{1}{2}(N\tau)\{[-{}^{\dagger}_{\xi}\hat{D}_z\,{}^{\dagger}_{\xi}\hat{D}_j X^{t(\alpha)(\beta)}X^p{}_{(\alpha)(\beta)}\hbar_{tr}+{}^{\dagger}_{\xi}\hat{D}_z\,{}^{\dagger}_{\xi}\hat{D}_r X^{t(\alpha)(\beta)}\hbar_{tj}X_{p(\alpha)(\beta)}+{}^{\dagger}_{\xi}\hat{D}_j X^{t(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_z X^p{}_{(\alpha)(\beta)}\hbar_{tr} \\ &\qquad -{}^{\dagger}_{\xi}\hat{D}_r X^{t(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_z X^p{}_{(\alpha)(\beta)}\hbar_{tj}+{}^{\dagger}_{\xi}\hat{D}_q X^{t(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_z X^q{}_{(\alpha)(\beta)}\hbar_{tj}\delta^p{}_r-{}^{\dagger}_{\xi}\hat{D}_j X^{t(\alpha)(\beta)}\,{}^{\dagger}_{\xi}\hat{D}_z X^q{}_{(\alpha)(\beta)}\hbar_{tq}\delta^p{}_r]W_p{}^{ij}W_i{}^{zr} \\ &\qquad +(\text{analogous term with } Y^{\cdots})-X^{t(\alpha)}{}_{(\omega)}Y^{q(\omega)(\beta)}X^p{}_{(\alpha)(\beta)}\hbar_{tz}\epsilon_{qjr}W_p{}^{ij}W_i{}^{zr}\}=0 . \end{aligned} \tag{3.33}$$

The coefficients of this equation are functions of reduced symplectic variables and their spatial derivatives.

This first-order linear equation on the manifold σ can be solved by the method of characteristics and it determines the function $\tau=\tau(x^k)$ up to its initial value f on a noncharacteristic two-dimensional surface $c_2\subset\sigma$. Later on we will discuss the meaning of f. Moreover, we assume that the solution τ of (3.33) is a positive function. We have the following situation. On the initial surface σ we fix 36 symplectic variables $X^k{}_{(\alpha)(\beta)}, \hat{\Gamma}_k{}^{(\alpha)(\beta)}$ satisfying 16 symplectic constraints (3.4), (3.6), (3.8), and (3.26). On spacetime we fix 13 gauge variables

$$N, N^k, \hat{\Gamma}_0{}^{(a)(b)}, n^{(a)}, \hat{\Gamma}_0{}^{(0)(b)} . \tag{3.34}$$

The metric on σ

$$\bar{g}_{ij}=\bar{\mathfrak{g}}_{ij}(\det[\bar{\mathfrak{g}}_{mn}])^{-1} \tag{3.35}$$

should be determined from (3.19) and (3.33). A function f of two spatial variables remains undetermined.

The time derivatives of the dynamical symplectic variables $X^k{}_{(\alpha)(\beta)}, \hat{\Gamma}_k{}^{(\alpha)(\beta)}$ can be computed from Eqs. (3.9) and (3.10). The time derivatives of the metric density $\bar{\mathfrak{g}}_{ij}$ can be determined by the following procedure. We time-differentiate Eqs. (3.19) and obtain

$$A^{kspq}\hat{D}_0\bar{\mathfrak{g}}_{pq}=-\hat{D}_0 A^{kspq}\,\bar{\mathfrak{g}}_{pq} . \tag{3.36}$$

Equations (3.36) have solutions for $\hat{D}_0\bar{\mathfrak{g}}_{pq}$ if and only if the following consistency condition holds:

$$-\bar{\mathfrak{g}}_{ks}\hat{D}_0 A^{kspq}\,\bar{\mathfrak{g}}_{pq}=0 . \tag{3.36a}$$

But (3.36a) is satisfied by virtue of skew symmetry of A^{kspq}, cf. (3.23). Let ${}_1\mathfrak{z}_{pq}$ be a fixed solution of (3.36). The components of ${}_1\mathfrak{z}_{pq}$ are functions of the symplectic and gauge variables, $\bar{\mathfrak{g}}_{ij}$ as well as of spatial derivatives of these variables.

A general solution of (3.36) is

$$\hat{D}_0\bar{\mathfrak{g}}_{pq}={}_1\mathfrak{z}_{pq}+{}_1\tau\hbar_{pq} \tag{3.37}$$

with an arbitrary function ${}_1\tau$ on σ.

In order to determine ${}_1\tau$ we compute the second time derivative of (3.26) and insert (3.37) into the result. It is easy to observe that the condition

$$\hat{D}_0\hat{D}_0(\epsilon_{qjr}W_z{}^{pq}W_p{}^{ij}W_i{}^{zr})=0 \tag{3.38}$$

gives rise to a first-order linear differential equation for ${}_1\tau$ on σ. The coefficients of this equation are functions of symplectic variables, $\bar{\mathfrak{g}}_{ij}$, spatial derivatives of those variables as well as of gauge variables, their spatial and time derivatives.

Equation (3.38) determines ${}_1\tau$ on the initial surface σ up to a function ${}_1f$ on the two-dimensional surface $c_2\subset\sigma$. The main terms in Eqs. (3.33) and (3.38) coincide. Therefore their characteristic surfaces are identical.

Remark: (3.33) is a homogeneous linear differential equation for τ but (3.38) is a nonhomogeneous linear equation for ${}_1\tau$.

Now we are able to compute first time derivatives of the symplectic variables (3.16) and of $\bar{\mathfrak{g}}_{ij}$ on the initial surface. The time differentiation of the dynamical equations (3.9) and (3.10) gives us second time derivatives of the symplectic variables. The second time derivative of $\bar{\mathfrak{g}}_{ij}$

can be computed from the condition

$$\hat{D}_0\hat{D}_0(A^{kspq}\bar{\mathscr{g}}_{pq})=0 \tag{3.39}$$

or equivalently

$$A^{kspq}\hat{D}_0\hat{D}_0\bar{\mathscr{g}}_{pq}=-(\hat{D}_0\hat{D}_0A^{kspq}\bar{\mathscr{g}}_{pq}+2\hat{D}_0A^{kspq}\hat{D}_0\bar{\mathscr{g}}_{pq})\ . \tag{3.39'}$$

The solvability condition for these equations reads

$$-\bar{\mathscr{g}}_{ks}(\hat{D}_0\hat{D}_0A^{kspq}\bar{\mathscr{g}}_{pq}+2\hat{D}_0A^{kspq}\hat{D}_0\bar{\mathscr{g}}_{pq})=0\ . \tag{3.39a}$$

It is satisfied by virtue of (3.36) and skew-symmetry properties of A^{kspq} (see Appendix D for a general proof of this fact).

If ${}_2\mathscr{z}_{pq}$ is a fixed solution of (3.39′) then

$$\hat{D}_0\hat{D}_0\bar{\mathscr{g}}_{pq}={}_2\mathscr{z}_{pq}+{}_2\tau\mathscr{h}_{pq} \tag{3.40}$$

with an arbitrary function ${}_2\tau$ on σ.

Again, the time-maintenance condition

$$\hat{D}_0{}^{(3)}(\epsilon_{qjr}W_z{}^{pq}W_p{}^{ij}W_i{}^{zr})=0 \tag{3.41}$$

leads to a first-order linear differential equation for ${}_2\tau$ on σ. The main terms of this equation coincide with the main terms of (3.33). We are able to determine ${}_2\tau$ up to a function ${}_2f$ on $c_2\subset\sigma$.

We may repeat this procedure and compute consecutive time derivatives of the symplectic variables and of $\bar{\mathscr{g}}_{ij}$. At each step the time-maintenance conditions

$$\hat{D}_0{}^{(n)}(A^{kspq}\bar{\mathscr{g}}_{pq})=0 \tag{3.42}$$

and

$$\hat{D}_0{}^{(n+1)}(\epsilon_{qjr}W_z{}^{pq}W_p{}^{ij}W_i{}^{zr})=0 \tag{3.43}$$

enable us to determine $\hat{D}_0{}^{(n)}\bar{\mathscr{g}}_{pq}$ up to a function ${}_nf$ on $c_2\subset\sigma$. In Appendix D we prove that the system of linear algebraic equations for $\hat{D}_0{}^{(n)}\bar{\mathscr{g}}_{pq}$ always has solutions.

The analysis of this section shows that in order to pose the formal Cauchy-Kowalewska initial-value problem for the vacuum SKY theory we have to fix (i) values of 36 dynamical symplectic variables (3.16) on an initial surface σ, which have to satisfy 16 symplectic constraints (3.4), (3.6), (3.8), and (3.26), (ii) 13 gauge variables (3.34) on spacetime, and (iii) a sequence $f,{}_1f,{}_2f,\ldots$ of functions of two spatial variables on a two-dimensional surface $c_2\subset\sigma$. (Such a sequence may represent a function of two spatial variables and of time.)

If such initial data are given then we are able to compute (a) the three-metric $\bar{g}_{ij}$ on σ and (b) all time derivatives of symplectic variables and of $\bar{g}_{ij}$.

The dynamical picture of the SKY gravity presented in this section gives us only necessary conditions that a well-posed initial-value problem has to satisfy. Whether the algebraic-differential-integral procedure described above can be used for an effective construction of solutions remains open. To our best knowledge systems of partial differential equations for which the Cauchy-Kowalewska procedure requires not only algebraic and differential operations but also integration of a sequence of linear partial differential equations of first order (for one function each) have not been investigated in the literature yet. Thus physics brings about new mathematical problems that apparently are worthy of a profound investigation.

IV. SYMPLECTIC CONSTRAINTS, DEGENERACY OF THE SYMPLECTIC TWO-FORM AND GAUGE TRANSFORMATIONS

In the vacuum SKY theory of gravity we have 16 symplectic constraints for 36 dynamical symplectic variables $X^k{}_{(\alpha)(\beta)},\hat{\Gamma}_k{}^{(\alpha)(\beta)}$:

$$(\hat{E}1)^0{}_p=-\epsilon_{pij}X^i{}_{(\alpha)(\beta)}Y^{j(\alpha)(\beta)}=0\ , \tag{4.1}$$

$$(\hat{E}2)^0{}_{(\alpha)(\beta)}={}^\dagger\hat{D}_kX^k{}_{(\alpha)(\beta)}=0\ , \tag{4.2}$$

$$C^{ks}=X^k{}_{(\alpha)(\beta)}X^{s(\alpha)(\beta)}+Y^{k(\alpha)(\beta)}Y^s{}_{(\alpha)(\beta)}=0\ , \tag{4.3}$$

$$Z=\epsilon_{qjr}W_z{}^{pq}W_p{}^{ij}W_i{}^{zr}=0\ . \tag{4.4}$$

Let u^p be a spatial vector field on the initial surface σ, $v^{(a)(b)}$ be a skew-symmetric SO(3)-tensor-valued function on σ, $v^{(a)(0)}=-v^{(0)(a)}$ be an SO(3)-vector-valued function on σ, χ_{ks} be a symmetric tensor density of weight -1 on σ, and w be a scalar density of weight -4 on σ.

Let V be a vector in the linear space of infinitesimal variations of geometric configurations of the system in question (i.e., a vector tangent to the space of geometric configurations). Such a vector represents a vector field V on the finite dimensional manifold parametrized by

$$(x^\mu,\Gamma_\mu{}^{(\alpha)(\beta)},e^{(\alpha)}{}_\mu)\ .$$

In terms of symplectic variables we write

$$V=\delta\hat{\Gamma}_k{}^{(\alpha)(\beta)}/\partial\hat{\Gamma}_k{}^{(\alpha)(\beta)}+\delta X^k{}_{(\alpha)(\beta)}/\partial X^k{}_{(\alpha)(\beta)}+\cdots\ .$$

The linearized version of constraints (4.1)–(4.4) reads

$$d[(\hat{E}1)^0{}_p]V=-\epsilon_{pij}\delta X^i{}_{(\alpha)(\beta)}Y^{j(\alpha)(\beta)}-X^v{}_{(\alpha)(\beta)}{}^\dagger\hat{D}_p\delta\hat{\Gamma}_v{}^{(\alpha)(\beta)}+X^v{}_{(\alpha)(\beta)}{}^\dagger\hat{D}_v\delta\hat{\Gamma}_p{}^{(\alpha)(\beta)}=0\ , \tag{4.1a}$$

$$d[(\hat{E}2)^0{}_{(\alpha)(\beta)}]V={}^\dagger\hat{D}_k\delta X^k{}_{(\alpha)(\beta)}-\delta\hat{\Gamma}_k{}^{(\epsilon)}{}_{(\alpha)}X^k{}_{(\epsilon)(\beta)}-\delta\hat{\Gamma}_k{}^{(\epsilon)}{}_{(\beta)}X^k{}_{(\alpha)(\epsilon)}=0\ , \tag{4.2a}$$

$$d[C^{ks}]V=(\delta X^k{}_{(\alpha)(\beta)}X^{s(\alpha)(\beta)}+\epsilon^{kuv}{}^\dagger\hat{D}_u\delta\hat{\Gamma}_v{}^{(\alpha)(\beta)}Y^s{}_{(\alpha)(\beta)})+(k\leftrightarrow s)=0\ , \tag{4.3a}$$

$$\begin{aligned}d[Z]V=\tfrac{3}{4}\epsilon_{qjr}[(&\epsilon^{pmn}{}^\dagger\hat{D}_z{}^\dagger\hat{D}_m\delta\hat{\Gamma}_n{}^{(\alpha)(\beta)}X^q{}_{(\alpha)(\beta)}+\delta\hat{\Gamma}_z{}^{(\alpha)}{}_{(\epsilon)}Y^{p(\epsilon)(\beta)}X^q{}_{(\alpha)(\beta)}+\delta\hat{\Gamma}_z{}^{(\beta)}{}_{(\epsilon)}Y^{p(\alpha)(\epsilon)}X^q{}_{(\alpha)(\beta)}+{}^\dagger\hat{D}_zY^{p(\alpha)(\beta)}\delta X^q{}_{(\alpha)(\beta)}\\&-\epsilon^{qmn}{}^\dagger\hat{D}_m\delta\hat{\Gamma}_n{}^{(\alpha)(\beta)}{}^\dagger\hat{D}_zX^p{}_{(\alpha)(\beta)}-Y^{q(\alpha)(\beta)}{}^\dagger\hat{D}_z\delta X^p{}_{(\alpha)(\beta)}+Y^{q(\alpha)(\beta)}\delta\hat{\Gamma}_z{}^{(\tau)}{}_{(\alpha)}X^p{}_{(\tau)(\beta)}\\&+Y^{q(\alpha)(\beta)}\delta\hat{\Gamma}_z{}^{(\tau)}{}_{(\beta)}X^p{}_{(\alpha)(\tau)})+(p\leftrightarrow q)]W_p{}^{ij}W_i{}^{zr}=0\ .\end{aligned} \tag{4.4a}$$

Hamiltonian vector fields of the constraints are defined according to the definitions

$$\begin{aligned} &d[-(\hat{E}1)^0{}_p u^p]V=-\Omega(Y_{H(1)}\wedge V), \quad d[-(\hat{E}2)^0{}_{(\alpha)(\beta)}v^{(\alpha)(\beta)}]V=-\Omega(Y_{H(2)}\wedge V)\ ,\\ &d[-C^{ks}\chi_{ks}]V=-\Omega(Y_{H(3)}\wedge V), \quad d[-Zw]V=-\Omega(Y_{H(4)}\wedge V)\ . \end{aligned} \tag{4.5}$$

Here V is a sample (arbitrary) vector and $Y_{H(i)}$, $i=1,\ldots,4$ are the corresponding Hamiltonian vectors on the space of geometric configurations of the system.

In local coordinates $Y_{H(1)}$ is represented by the variations

$$\delta_{H(1)}X^k{}_{(\alpha)(\beta)}=-{}^{\dagger}\hat{D}_p(X^p{}_{(\alpha)(\beta)}u^k-X^k{}_{(\alpha)(\beta)}u^p)\ , \tag{4.6a}$$

$$\delta_{H(1)}\hat{\Gamma}_k{}^{(\alpha)(\beta)}=-\epsilon_{kij}u^iY^{j(\alpha)(\beta)}\ , \tag{4.6b}$$

$$\delta_{H(1)}Y^{k(\alpha)(\beta)}=-{}^{\dagger}\hat{D}_p(Y^{p(\alpha)(\beta)}u^k-Y^{k(\alpha)(\beta)}u^p)\ ; \tag{4.6b$'$}$$

$Y_{H(2)}$ is represented by the variations

$$\delta_{H(2)}X^k{}_{(\alpha)(\beta)}=-X^k{}_{(\alpha)(\sigma)}v_{(\beta)}{}^{(\sigma)}+X^k{}_{(\beta)(\sigma)}v_{(\alpha)}{}^{(\sigma)}\ , \tag{4.7a}$$

$$\delta_{H(2)}\hat{\Gamma}_k{}^{(\alpha)(\beta)}={}^{\dagger}\hat{D}_k v^{(\alpha)(\beta)}\ , \tag{4.7b}$$

$$\delta_{H(2)}Y^{k(\alpha)(\beta)}=-Y^{k(\alpha)(\sigma)}v^{(\beta)}{}_{(\sigma)}+Y^{k(\beta)(\sigma)}v^{(\alpha)}{}_{(\sigma)}\ ; \tag{4.7b$'$}$$

$Y_{H(3)}$ is represented by the variations

$$\delta_{H(3)}X^k{}_{(\alpha)(\beta)}=\epsilon^{kij}\,{}^{\dagger}\hat{D}_i(Y^z{}_{(\alpha)(\beta)}\chi_{zj})\ , \tag{4.8a}$$

$$\delta_{H(3)}\hat{\Gamma}_k{}^{(\alpha)(\beta)}=-X^{z(\alpha)(\beta)}\chi_{zk}\ , \tag{4.8b}$$

$$\delta_{H(3)}Y^{k(\alpha)(\beta)}=-\epsilon^{kij}\,{}^{\dagger}\hat{D}_i(X^{z(\alpha)(\beta)}\chi_{zj})\ ; \tag{4.8b$'$}$$

$Y_{H(4)}$ is represented by the variations

$$\begin{aligned} \delta_{H(4)}X^k{}_{(\alpha)(\beta)}=-\tfrac{3}{4}\epsilon_{qjr}[&\epsilon^{pmk}\,{}^{\dagger}\hat{D}_m\,{}^{\dagger}\hat{D}_z(X^q{}_{(\alpha)(\beta)}W_p{}^{ij}W_i{}^{zr}w)+(Y^p{}_{(\beta)(\tau)}X^q{}_{(\alpha)}{}^{(\tau)}-Y^p{}_{(\alpha)(\tau)}X^q{}_{(\beta)}{}^{(\tau)})W_p{}^{ij}W_i{}^{kr}w\\ &+\epsilon^{qmk}\,{}^{\dagger}\hat{D}_m({}^{\dagger}\hat{D}_zX^p{}_{(\alpha)(\beta)}W_p{}^{ij}W_i{}^{zr}w)+(Y^{q(\sigma)}{}_{(\beta)}X^p{}_{(\sigma)(\alpha)}-Y^{q(\sigma)}{}_{(\alpha)}X^p{}_{(\sigma)(\beta)})W_p{}^{ij}W_i{}^{kr}w\\ &+\epsilon^{qmk}\,{}^{\dagger}\hat{D}_m\,{}^{\dagger}\hat{D}_z(X^p{}_{(\alpha)(\beta)}W_p{}^{ij}W_i{}^{zr}w)+(Y^q{}_{(\beta)(\tau)}X^p{}_{(\alpha)}{}^{(\tau)}-Y^q{}_{(\alpha)(\tau)}X^p{}_{(\beta)}{}^{(\tau)})W_p{}^{ij}W_i{}^{kr}w\\ &+\epsilon^{pmk}\,{}^{\dagger}\hat{D}_m({}^{\dagger}\hat{D}_zX^q{}_{(\alpha)(\beta)}W_p{}^{ij}W_i{}^{zr}w)+(Y^p{}_{(\beta)(\sigma)}X^q{}_{(\alpha)}{}^{(\sigma)}-Y^p{}_{(\alpha)(\sigma)}X^q{}_{(\beta)}{}^{(\sigma)})W_p{}^{ij}W_i{}^{kr}w]\ , \end{aligned} \tag{4.9a}$$

$$\begin{aligned} \delta_{H(4)}\hat{\Gamma}_k{}^{(\alpha)(\beta)}=\tfrac{3}{4}[&\epsilon_{kjr}\,{}^{\dagger}\hat{D}_zY^{p(\alpha)(\beta)}W_p{}^{ij}W_i{}^{zr}w+\epsilon_{qjr}\,{}^{\dagger}\hat{D}_zY^{q(\alpha)(\beta)}W_k{}^{ij}W_i{}^{zr}w\\ &+\epsilon_{qjr}\,{}^{\dagger}\hat{D}_z(Y^{q(\alpha)(\beta)}W_k{}^{ij}W_i{}^{zr}w)+\epsilon_{kjr}\,{}^{\dagger}\hat{D}_z(Y^{p(\alpha)(\beta)}W_p{}^{ij}W_i{}^{zr}w)]\ , \end{aligned} \tag{4.9b}$$

$$\delta_{H(4)}Y^{k(\alpha)(\beta)}=\epsilon^{kuv}\,{}^{\dagger}\hat{D}_u\delta\hat{\Gamma}_v{}^{(\alpha)(\beta)}\ . \tag{4.9b$'$}$$

If the constraints (4.1)–(4.4) hold then, for $i=1,2$,

$$d[(\hat{E}1)^0{}_p]Y_{H(i)}=0\ , \tag{4.10a}$$

$$d[(\hat{E}2)^0{}_{(\alpha)(\beta)}]Y_{H(i)}=0\ , \tag{4.10b}$$

$$d[C^{ks}]Y_{H(i)}=0\ , \tag{4.10c}$$

$$d[Z]Y_{H(i)}=0\ . \tag{4.10d}$$

Moreover, (4.10a) and (4.10b) hold also for $i=3,4$.

The relations (4.10c) for $i=3$ hold if and only if

$$A^{kspq}\chi_{pq}=0\ , \tag{4.11}$$

that is, if

$$\chi_{pq}=\beta\overline{\mathcal{G}}_{pq}\ , \tag{4.12}$$

where β is a function on the initial surface σ.

The condition (4.10d) for $i=3$ gives rise to a differential equation for β and this equation is analogous to (3.33), where N is replaced with β. We conclude that (4.10a)–(4.10d) hold for $i=3$ if and only if

$$\beta=\theta N \ , \tag{4.13}$$

where θ is independent of x^k but may depend on x^0.

Remark: We know[62] that N is a time density of weight 1; therefore $\theta=\theta(x^0)$ is a time density of weight -1.

The relations (4.10c) and (4.10d) hold for $i=4$ if and only if $w=0$, that is $Y_{H(4)}=0$.

In general, Hamiltonian vectors of constraints do not satisfy the linearized constraints [Eqs. (4.10)]. For the SKY theory the linear space of Hamiltonian vectors satisfying the linearized constraints is parametrized by 10 quantities on the initial surface σ, u^p, $v^{(\alpha)(\beta)}$, and β.

It follows from the very definition of Hamiltonian vectors that their subspace satisfying the linearized constraints determines the gauge distribution of the symplectic two-form Ω. The gauge distribution W_Ω of Ω is such a distribution of subspaces in the space tangent to the space of symplectic variables that for each vector X belonging to W_Ω and every vector V tangent to the space of symplectic variables $\Omega(X\wedge V)=0$.

We expect that the gauge distribution of Ω is the image of the action of all infinitesimal transformations, that is, such infinitesimal transformations in the set of symplectic variables that preserve the symplectic constraints. In subsequent considerations we present the complete set of infinitesimal gauge transformations for the SKY theory and prove that they can be extended to other field variables in such a way that the field equations are preserved. Moreover, these infinitesimal gauge transformations can be integrated to actions of corresponding gauge groups.

The Hamiltonian vector $Y_{H(1)}$ defines the infinitesimal transformations

$$\begin{aligned} &X^k{}_{(\alpha)(\beta)}\to X^k{}_{(\alpha)(\beta)}+\epsilon[{}^\dagger\hat{D}_p(u^pX^k{}_{(\alpha)(\beta)})-\hat{D}_pu^kX^p{}_{(\alpha)(\beta)}] \ ,\\ &\hat{\Gamma}_k{}^{(\alpha)(\beta)}\to\hat{\Gamma}_k{}^{(\alpha)(\beta)}+\epsilon(u^p\,{}^\dagger\hat{D}_p\hat{\Gamma}_k{}^{(\alpha)(\beta)}) \ . \end{aligned} \tag{4.14}$$

That is to say, the infinitesimal transformations generated by $Y_{H(1)}$ coincide with the transformations given by the dagger-covariant Lie derivatives ${}^{\dagger3}L$ on the manifold σ.

Remark: The dagger-covariant Lie derivative on σ is a modification of the SO(3)-covariant Lie derivative defined in Appendix D of Ref. 64. For example,

$${}^{\dagger3}L_uX^p{}_{(\alpha)(\beta)}={}^\dagger\hat{D}_s(u^sX^p{}_{(\alpha)(\beta)})-\hat{D}_su^pX^s{}_{(\alpha)(\beta)} \ .$$

The Hamiltonian vector field $Y_{H(3)}$ defines a transformation consistent with constraints (4.1)–(4.4) if and only if

$$\chi_{pq}=\beta\bar{\mathscr{g}}_{pq} \ , \tag{4.15}$$

where β is a function on σ that is proportional to the lapse N:

$$\beta=\theta N \ . \tag{4.16}$$

$\theta=\theta(x^0)$ is a time density of weight -1 on the x^0 line.

The transformations (4.14) perserve the constraints. We expect that there exists a natural prolongation of these transformations to the space of gauge variables. In fact, if we postulate

$$\delta_{H(1)}N=u^p\partial_pN, \quad \delta_{H(1)}N^s=N\hat{D}_0u^s \ , \tag{4.17}$$

$$\begin{aligned}\delta_{H(1)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}=u^p\,{}^\dagger\hat{D}_p\hat{\Gamma}_0{}^{(\alpha)(\beta)}-(1/N)\delta_{H(1)}N\hat{\Gamma}_0{}^{(\alpha)(\beta)}\\ -(1/N)\delta_{H(1)}N^s\hat{\Gamma}_s{}^{(\alpha)(\beta)} \ ,\end{aligned} \tag{4.18}$$

$$\delta_{H(1)}\bar{\mathscr{g}}_{ij}={}^3L_u\bar{\mathscr{g}}_{ij}=-\hat{D}_pu^p\bar{\mathscr{g}}_{ij}+\hat{D}_iu^p\bar{\mathscr{g}}_{pj}+\hat{D}_ju^p\bar{\mathscr{g}}_{ip} \ , \tag{4.19}$$

then the field equations are invariant with respect to transformations (4.14) and (4.17)–(4.19).

We note the following.

(i) Relations (4.17) were derived in Refs. 77 and 64. They correspond to the action of the diffeomorphism group on the variables N and N^k.

(ii) Relations (4.18) can be obtained if we consider the SO(3,1)-covariant action of the diffeomorphism group of spacetime in the set of SO(3,1) connections[12,64,65] and then pass to the $(3+1)$ picture. At the present moment we simply postulate them.

(iii) Taking into account the relations between the Hamiltonian vector $Y_{H(1)}$ and the action of Diff$M$ we see that formulas (4.17)–(4.19) are natural consequences of previous considerations.

If the conditions (4.15) and (4.16) hold then the Hamiltonian vector $Y_{H(3)}$ generates the infinitesimal transformations

$$\begin{aligned} &X^k{}_{(\alpha)(\beta)}\to X^k{}_{(\alpha)(\beta)}+\epsilon\beta\,{}^\dagger\hat{D}_0X^k{}_{(\alpha)(\beta)} \ ,\\ &\hat{\Gamma}_k{}^{(\alpha)(\beta)}\to\hat{\Gamma}_k{}^{(\alpha)(\beta)}+\epsilon\beta\,{}^\dagger\hat{D}_0\hat{\Gamma}_k{}^{(\alpha)(\beta)} \ . \end{aligned} \tag{4.20}$$

These transformations preserve the constraints.

The transformations (6.20) can be extended to the set of gauge variables. Taking into account the results of Refs. 64 and 77 we postulate

$$\begin{aligned} &\delta_{H(3)}N=N\hat{D}_0\beta \ ,\\ &\delta_{H(3)}N^k=N(\beta\bar{\nabla}^k\ln N-\bar{\nabla}^k\beta) \ , \end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\delta_{H(3)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}=-(1/N)\delta_{H(3)}N\hat{\Gamma}_0{}^{(\alpha)(\beta)}\\ -(1/N)\delta_{H(3)}N^s\hat{\Gamma}_s{}^{(\alpha)(\beta)}+\beta\,{}^\dagger\hat{D}_0\hat{\Gamma}_0{}^{(\alpha)(\beta)} \ .\end{aligned} \tag{4.22}$$

But ${}^\dagger\hat{D}_0\hat{\Gamma}_0{}^{(\alpha)(\beta)}=\hat{R}^{(\alpha)(\beta)}{}_{00}=0$, and by virtue of (4.16) $\delta_{H(3)}N^k=0$. Therefore

$$\delta_{H(3)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}=-\hat{D}_0\beta\hat{\Gamma}_0{}^{(\alpha)(\beta)} \ . \tag{4.22'}$$

For the metric density $\bar{\mathscr{g}}_{ij}$ we have

$$\delta_{H(3)}\bar{\mathscr{g}}_{ij}=\beta\hat{D}_0\bar{\mathscr{g}}_{ij} \ . \tag{4.23}$$

The transformations (4.20)–(4.23) preserve the field equations.

Though the transformations (4.14), (4.17)–(4.19), and (4.20)–(4.23) are generated by the diffeomorphism group of spacetime nevertheless they cannot be integrated to an action of Diff$M$. The reason is that some integrability conditions (Jacobi identities) do not hold. We can, however, modify the system of constraints taking their appropriate linear combinations and the Hamiltonian vectors of such a modified system are the generators of an action of Diff$M$. To accomplish this program we have to define an auxiliary connection $\zeta_\lambda{}^{(\alpha)(\beta)}$ satisfying the condition

$$\hat{\zeta}_\lambda{}^{(a)(0)}=0 \ . \tag{4.24}$$

Then the Hamiltonian vectors $Y_{H'(1)}$ and $Y_{H'(2)}$ of the functions

$$[-(\hat{E}1)^0{}_p-(\hat{E}2)^0{}_{(\alpha)(\beta)}(\hat{\Gamma}_p{}^{(\alpha)(\beta)}-\hat{\zeta}_p{}^{(\alpha)(\beta)})]u^p \ , \tag{4.25a}$$

$$[-C^{ks}\bar{\mathscr{g}}_{ks}-(\hat{E}2)^0{}_{(\alpha)(\beta)}(\hat{\Gamma}_0{}^{(\alpha)(\beta)}-\hat{\zeta}_0{}^{(\alpha)(\beta)})]\beta \ , \tag{4.25b}$$

respectively, are the generators of the SO(3,1)-covariant action of the diffeomorphism group.[12,65]

The connection $\hat{\zeta}_\lambda{}^{(\alpha)(\beta)}$ generates the covariant derivatives ${}^{\dagger}_{\zeta}\hat{D}_\lambda$. To compute them we have to replace $\hat{\Gamma}_\lambda{}^{(\alpha)(\beta)}$ with $\hat{\zeta}_\lambda{}^{(\alpha)(\beta)}$ in (A10) and (A12). We have

$$\delta_{H'(1)}X^k{}_{(\alpha)(\beta)}={}^{\dagger}_{\zeta}\hat{D}_p(u^pX^k{}_{(\alpha)(\beta)})-\hat{D}_pu^kX^p{}_{(\alpha)(\beta)}={}^{\dagger 3}_{\zeta}L_uX^k{}_{(\alpha)(\beta)} \ , \tag{4.26a}$$

$$\begin{aligned}\delta_{H'(1)}\hat{\Gamma}_k{}^{(\alpha)(\beta)}&=u^p{}_{\zeta}\hat{R}{}^{(\alpha)(\beta)}{}_{pk}+u^p{}^{\dagger}_{\zeta}\hat{D}_p(\hat{\Gamma}_k{}^{(\alpha)(\beta)}-\hat{\zeta}_k{}^{(\alpha)(\beta)})+\hat{D}_ku^p(\hat{\Gamma}_p{}^{(\alpha)(\beta)}-\hat{\zeta}_p{}^{(\alpha)(\beta)})\\&={}^{\dagger 3}_{\zeta}L_u\hat{\Gamma}_k{}^{(\alpha)(\beta)} \ ,\end{aligned} \tag{4.26b}$$

$$\begin{aligned}\delta_{H'(1)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}&=\delta_{H(1)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}+{}^{\dagger}\hat{D}_0((\hat{\Gamma}_p{}^{(\alpha)(\beta)}-\hat{\zeta}_p{}^{(\alpha)(\beta)})u^p)\\&=-u^p\partial_p\ln N\hat{\zeta}_0{}^{(\alpha)(\beta)}-\hat{D}_0u^p\hat{\zeta}_p{}^{(\alpha)(\beta)}-u^p{}_{\zeta}\hat{R}{}^{(\alpha)(\beta)}{}_{0p}+u^p{}^{\dagger}_{\zeta}\hat{D}_p(\hat{\Gamma}_0{}^{(\alpha)(\beta)}-\hat{\zeta}_0{}^{(\alpha)(\beta)}) \ ,\end{aligned} \tag{4.27a}$$

$$\delta_{H'(1)}N=\delta_{H(1)}N,\ \delta_{H'(1)}N^k=\delta_{H(1)}N^k,\ \delta_{H'(1)}\bar{\mathscr{g}}_{ij}=\delta_{H(1)}\bar{\mathscr{g}}_{ij} \ , \tag{4.27b}$$

$$\delta_{H'(3)}X^k{}_{(\alpha)(\beta)}=\beta{}^{\dagger}_{\zeta}\hat{D}_0X^k{}_{(\alpha)(\beta)},\ \delta_{H'(3)}\hat{\Gamma}_k{}^{(\alpha)(\beta)}=\beta{}^{\dagger}_{\zeta}\hat{D}_0\hat{\Gamma}_k{}^{(\alpha)(\beta)} \ , \tag{4.28}$$

$$\begin{aligned}\delta_{H'(3)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}&=\delta_{H(3)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}+{}^{\dagger}\hat{D}_0((\hat{\Gamma}_0{}^{(\alpha)(\beta)}-\hat{\zeta}_0{}^{(\alpha)(\beta)})\beta)\\&=-\hat{D}_0\beta\hat{\zeta}_0{}^{(\alpha)(\beta)}+\beta{}^{\dagger}_{\zeta}\hat{D}_0(\hat{\Gamma}_0{}^{(\alpha)(\beta)}-\hat{\zeta}_0{}^{(\alpha)(\beta)}) \ ,\end{aligned} \tag{4.29a}$$

$$\delta_{H'(3)}N=\delta_{H(3)}N,\quad \delta_{H'(3)}N^k=\delta_{H(3)}N^k=0,\quad \delta_{H'(3)}\bar{\mathscr{g}}_{ij}=\delta_{H(3)}\bar{\mathscr{g}}_{ij} \ . \tag{4.29b}$$

In the above formulas ${}_{\zeta}R{}^{(\alpha)(\beta)}{}_{\mu\nu}$ is the curvature tensor of $\zeta_\lambda{}^{(\alpha)(\beta)}$ and ${}^{\dagger 3}_{\zeta}L$ is the covariant Lie derivative (with respect to the connection $\hat{\zeta}$) on σ. Moreover we observe that (4.24) gives rise to the relations

$${}^{\dagger}_{\zeta}\hat{D}_\lambda={}_{\zeta}\hat{D}_\lambda,\quad {}^{\dagger 3}_{\zeta}L={}^{3}_{\zeta}L \ .$$

We would like to emphasize that the generators (4.25) play a very important role in the SO(3,1)-covariant Hamiltonian formulation of gauge theories of gravity as well as in their SO(3)-covariant Hamiltonian formulation. For a profound discussion of this subject we refer the reader to Refs. 12, 65, and 66.

The Hamiltonian vector $Y_{H(2)}$ generates the infinitesimal transformations

$$X^k{}_{(\alpha)(\beta)}\to X^k{}_{(\alpha)(\beta)}+\epsilon(X^k{}_{(\sigma)(\beta)}v^{(\sigma)}{}_{(\alpha)}+X^k{}_{(\alpha)(\sigma)}v^{(\sigma)}{}_{(\beta)}) \ , \tag{4.30a}$$

$$\hat{\Gamma}_k{}^{(\alpha)(\beta)}\to\hat{\Gamma}_k{}^{(\alpha)(\beta)}+\epsilon{}^{\dagger}\hat{D}_kv^{(\alpha)(\beta)} \ . \tag{4.30b}$$

These transformations extend to the gauge variables according to the following rules:

$$\delta_{H(2)}N=0,\quad \delta_{H(2)}N^k=0 \ , \tag{4.31}$$

$$\delta_{H(2)}\hat{\Gamma}_0{}^{(\alpha)(\beta)}={}^{\dagger}\hat{D}_0v^{(\alpha)(\beta)} \ . \tag{4.32}$$

For the metric density we have

$$\delta_{H(2)}\bar{\mathscr{g}}_{ij}=0 \ . \tag{4.33}$$

It is easy to check that these transformations preserve the field equations. The transformations (4.30)–(4.33) are generated by the left-hand sides of constraints (4.2). It follows from the general Hamiltonian theory of SO(3,1) gauge theories of gravity that those constraints are related to local SO(3,1) rotations. Let us examine these transformations more profoundly.

We consider two cases. (i) $v^{(a)(0)}=0$, then

$$\begin{aligned}&\delta_{H(2)}X^k{}_{(a)(0)}=X^k{}_{(c)(0)}v^{(c)}{}_{(a)} \ ,\\&\delta_{H(2)}X^k{}_{(a)(b)}=X^k{}_{(c)(b)}v^{(c)}{}_{(a)}+X^k{}_{(a)(c)}v^{(c)}{}_{(b)} \ ,\\&\delta_{H(2)}\hat{\Gamma}_k{}^{(a)(b)}=\hat{D}_kv^{(a)(b)} \ ,\\&\delta_{H(2)}\hat{\Gamma}_k{}^{(a)(0)}=\hat{\Gamma}_k{}^{(0)(c)}v^{(a)}{}_{(c)} \ .\end{aligned} \tag{4.34}$$

(ii) $v^{(a)(b)}=0$, then

$$\begin{aligned}&\delta_{H(2)}X^k{}_{(a)(0)}=X^k{}_{(a)(c)}v^{(c)}{}_{(0)} \ ,\\&\delta_{H(2)}X^k{}_{(a)(b)}=X^k{}_{(0)(b)}v^{(0)}{}_{(a)}+X^k{}_{(a)(0)}v^{(0)}{}_{(b)} \ ,\\&\delta_{H(2)}\hat{\Gamma}_k{}^{(a)(b)}=\hat{\Gamma}_k{}^{(a)}{}_{(0)}v^{(0)(b)}+\hat{\Gamma}_k{}^{(b)}{}_{(0)}v^{(a)(0)} \ ,\\&\delta_{H(2)}\hat{\Gamma}_k{}^{(a)(0)}=\hat{D}_kv^{(a)(0)} \ .\end{aligned} \tag{4.35}$$

If we compare (4.34) with the formulas (C12) of Ref. 11 then we observe that they coincide. It is understandable because in the case (i) we deal with the transformations generated by the left-hand sides of the constraints

$$(\hat{E}2)^0{}_{(a)(b)}=0 \ . \tag{4.36}$$

It is known[11] that in a general case the transformations generated by these constraints correspond to the action of the local SO(3) group. On the other hand, if we compare (4.35) with (C12) of Ref. 11 then we see that the case (ii) does not correspond to the standard transformations generated by boosts. Now it becomes clear that the case (ii) defines a set of new transformations unknown previously.

In order to explain this situation we recall that in a general case we have three SO(3)-rotational constraints (4.36) as well as three boost constraints

$$M_{(a)}=0 \tag{4.37}$$

which together generate the standard action of the local SO(3,1) group in the space of the dynamical variables. In the SKY theory we have three additional symplectic constraints

$$(\hat{E}2)^{0}{}_{(a)(0)}=0 \tag{4.38}$$

which generate the transformations (4.35). In the next section we show that these transformations correspond to a nonstandard action of boosts in the space of the dynamical variables. This action is typical for the SKY theory and its Lagrangian is invariant with respect to it.

For the infinitesimal SO(3) gauge transformations (4.34) we have three SO(3) gauge variables $\hat{\Gamma}_0{}^{(a)(b)}$. It follows from (4.32) that the action of SO(3) in the space of these variables is transitive. The nonstandard boost gauge transformations define three new gauge variables $\hat{\Gamma}_0{}^{(a)(0)}$. We see from (4.32) that the nonstandard boost transformations act transitively in the space of their gauge variables. We complete these considerations recalling that for the standard boost transformations the gauge variables are $n^{(a)}$ (see Refs. 11 and 12).

V. NONSTANDARD ACTIONS OF THE LOCAL LORENTZ GROUP IN THE SPACE OF FIELD VARIABLES

The standard action of the local SO(3,1) group in the space of field potentials consists in their rotations according to the following formulas:[11]

$$\begin{aligned} e^{(\alpha)}{}_{\mu} &\to 'e^{(\alpha)}{}_{\mu}=L^{-1(\alpha)}{}_{(\beta)}e^{(\beta)}{}_{\mu}\ ,\\ \Gamma_{\lambda}{}^{(\alpha)}{}_{(\beta)} &\to '\Gamma_{\lambda}^{(\alpha)}{}_{(\beta)}= L^{-1(\alpha)}{}_{(\tau)}\Gamma_{\lambda}{}^{(\tau)}{}_{(\epsilon)}L^{(\epsilon)}{}_{(\beta)}\\ &\qquad -\partial_{\lambda}L^{-1(\alpha)}{}_{(\tau)}L^{(\tau)}{}_{(\beta)}\ . \end{aligned} \tag{5.1}$$

We recall that the careted variables are constructed by means of the composition of two operations—boost rotations denoted by a tilde and the (3 + 1) decomposition denoted by an overbar. Therefore the standard action of the local Lorentz group generates transformations in the space of careted variables by means of the following diagram:

$$\begin{tikzcd} e^{(\alpha)}{}_{\lambda},\Gamma_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[r, "L"] \arrow[d, "B(n^{(a)})"] & 'e^{(\alpha)}{}_{\lambda},'\Gamma_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[d, "B('n^{(a)})"'] \\ \tilde{e}^{(\alpha)}{}_{\lambda},\tilde{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[d] & '\tilde{e}^{(\alpha)}{}_{\lambda},'\tilde{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[d] \\ \hat{e}^{(\alpha)}{}_{\lambda},\hat{\Gamma}_{\lambda}{}^{(\alpha)}{}_{\beta} \arrow[r, "\hat{L}"] & '\hat{e}^{(\alpha)}{}_{\lambda},'\hat{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)}\ . \end{tikzcd} \tag{5.2}$$

Here $B(n^{(a)})$ and $B('n^{(a)})$ are the boost matrices corresponding to boost parameters $n^{(a)}$ and $'n^{(a)}$, respectively;

$$\begin{aligned} &n_{(a)}=e_{(a)}\mathcal{N},\quad n_{(a)}=n^{(a)}\ ,\\ &'n_{(a)}='e_{(a)}\mathcal{N},\quad 'n_{(a)}='n^{(a)}\ , \end{aligned}$$

where $\mathcal{N}$ is a unit vector field normal to the slicing. A profound discussion of the operator $\hat{L}$ was presented in Ref. 11.

The SKY Lagrangian, however, is also invariant with respect to the following nonstandard action of the local SO(3,1) group:

$$\begin{aligned} &e^{(\alpha)}{}_{\mu}\to 'e^{(\alpha)}{}_{(\mu)}=e^{(\alpha)}{}_{\mu}\ ,\\ &\Gamma_{\lambda}{}^{(a)}{}_{(\beta)}\to '\Gamma_{\lambda}{}^{(\alpha)}{}_{(\beta)}\\ &\qquad =L^{-1(\alpha)}{}_{(\tau)}\Gamma_{\lambda}{}^{(\tau)}{}_{(\epsilon)}L^{(\epsilon)}{}_{(\beta)}-\partial_{\lambda}L^{-1(\alpha)}{}_{(\tau)}L^{(\tau)}{}_{(\beta)}\ . \end{aligned} \tag{5.3}$$

(This invariance was observed by Schweitzer in Ref. 25.)

The transformations (5.3) generate the diagram

$$\begin{tikzcd} e^{(\alpha)}{}_{\lambda},\Gamma_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[r, "L_{\text{new}}"] \arrow[d, "B(n^{(a)})"] & 'e^{(\alpha)}{}_{\lambda},'\Gamma_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[d, "B('n^{(a)})"'] \\ \tilde{e}^{(\alpha)}{}_{\lambda},\tilde{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[d] & '\tilde{e}^{(\alpha)}{}_{\lambda},'\tilde{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[d] \\ \hat{e}^{(\alpha)}{}_{\lambda},\hat{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)} \arrow[r, "\hat{L}_{\text{new}}"] & '\hat{e}^{(\alpha)}{}_{\lambda},'\hat{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)}\ . \end{tikzcd} \tag{5.4}$$

We observe that $'\tilde{e}^{(\alpha)}{}_{\lambda}=\tilde{e}^{(\alpha)}{}_{\lambda}$ and $'n^{(a)}=n^{(a)}$. Let

$$\hat{L}^{(\beta)}{}_{(\alpha)}=B^{-1(\beta)}{}_{(\tau)}L^{(\tau)}{}_{(\epsilon)}B^{(\epsilon)}{}_{(\alpha)}\ ,$$

then

$$'\hat{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)}=\hat{L}^{-1(\alpha)}{}_{(\tau)}\hat{\Gamma}_{\lambda}{}^{(\tau)}{}_{(\epsilon)}\hat{L}^{(\epsilon)}{}_{(\beta)}-\bar{\partial}_{\lambda}\hat{L}^{-1(\alpha)}{}_{(\tau)}\hat{L}^{(\tau)}{}_{(\beta)}\ . \tag{5.5}$$

The infinitesimal version of (5.5) reads

$$\delta\hat{\Gamma}_{\lambda}{}^{(\alpha)}{}_{(\beta)}={}^{\dagger}\hat{D}_{\lambda}\delta\hat{L}^{(\alpha)}{}_{(\beta)} \tag{5.6}$$

and

$$\delta\hat{e}^{(\alpha)}{}_{\lambda}=0\ . \tag{5.7}$$

It follows from (5.7) that

$$\delta N=0,\quad \delta N^{k}=0\ . \tag{5.8}$$

Comparing (5.6) and (5.8) with (4.34) and (4.35) we observe that the transformations generated by the constraints $(\hat{E}2)^{0}{}_{(\alpha)(\beta)}=0$ are just the transformations given by the nonstandard action L_{new} of the local Lorentz group. As we have already pointed out, for local SO(3) rotations we have a one-to-one correspondence between standard and nonstandard transformations but for boosts the transformations L and L_{new} are different. Now it is clear that the invariance of the SKY theory with respect to local Lorentz rotations gives rise to 9 rotational gauge variables: 3 SO(3) gauge variables $\hat{\Gamma}_0{}^{(a)(b)}$, 3 standard boost-gauge variables $n^{(a)}$, and 3 nonstandard boost gauge variables $\hat{\Gamma}_0{}^{(a)(0)}$.

The SKY Lagrangian is also invariant with respect to the following action of the local Lorentz group:

$$e^{(\alpha)}{}_\lambda \to {}'e^{(\alpha)}{}_\lambda = L^{-1(\alpha)}{}_{(\tau)} e^{(\tau)}{}_\lambda \ ,$$
$$\Gamma_\lambda{}^{(\alpha)}{}_{(\beta)} \to {}'\Gamma_\lambda{}^{(\alpha)}{}_{(\beta)} = \Gamma_\lambda{}^{(\alpha)}{}_{(\beta)} \ . \tag{5.9}$$

This action may be considered as a complementary to (5.3).

The transformations (5.9) generate the corresponding action in the space of careted variables. Here we present only their infinitesimal version, that is, the action of the Lie algebra of the local Lorentz group.

Let $\delta L^{(\mu)}{}_{(\nu)}$ be an element of the Lie algebra of the local Lorentz group; that is, $\delta L^{(\mu)(\nu)}$ is a field of skew-symmetric matrices on spacetime. If $B^{(\alpha)}{}_{(\beta)}(n^{(a)})$ is the field of boost matrices corresponding to the boost coefficients $n^{(a)}$ then we define

$$\hat{\delta} L^{(\mu)}{}_{(\nu)} = B^{-1(\mu)}{}_{(\sigma)} \delta L^{(\sigma)}{}_{(\rho)} B^{(\rho)}{}_{(\nu)} \ . \tag{5.10}$$

The transformations (5.9) give rise to the relations

$$\delta n_{(a)} = -\hat{\delta} L^{(0)}{}_{(c)} B^{-1(c)}{}_{(a)} \ , \tag{5.11}$$
$$\delta \hat{\Gamma}_\lambda{}^{(\alpha)}{}_{(\beta)} = -{}^{\dagger}\hat{D}_\lambda (\delta B^{-1(\alpha)}{}_{(\tau)} B^{(\tau)}{}_{(\beta)}) \ , \tag{5.12a}$$
$$\delta X^k{}_{(\alpha)(\beta)} = -\delta B^{-1(\mu)}{}_{(\tau)} B^{(\tau)}{}_{(\alpha)} X^k{}_{(\mu)(\beta)} - \delta B^{-1(\nu)}{}_{(\tau)} B^{(\tau)}{}_{(\beta)} X^k{}_{(\alpha)(\nu)} \ , \tag{5.12b}$$
$$\delta Y^{k(\alpha)(\beta)} = \delta B^{-1(\alpha)}{}_{(\tau)} B^{(\tau)}{}_{(\mu)} X^{k(\mu)(\beta)} + \delta B^{-1(\beta)}{}_{(\tau)} B^{(\tau)}{}_{(\nu)} Y^{k(\alpha)(\nu)} \ , \tag{5.12c}$$
$$\delta N = 0, \quad \delta N^k = 0, \quad \delta \bar{\mathscr{g}}_{ij} = 0 \ , \tag{5.13}$$
$$\delta \hat{e}^{(a)}{}_k = \delta B^{-1(a)}{}_{(\mu)} B^{(\mu)}{}_{(c)} \hat{e}^{(c)}{}_k - \hat{\delta} L^{(a)}{}_{(c)} \hat{e}^{(c)}{}_k \ . \tag{5.14}$$

We have the following formulas for $B^{(\alpha)}{}_{(\beta)}$ (Ref. 11):

$$\begin{aligned} &B^{(0)}{}_{(0)} = n^{(0)}, \quad B^{(a)}{}_{(0)} = n^{(a)}, \quad B^{(0)}{}_{(a)} = n_{(a)} \ , \\ &B^{(a)}{}_{(b)} = \delta^{(a)}{}_{(b)} + (1+n^{(0)})^{-1} n^{(a)} n_{(b)} \ , \\ &n_{(a)} = e_{(a)} \mathcal{N}, \quad n^{(a)} = n_{(a)}, \quad n^{(0)} = (1+n^{(c)} n_{(c)})^{1/2} \ . \end{aligned} \tag{5.15}$$

Taking into account (5.15) we get

$$\begin{aligned} &\delta B^{-1(0)}{}_{(\tau)} B^{(\tau)}{}_{(0)} = 0 \ , \\ &\delta B^{-1(0)}{}_{(\tau)} B^{(\tau)}{}_{(b)} = {}^3B^{-1(c)}{}_{(b)} \delta n_{(c)} = -\hat{\delta} L_{(b)}{}^{(0)} \ , \\ &\delta B^{-1(a)}{}_{(\tau)} B^{(\tau)}{}_{(0)} = {}^3B^{-1(a)}{}_{(c)} \delta n^{(c)} = -\hat{\delta} L^{(a)(0)} \ , \\ &\delta B^{-1(a)}{}_{(\tau)} B^{(\tau)}{}_{(b)} = (1+n^{(0)})^{-1} ({}^3B^{-1(c)}{}_{(b)} n^{(a)} \delta n_{(c)} - {}^3B^{-1(a)}{}_{(c)} n_{(b)} \delta n^{(c)}) \\ &\qquad\qquad = (1+n^{(0)})^{-1} (n^{(a)} \hat{\delta} L_{(b)}{}^{(0)} - \hat{\delta} L^{(a)(0)} n_{(b)}) \ , \end{aligned} \tag{5.16}$$

where

$${}^3B^{-1(a)}{}_{(b)} = \delta^{(a)}{}_{(b)} - n^{(a)} n_{(b)} [n^{(0)} (1+n^{(0)})]^{-1} \ .$$

Now we are able to compute the right-hand sides of (5.12) and (5.14). Particularly,

$$\delta \hat{e}^{(a)}{}_k = -[\hat{\delta} L^{(a)}{}_{(b)} + (1+n^{(0)})^{-1} (\hat{\delta} L^{(a)(0)} n_{(b)} - \hat{\delta} L_{(b)}{}^{(0)} n^{(a)})] \hat{e}^{(b)}{}_k \tag{5.14'}$$

and this formula coincides with that given in (C12a) of Ref. 11. It is also clear that formulas (5.13) differ from the corresponding ones in (C12) of Ref. 11.

We consider two special cases of transformations (5.12)–(5.14):

$$\text{(i)} \quad \hat{\delta} L^{(a)(0)} = 0 \ . \tag{5.17}$$

Then we have infinitesimal SO(3) rotations of triads $\hat{e}^{(a)}{}_k$ generated by arbitrary skew-symmetric matrices $\hat{\delta} L^{(a)(b)}$ [cf. (5.14′)] and the trivial transformations in the space of the connections coefficients.

$$\text{(ii)} \quad \hat{\delta} L^{(a)(b)} = 0 \ . \tag{5.18}$$

Then the transformations (5.12)–(5.14) are given by the composition of the transformations (4.35) [or equivalently (5.6)–(5.8) with $\hat{\delta} L^{(a)(b)} = 0$] and of the infinitesimal SO(3) rotation corresponding to the 3×3 matrix $\delta B^{-1(a)}{}_{(\tau)} B^{(\tau)}{}_{(b)}$. Now it is clear that this case is entirely covered by the standard action of the local Lorentz group and the nonstandard action (5.6)–(5.8).

The gauge transformations given by case (i) of (5.12)–(5.14) are related to the following property of the SKY theory. From 9 triad components $\hat{e}^{(a)}{}_k$, 6 are determined by the metric $\bar{g}_{ij}$ and that, in turn, can be computed by means of the field equations. The remaining three components of triads are rotational gauge variables of the action (5.12)–(5.14) (i) of the local SO(3) group. We know that the three-metric density $\bar{\mathscr{g}}_{ij}$ is determined by (3.19) up to a conformal factor τ. Let us observe that multiplying τ by a function λ and simultaneously multiplying N by λ^{-1} we do not change Eqs. (3.19) and Eq. (3.33), which determines τ, is not changed either. Therefore we have the transformations

$$\begin{aligned}
&\bar{\mathscr{g}}_{ij}\rightarrow{}'\bar{\mathscr{g}}_{ij}=\lambda\bar{\mathscr{g}}_{ij},\quad \bar{g}_{ij}\rightarrow{}'\bar{g}_{ij}=\lambda^{-2}\bar{g}_{ij}\ ,\\
&\hat{\Gamma}_k^{(\alpha)(\beta)}\rightarrow{}'\hat{\Gamma}_k{}^{(\alpha)(\beta)}=\hat{\Gamma}_k{}^{(\alpha)(\beta)}\ ,\\
&X^k{}_{(\alpha)(\beta)}\rightarrow{}'X^k{}_{(\alpha)(\beta)}=X^k{}_{(\alpha)(\beta)}\ ,\\
&Y^{k(\alpha)(\beta)}\rightarrow{}'Y^{k(\alpha)(\beta)}=Y^{k(\alpha)(\beta)}\ ,\\
&N\rightarrow{}'N=\lambda^{-1}N\ .
\end{aligned}\tag{5.19}$$

If we extend these transformations according to the formulas

$$\begin{aligned}
&N^k\rightarrow{}'N^k=N^k\ ,\\
&\hat{\Gamma}_0{}^{(\alpha)}{}_{(\beta)}\rightarrow{}'\hat{\Gamma}_0{}^{(\alpha)}{}_{(\beta)}=\lambda\hat{\Gamma}_0{}^{(\alpha)}{}_{(\beta)}\ ,
\end{aligned}\tag{5.20}$$

then ${}'^{\dagger}\hat{D}_0=\lambda\,{}^{\dagger}\hat{D}_0$, ${}'^{\dagger}\hat{D}_k={}^{\dagger}\hat{D}_k$ and the field equations are invariant with respect to the transformations (5.19) and (5.20).

Translating formulas (5.19) and (5.20) into the four-dimensional picture we observe that they correspond to the conformal (scale) transformations[78]

$$\begin{aligned}
&e^{(\alpha)}{}_\mu\rightarrow{}'e^{(\alpha)}{}_\mu=\lambda^{-1}e^{(\alpha)}{}_\mu\ ,\\
&g_{\mu\nu}\rightarrow{}'g_{\mu\nu}=\lambda^{-2}g_{\mu\nu}\ ,\\
&\Gamma_\mu{}^{(\alpha)}{}_{(\beta)}\rightarrow{}'\Gamma_\mu{}^{(\alpha)}{}_{(\beta)}=\Gamma_\mu{}^{(\alpha)}{}_{(\beta)}\ .
\end{aligned}\tag{5.21}$$

It is obvious that the SKY Lagrangian is invariant with respect to these transformations.

In the SKY theory the metric density $\bar{\mathscr{g}}_{ij}$ is determined by algebraic functions of symplectic variables and their spatial derivatives up to a conformal factor τ. On the other hand, the conformal gauge transformations (5.21) [or equivalently (5.19) and (5.20)] can reduce τ to the trivial function $\tau=1$. But then we have to multiply the lapse N by τ. Therefore though τ is the gauge variable of the transformations (5.21) it is not independent of other gauge variables. A gauge change of τ generates a corresponding change of N so that the product τN satisfies Eq. (3.33).

In order to sum up the discussion on gauge transformations and gauge variables we should give precise definitions of these notations.

A gauge transformation in a field theory is such a transformation in the space of field variables (potentials, field strengths, momenta, etc.) that preserves the field equations of the theory in question.

Gauge variables corresponding to given gauge transformations are such quantities that can be changed arbitrarily under the action of the gauge transformations. That is to say, the action of gauge transformations in the space of gauge variables is transitive.

For the SKY theory we have the following independent gauge transformations in the set of the gravitational potentials: (i) 10-parameter standard action of the local Lorentz group and of the diffeomorphism group of space-time; (ii) 3-parameter set of transformations generated by a nonstandard action of boosts; (iii) a natural 3-parameter action of the local SO(3) group in the space of triads (5.14′).

Remark: From an SO(3,1)-covariant point of view it is more natural to consider a combined action of the standard transformations given by the local Lorentz group with an SO(3,1)-covariant action of the group of diffeomorphisms. Such an action is generated by a 10-parameter (local) group having the bundle structure over Diff$M$ with fibers isomorphic to the local Lorentz group. In this approach the generators of translations are SO(3,1)-covariant Lie derivatives and the generators of rotations are the standard generators of the local Lorentz group. For details see Refs. 12, 65, and also 79.

If we reduce the transformations (i)–(iii) to the complete set of symplectic variables $X^k{}_{(\alpha)(\beta)}$, $\hat{\Gamma}_k{}^{(\alpha)(\beta)}$, M_a, $n^{(a)}$ then we have 13-parameter gauge transformations (i) and (ii) as well as 13 gauge variables $N,N^k,n^{(a)},\hat{\Gamma}_0{}^{(a)(b)},\hat{\Gamma}_0{}^{(a)(0)}$.

It is known from Ref. 11 that for solutions $M_{(a)}=0$ and that $n^{(a)}$ are the gauge variables for standard boost transformations. Therefore in the set of the dynamical symplectic variables (3.16) we have only ten independent gauge transformations—seven of (i) and three of (ii). Their gauge variables are $N,N^k,\hat{\Gamma}_0{}^{(a)(b)},\hat{\Gamma}_0{}^{(a)(0)}$.

Three gauge transformations (iii) cause us to have three arbitrary components of triads on slices. The remaining 6 are determined by means of the field equations.

Discussing the problem of independent degrees of freedom we consider initial values of 36 dynamical symplectic variables $X^k{}_{(\alpha)(\beta)}$ and $\hat{\Gamma}_k{}^{(\alpha)(\beta)}$ satisfying 16 constraints and subject to the action of 10-parameter gauge transformations. Therefore we expect $36-(16+10)=10$ independent degrees of freedom in the phase space.

VI. THE SKY GRAVITY COUPLED TO YANG-MILLS FIELDS

We know that the vacuum SKY theory has a very elegant canonical structure. The situation becomes less elegant and much more complicated if we couple it to a tensor (spinor) matter field. Then the field equations (2.2) read

$$(E1)^\lambda{}_{(\alpha)}=V^\lambda{}_{(\alpha)}+T^\lambda{}_{(\alpha)}\ ,\tag{6.1a}$$

$$(E2)^\lambda{}_{(\alpha)(\beta)}=-D_\mu(eR_{(\alpha)(\beta)}{}^{\mu\lambda})+s^\lambda{}_{(\alpha)(\beta)}\ ,\tag{6.1b}$$

where

$$T^\lambda{}_{(\alpha)}=\partial L_{\rm mat}/\partial e^{(\alpha)}{}_\lambda\ ,\tag{6.2a}$$

$$s^\lambda{}_{(\alpha)(\beta)}=\partial L_{\rm mat}/\partial\Gamma_\lambda{}^{(\alpha)(\beta)}\tag{6.2b}$$

are the matter canonical energy-momentum and spin tensors, respectively. It follows from the properties of the gravitational energy-momentum tensor (2.3) that the dynamics of the coupled system requires symmetry and tracelessness of the matter energy-momentum tensor (6.2a). These two conditions give rise to supplementary constraint equations, which are not symplectic in general. It is clear that such a system can be very complicated. It would be interesting, however, to couple the SKY Lagrangian to matter fields whose canonical energy-momentum tensors are *a priori* symmetric and traceless. These conditions are satisfied by Yang-Mills fields.

Let H be a semisimple p-dimensional Lie group and h be its Lie algebra equipped with a nondegenerate Killing form $k(\ ,\ )$. Let

$$A^I = A^I{}_\mu dx^\mu$$

be a one-form with values in h. Its covariant exterior derivative, that is, the curvature of A^I,

$$f^I = DA^I = \tfrac{1}{2} f^I{}_{\mu\nu} dx^\mu \wedge dx^\nu$$
$$= \tfrac{1}{2}(\partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu + c^I{}_{JK} A^J{}_\mu A^K{}_\nu) dx^\mu \wedge dx^\nu$$

plays the role of the field strength. Here $c^I{}_{JK}$ are the structure constants of h.

The Yang-Mills Lagrangian reads

$$L_{\rm YM} = -\tfrac{1}{4} e f^I{}_{\mu\nu} f^{J\mu\nu} k_{IJ} \ , \tag{6.3}$$

where $k_{IJ} = -c^U{}_{IV} c^V{}_{JU}$ are the components of the Killing form $k(\ ,\ )$. The spin tensor for the Yang-Mills Lagrangian vanishes and the canonical energy-momentum tensor is given by the formula

$$T^\lambda{}_{(\alpha)} = F^{I\lambda\nu} f^J{}_{\epsilon\nu} e^\epsilon{}_{(\alpha)} k_{IJ} - \tfrac{1}{4} e^\lambda{}_{(\alpha)} F^{I\mu\nu} f^J{}_{\mu\nu} k_{IJ} \ , \tag{6.4}$$

where

$$F^{I\lambda\nu} = e f^{I\lambda\nu} \tag{6.5}$$

are the momenta of the Yang-Mills field.

It is obvious that the the tensor (6.4) is symmetric and traceless. We define the electric field (an h-valued density of slices)

$$E_I{}^k = \hat{F}_I{}^{0k} \tag{6.6a}$$

and the magnetic induction (an h-valued density on slices)

$$B^{Ik} = \tfrac{1}{2} \epsilon^{kab} \hat{f}^I{}_{ab} \ . \tag{6.6b}$$

The symplectic variables for the Yang-Mills field are $E_I{}^k$ and $\hat{A}^I{}_k$. Therefore the electric field and the magnetic induction are functions of the symplectic variables and their spatial derivatives.

We have the important formulas

$$\begin{aligned} &\hat{T}^0{}_k = \epsilon_{kmn} E_I{}^m B^{In} \ , \\ &\sqrt{g}\,(\hat{T}^{ks} + \hat{g}^{ks} \hat{T}^0{}_0) = -(E^{Jk} E_J{}^s + B^{Jk} B_J{}^s) \ . \end{aligned} \tag{6.7}$$

The Yang-Mills equations

$$-D_\lambda F^{I\lambda\tau} = 0 \ , \tag{6.8}$$

rewritten in the (3 + 1) form are

$$\hat{D}_0 E_I{}^k = \epsilon^{kab} (\hat{D}_a + \partial_a \ln N)(B_I{}^r \bar{\mathscr{g}}_{rb}) \ , \tag{6.9a}$$

$$\hat{D}_k E_I{}^k = 0 \ , \tag{6.9b}$$

$$\hat{D}_0 B^{Ik} = -\epsilon^{kab} (\hat{D}_a + \partial_a \ln N)(E^{Ir} \bar{\mathscr{g}}_{rb}) \ , \tag{6.10a}$$

$$\hat{D}_k B^{Ik} = 0 \ . \tag{6.10b}$$

In the above formulas D_λ and $\hat{D}_\lambda$ are the Yang-Mills covariant derivatives defined in Appendix A.

Remark: Equations (6.10b) follow from the Bianchi identities for the curvature f^I.

For the SKY gravity coupled to a Yang-Mills field we have the dynamical gravitational equations (3.9) and (3.10), the dynamical Yang-Mills equations (6.9a) and (6.10a), and the following relations for the dynamics of potentials [see (A27)]:

$$\hat{D}_0 \hat{A}^I{}_m = -E^{Ik} \bar{\mathscr{g}}_{km} \ . \tag{6.10a$'$}$$

In the presence of a Yang-Mills matter field the gravitational constraints (4.2) do not change their form. The left-hand sides of constraints (4.1) and (4.3) acquire additional terms due to energy-momentum tensor of matter. These terms are given by formulas (6.7a) and (6.7b), respectively. The time-maintenance condition for constraints (4.2) can again be written in the form of a homogeneous linear system (3.19) with the quantity A^{kspq} defined by means of the modified tensor density $W_z{}^{pq}$. The right-hand side of the formula (3.21′) has to be supplemented by the term

$$-\tfrac{1}{2}({}_\xi\hat{D}_z B^{iJ} E^j{}_J - B^{jJ}\,{}_\xi\hat{D}_z E^i{}_J) \ . \tag{6.11}$$

The solvability condition for the modified system (3.19) gives rise to the gravitational constraint (4.4). The time-maintenance condition for the modified constraint (4.4) leads to a partial differential equation of type (3.33) that enables us to determine the conformal factor τ.

For the coupled system we have the same gravitational gauge transformations and the same gravitational gauge variables as in the vacuum case. Additionally we have infinitesimal Yang-Mills gauge transformations generated by constraints (6.9). They read

$$\begin{aligned} &\delta \hat{A}^I{}_k = \hat{D}_k \psi^I, \quad \delta E_I{}^k = -c^J{}_{IV} \psi^V E_J{}^k \ , \\ &\delta B^{Ik} = c^I{}_{JV} \psi^V B^{Jk} \ . \end{aligned} \tag{6.12}$$

Here ψ^I is a field of elements of the Lie algebra h, that is, an element of the local gauge algebra.

If we define

$$\begin{aligned} &\delta \hat{A}^I{}_0 = \hat{D}_0 \psi^I, \quad \delta N = 0 \ , \\ &\delta N^k = 0, \quad \delta \bar{\mathscr{g}}_{ij} = 0 \ , \end{aligned} \tag{6.13}$$

then the transformations (6.11) and (6.12) can be integrated to an action of the local Yang-Mills group. The corresponding formulas can be found in Refs. 80 and 81.

For Yang-Mills fields we have $6p$ symplectic matter variables $E_I{}^k, \hat{A}^I{}_k$ ($p = \dim h$), p symplectic matter constraints (6.9b), and a p parameter group of transformations. Therefore we have $6p - 2p = 4p$ independent degrees of freedom in the phase space.

At the end of the present section we briefly discuss the SKY gravity in the presence of a vector field. Our analysis is more precise than that given by Schweitzer in Ref. 25. He claimed that nonstandard symmetries of the SKY Lagrangian make it impossible to couple this theory to matter fields. This is not true, however. With much more sophisticated techniques at hand than those available to Schweitzer seven years ago we can handle the situation.

Let us take the vector field $f^{(\alpha)}$ with the Lagrangian

$$\begin{aligned} L = {} & (e/2) D_\mu f^{(\alpha)} D^\mu f_{(\alpha)} + (e/2)(e^\nu{}_{(\beta)} D_\nu f^{(\beta)})^2 \\ & + (em^2/2) f_{(\alpha)} f^{(\alpha)} \ , \end{aligned} \tag{6.14}$$

which breaks the nonstandard symmetries of the vacuum SKY theory. The dynamics of the matter field is given by the Euler-Lagrangian (EL) equations of (6.14) whose explicit form is given in Appendix A. For the coupled system we have the following situation.

(i) Equations (3.4) and (3.8b),

$$(\hat{E}1)^0{}_p = -\epsilon_{pij} X^i{}_{(\alpha)(\beta)} Y^{j(\alpha)(\beta)} + \hat{T}^0{}_p = 0 \ , \tag{6.15}$$

$$(\hat{E}2)^0{}_{(a)(b)} = {}^\dagger \hat{D}_k X^k{}_{(a)(b)} + \hat{s}^0{}_{(a)(b)} = 0 \ , \tag{6.16}$$

are still first-class constraints and their left-hand sides are expressed by the gravitational symplectic variables (3.16) and the matter symplectic variables $\hat{p}^0{}_{(\alpha)}, \hat{f}^{(\alpha)}$ (cf. Appendix A).

(ii) Equations (3.6),

$$C^{ks} = -\hat{e}(\hat{T}^{ks} + \bar{g}^{ks} \hat{T}^0{}_{(0)}) \ , \tag{6.17}$$

are not symplectic constraints any more.

The right-hand sides of (6.17) are algebraic functions of the triad components $\hat{e}^k{}_{(a)}$ as well as of the gravitational and matter symplectic variables and their spatial derivatives [cf. (A35)]. Thus, the triad components $\hat{e}^k{}_{(a)}$ as well as the metric components $\bar{g}_{ij}$ are algebraic functions of the symplectic variables and their x^k derivatives (see Appendix A for a more detailed discussion).

(iii) The energy constraint reads

$$(\hat{E}1)^0{}_{(0)} = \tfrac{1}{2} C^{ks} \bar{\varphi}_{ks} + \hat{T}^0{}_{(0)} = 0 \ . \tag{6.18}$$

It follows from (ii) and Refs. 11 and 12 that Eq. (6.18) is a symplectic first-class constraint.

(iv) Equations (3.8a),

$$(\hat{E}2)^0{}_{(a)(0)} = {}^\dagger \hat{D}_k X^k{}_{(a)(0)} + \hat{s}^0{}_{(a)(0)} = 0 \ , \tag{6.19}$$

are now second-class constraints. Their time-conservation conditions lead to equations $(\hat{E}1)^k{}_{(0)} = 0$ (Refs. 11 and 12) which by virtue of (6.15) are equivalent to

$$\hat{e}\, {}^\dagger \hat{D}_k \hat{f}_{(0)} + \hat{e}^{(c)}{}_k \hat{p}^0{}_{(c)} = 0 \ . \tag{6.20}$$

These equations are second-class constraints. (In fact, they are the symmetry conditions $\hat{T}^{0k} = \hat{T}^{k0}$ for the energy-momentum tensor.) The time-conservation conditions for (6.20) give rise to three algebraic equations for $\hat{\Gamma}_0{}^{(0)}{}_{(c)}$ [see (A37)].

(v) The dynamics of the gravitational variables $X^k{}_{(\alpha)(\beta)}, Y^{k(\alpha)(\beta)}$ is given by Eqs. (3.9) with the term $\hat{s}^k{}_{(\alpha)(\beta)}$ added to their right-hand sides and Eqs. (3.10).

For $36+8=44$ symplectic variables $X^k{}_{(\alpha)(\beta)}$, $Y^{k(\alpha)(\beta)}, \hat{p}^0{}_{(\alpha)}, \hat{f}^{(\alpha)}$ we have seven first-class constraints and six second-class constraints (6.19) and (6.20). Thus we have $44-(2\times 7+6)=24$ degrees of freedom in the phase space. Sixteen of them are the gravitational degrees of freedom and eight are matter ones.

It is worthwhile to mention that the number of 16 gravitational degrees of freedom (in the phase space) is generic for Lagrangians at most quadratic in components of curvature (no torsion terms). A generic example is the Fairchild theory[17] whose canonical analysis has recently been presented in Ref. 62. Thus the coupling with matter leads to a generic $R+R^2$ theory breaking extraordinary symmetries of vacuum SKY gravity.

Let us observe that Eqs. (6.17), (6.18), and (6.20) coerce the symmetry and tracelessness of the matter energy-momentum tensor for a solution of the system. If, however, for a matter Lagrangian $T^{\mu\nu}$ is symmetric *a priori* then we have only six independent equations for $\hat{e}^k{}_{(a)}$ in (6.17) and we are able to determine only $\bar{\varphi}_{ij}$ in terms of symplectic variables. Equations (6.20) are identities and we have no equations for $\hat{\Gamma}_0{}^{(0)}{}_{(c)}$. Such a situation occurs for the vector field Lagrangian whose kinematic term is given by the first summand in (6.14). It is almost obvious that this Lagrangian is invariant with respect to the nonstandard Lorentz transformations.

Another example of an extrasymmetric gravity Lagrangian coupled to a matter field has recently been discussed in Ref. 65. It has been shown there how such a coupling breaks the projective invariance of the Einstein Lagrangian in the scheme of GL(4,R) theory of gravity.

The method presented above should, in principle, enable us to discuss the coupled SKY Dirac system. A necessary Hamiltonian structure for SL(2,C) gauge theories of gravity has been presented in Ref. 12.

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APPENDIX A

The relations between tetrads and the metric are

$$\begin{aligned} g_{\mu\nu} &= e^{(\alpha)}{}_\mu e^{(\beta)}{}_\nu \eta_{(\alpha)(\beta)} \ , \\ g^{\mu\nu} &= e^\mu{}_{(\alpha)} e^\nu{}_{(\beta)} \eta^{(\alpha)(\beta)}, \quad e = \det[e^{(\alpha)}{}_\lambda] \ , \end{aligned} \tag{A1}$$

where $\eta_{(\alpha)(\beta)}$ and $\eta^{(\alpha)(\beta)}$ are the constant, diagonal Minkowski metrics with the signature $(-1,+1,+1,+1)$.

ADM's lapse and shift are

$$N = (-g^{00})^{-1/2}, \ N^k = \bar{g}^{ks} g_{0s} \ . \tag{A2}$$

The basic relations for the $(3+1)$ decomposition are

$$\begin{aligned} & g^{pq} = \bar{g}^{pq} - (N^p N^q)/N^2, \quad g_{00} = -N^2 + N^s N_s \ , \\ & g_{pq} = \bar{g}_{pq}, \quad g^{0p} = N^p/N^2 \ , \\ & g_{0s} = N_s, \quad \bar{g}_{ik} \bar{g}^{kj} = \delta^j{}_i \ , \\ & e = \sqrt{-g} = (-\det[g_{\mu\nu}])^{1/2} \\ & \qquad = N\sqrt{\bar{g}} = N(\det[\bar{g}_{pq}])^{1/2} = N\hat{e} \ . \end{aligned} \tag{A3}$$

$\bar{\gamma}_k{}^p{}_q$ are the coefficients of the Levi-Civita connection for the metric $\bar{g}_{pq}$. For a tensor density $F^{\mu_1}{}_{\nu_1}{}^{\cdots\mu_k}{}_{\cdots\nu_s}$ of weight $+r$ on spacetime we define its bar components by means of the relations

$$\begin{aligned} \bar{F}^{\mu_1}{}_{\nu_1}{}^{\cdots\mu_k}{}_{\cdots\nu_s} = N^{-r} A^{\bar{\mu}_1}{}_{\alpha_1} \cdots A^{\bar{\mu}_k}{}_{\alpha_k} A^{-1\beta_1}{}_{\bar{\nu}_1} \cdots \\ \times A^{-1\beta_s}{}_{\bar{\nu}_s} F^{\alpha_1}{}_{\beta_1}{}^{\cdots\alpha_k}{}_{\cdots\beta_s} \ , \end{aligned} \tag{A4}$$

where

$$A^{\bar{0}}{}_{0}=N, \quad A^{\bar{k}}{}_{0}=N^{k}, \quad A^{\bar{\alpha}}{}_{s}=\delta^{\alpha}{}_{s} ,$$
$$A^{-1 0}{}_{\bar{0}}=1/N, \quad A^{-1k}{}_{\bar{0}}=-N^{k}/N ,$$
$$A^{-1\alpha}{}_{\bar{s}}=\delta^{\alpha}{}_{s} \quad \text{(cf. Ref. 64)} .$$

In particular, we define the differential operators

$$\bar{\partial}_0=(1/N)\partial_0-(N^s/N)\partial_s, \quad \bar{\partial}_k=\partial_k \ . \tag{A5}$$

For a spacetime tensor field (tensor density) $F^{(A)B}$ with values in the spaces of corresponding representations of SO(3,1), the composition of the boost operation $B^{(\alpha)}{}_{(\beta)}$ and the bar operation defines the caret components of $F^{(A)B}$ that are spatial tensors (tensor densities) with values in the spaces of corresponding representations of SO(3).

For the tetrad field we have

$$\begin{gathered}\hat{e}^{(0)}{}_{0}=1, \quad \hat{e}^{(0)}{}_{k}=0, \quad \hat{e}^{(a)}{}_{0}=0 ,\\ \hat{e}^{0}{}_{(0)}=1, \quad \hat{e}^{k}{}_{(0)}=0, \quad \hat{e}^{0}{}_{(a)}=0 ;\end{gathered} \tag{A6}$$

the quantities $\hat{e}^{(a)}{}_{k}$ give us triad fields of covectors on the surfaces of a slicing and $\hat{e}^{k}{}_{(a)}$ are the dual triads of vectors tangent to these surfaces. The decomposition of the connection coefficients $\Gamma_{\lambda}{}^{(\alpha)(\beta)}$ on M gives rise to the following quantities:

$$\begin{aligned}&\hat{\Gamma}_{k}{}^{(0)(b)}, \text{ which represent the second fundamental form of slices} ,\\ &\hat{\Gamma}_{k}{}^{(a)(b)}, \text{ defining a metric compatible connection on slices} ,\\ &\hat{\Gamma}_{0}{}^{(a)(b)}, \text{ defining the coefficients of a } \bar{\partial}_0 \text{ connection on slices} ,\\ &\hat{\Gamma}_{0}{}^{(0)(b)}, \text{ representing an SO(3)-valued scalar field on slices} .\end{aligned} \tag{A7}$$

We have the following decomposition of the Riemann tensor:

$$\begin{aligned}\hat{R}^{(a)}{}_{(0)0k}&=\hat{D}_0\hat{\Gamma}_{k}{}^{(a)}{}_{(0)}-(\hat{D}_k+\partial_k \ln N)\hat{\Gamma}_{0}{}^{(a)}{}_{(0)} ,\\ \hat{R}^{(a)}{}_{(b)0k}&=\hat{D}_0\hat{\Gamma}_{k}{}^{(a)}{}_{(b)}+\hat{\Gamma}_{0}{}^{(a)}{}_{(0)}\hat{\Gamma}_{k}{}^{(0)}{}_{(b)}-\hat{\Gamma}_{k}{}^{(a)}{}_{(0)}\hat{\Gamma}_{0}{}^{(0)}{}_{(b)} ,\end{aligned} \tag{A8a}$$

$$\begin{aligned}\hat{R}^{(a)}{}_{(0)pq}&=\hat{D}_p\hat{\Gamma}_{q}{}^{(a)}{}_{(0)}-\hat{D}_q\hat{\Gamma}_{p}{}^{(a)}{}_{(0)} ,\\ \hat{R}^{(a)}{}_{(b)pq}&={}^3\hat{R}^{(a)}{}_{(b)pq}+\hat{\Gamma}_{p}{}^{(a)}{}_{(0)}\hat{\Gamma}_{q}{}^{(0)}{}_{(b)}-\hat{\Gamma}_{q}{}^{(a)}{}_{(0)}\hat{\Gamma}_{p}{}^{(0)}{}_{(b)} .\end{aligned} \tag{A8b}$$

Here ${}^3\hat{R}^{(a)}{}_{(b)pq}$ is the Riemann tensor of the SO(3) connection $\hat{\Gamma}_{k}{}^{(a)(b)}$ on slices.

For a spacetime tensor density $F^{(A)B}$ of weight $+r$ with values in the space of a tensor representation of SO(3,1) the covariant derivative is given by the formula

$$\begin{aligned}D_\lambda F^{(A)B}=\ &\partial_\lambda F^{(A)B}-r\gamma_{\lambda}{}^{\tau}{}_{\tau}F^{(A)B}+f_{(\alpha)}{}^{(\beta)(A)}{}_{(E)}\Gamma_{\lambda}{}^{(\alpha)}{}_{(\beta)}F^{(E)B}\\ &+h_{\sigma}{}^{\rho B}{}_{J}\gamma_{\lambda}{}^{\sigma}{}_{\rho}F^{(A)J} ,\end{aligned} \tag{A9a}$$

where $f_{(\alpha)}{}^{(\beta)(A)}{}_{(E)}$ and $h_{\sigma}{}^{\rho B}{}_{J}$ are the generators of corresponding tensor representations of SO(3,1) and GL(4,R), respectively.

For the connection we define

$$D_\lambda \Gamma_{\tau}{}^{(\alpha)}{}_{(\beta)}=R^{(\alpha)}{}_{(\beta)\lambda\tau} \ . \tag{A9b}$$

For a spatial tensor density $\hat{F}^{(A)B}$ of weight $+r$ with values in the space of a tensor representation of SO(3) the covariant derivative is

$$\begin{aligned}\hat{D}_k\hat{F}^{(A)B}=\ &\bar{\partial}_k\hat{F}^{(A)B}-r\bar{\gamma}_{k}{}^{z}{}_{z}\hat{F}^{(A)B}+\hat{f}_{(a)}{}^{(b)(A)}{}_{(E)}\hat{\Gamma}_{(k)}{}^{(a)}{}_{(b)}\hat{F}^{(E)B}\\ &+\bar{h}_{p}{}^{qB}{}_{J}\bar{\gamma}_{k}{}^{p}{}_{q}\hat{F}^{(A)J} ,\end{aligned} \tag{A10a}$$

where $\hat{f}_{(a)}{}^{(b)(A)}{}_{(E)}$ and $\bar{h}_{p}{}^{qB}{}_{J}$ are the generators of corresponding representations of SO(3) and GL(3,R), respectively.

For the SO(3) connection on slices we define

$$\hat{D}_k\hat{\Gamma}_{s}{}^{(a)}{}_{(b)}={}^3\hat{R}^{(a)}{}_{(b)ks} \ . \tag{A10b}$$

In order to define SO(3)-covariant time derivatives we need $\bar{\partial}_0$ connection coefficients for both the anholonomic and holonomic indices. For the anholonomic indices these coefficients are $\hat{\Gamma}_{0}{}^{(a)}{}_{(b)}$ and for the holonomic indices we take as connection coefficients the quantities

$$\sigma^{q}{}_{p}=(1/N)\partial_p N^q \ . \tag{A11}$$

We refer the reader to Refs. 64, 12, and 65 where it was proved that such a choice gives rise to a time derivative $\hat{D}_0$ covariant with respect to local SO(3) rotations of triads and the reparametrizations of the slicing

$$\begin{aligned}&x^0\to x^{0'}=x^{0'}(x^0) ,\\ &x^k\to x^{k'}=x^{k'}(x^0,x^k) .\end{aligned}$$

Therefore we have

$$\begin{aligned}\hat{D}_0\hat{F}^{(A)(B)}=\ &\bar{\partial}_0\hat{F}^{(A)B}-r\sigma^{p}{}_{p}\hat{F}^{(A)B}\\ &+\hat{f}_{(a)}{}^{(b)(A)}{}_{(E)}\hat{\Gamma}_{0}{}^{(a)}{}_{(b)}\hat{F}^{(E)B}+\bar{h}_{p}{}^{qB}{}_{J}\sigma^{p}{}_{q}\hat{F}^{(A)J} .\end{aligned} \tag{A12a}$$

For the SO(3) connection we have

$$\hat{D}_0\hat{\Gamma}_{k}{}^{(a)}{}_{(b)}=\bar{\partial}_0\hat{\Gamma}_{k}{}^{(a)}{}_{(b)}-\bar{\partial}_k\hat{\Gamma}_{0}{}^{(a)}{}_{(b)}+\hat{\Gamma}_{0}{}^{(a)}{}_{(c)}\hat{\Gamma}_{k}{}^{(c)}{}_{(b)}-\hat{\Gamma}_{0}{}^{(d)}{}_{(b)}\hat{\Gamma}_{k}{}^{(a)}{}_{(d)}-\partial_k \ln N\,\hat{\Gamma}_{0}{}^{(a)}{}_{(b)}-(1/N)\partial_k N^p\hat{\Gamma}_{p}{}^{(a)}{}_{(b)} \ . \tag{A12b}$$

We have the following commutation relations for $\hat{D}_0$ and $\hat{D}_k$ derivatives:

$$[\hat{D}_0,\hat{D}_k]\hat{f}=\partial_k \ln N\hat{D}_0\hat{f} , \tag{A13a}$$

where $\hat{f}$ is a scalar function on slices;

$$[\hat{D}_0,\hat{D}_k]\hat{F}=\partial_k \ln N[\hat{D}_0\hat{F}-(1/\hat{e})\hat{D}_0(\hat{e})\hat{F}]-(1/\hat{e})(\hat{D}_k\hat{D}_0\hat{e})\hat{F} , \tag{A13b}$$

where $\hat{F}$ is a scalar density on slices of weight 1, and $\hat{e}=\sqrt{\bar{g}}$;

$$[\hat{D}_0,\hat{D}_k]\hat{f}^{(a)}{}_{(b)}=\partial_k \ln N\hat{D}_0\hat{f}^{(a)}{}_{(b)}+\hat{D}_0\hat{\Gamma}_k{}^{(a)}{}_{(c)}\hat{f}^{(c)}{}_{(b)}-\hat{D}_0\hat{\Gamma}_k{}^{(d)}{}_{(b)}\hat{f}^{(a)}{}_{(d)} , \tag{A13c}$$

where $\hat{f}^{(a)}{}_{(b)}$ is an SO(3) tensor field on slices.

The following formulas can be treated as commutation relations:

$$\begin{aligned}
&\hat{D}_0\,{}^3\hat{R}^{(a)}{}_{(b)pq}=(\hat{D}_p+\partial_p\ln N)\hat{D}_0\hat{\Gamma}_q{}^{(a)}{}_{(b)}-(\hat{D}_q+\partial_q\ln N)\hat{D}_0\hat{\Gamma}_p{}^{(a)}{}_{(b)} ,\\
&\hat{D}_0\hat{R}^{(a)}{}_{(0)pq}=(\hat{D}_p+\partial_p\ln N)\hat{D}_0\hat{\Gamma}_q{}^{(a)}{}_{(0)}-(\hat{D}_q+\partial_q\ln N)\hat{D}_0\hat{\Gamma}_p{}^{(a)}{}_{(0)}+\hat{D}_0\hat{\Gamma}_p{}^{(a)}{}_{(c)}\hat{\Gamma}_q{}^{(c)}{}_{(0)}-\hat{D}_0\hat{\Gamma}_q{}^{(a)}{}_{(c)}\hat{\Gamma}_p{}^{(c)}{}_{(0)} ,\\
&\hat{D}_0\hat{R}^{(a)}{}_{(b)pq}=\hat{D}_0\,{}^3\hat{R}^{(a)}{}_{(b)pq}+\hat{D}_0\hat{\Gamma}_p{}^{(a)}{}_{(0)}\hat{\Gamma}_q{}^{(0)}{}_{(b)}+\hat{\Gamma}_p{}^{(a)}{}_{(0)}\hat{D}_0\hat{\Gamma}_q{}^{(0)}{}_{(b)}-\hat{D}_0\hat{\Gamma}_q{}^{(a)}{}_{(0)}\hat{\Gamma}_p{}^{(0)}{}_{(b)}-\hat{\Gamma}_q{}^{(a)}{}_{(0)}\hat{D}_0\hat{\Gamma}_p{}^{(0)}{}_{(b)} .
\end{aligned} \tag{A14}$$

Making use of the field equations (3.10′) we get

$$\begin{aligned}
\hat{D}_0\hat{R}^{(a)}{}_{(0)pq}=&\ \hat{R}^{(a)}{}_{(c)pq}\hat{\Gamma}_0{}^{(c)}{}_{(0)}+(\hat{D}_p+\partial_p\ln N)X^{k(a)(0)}\bar{\mathscr{g}}_{kq}-(\hat{D}_q+\partial_q\ln N)X^{k(a)(0)}\bar{\mathscr{g}}_{kp}\\
&-X^{k(a)(c)}\bar{\mathscr{g}}_{kp}\hat{\Gamma}_q{}^{(0)}{}_{(c)}+X^{k(a)(c)}\bar{\mathscr{g}}_{kq}\hat{\Gamma}_p{}^{(0)}{}_{(c)} ,\\
\hat{D}_0\hat{R}^{(a)}{}_{(b)pq}=&-\hat{\Gamma}_0{}^{(a)}{}_{(0)}\hat{R}^{(0)}{}_{(b)pq}+\hat{\Gamma}_0{}^{(0)}{}_{(b)}\hat{R}^{(a)}{}_{(0)pq}-(\hat{D}_p+\partial_p\ln N)X^{k(a)}{}_{(b)}\bar{\mathscr{g}}_{kq}\\
&+(\hat{D}_q+\partial_q\ln N)X^{k(a)}{}_{(b)}\bar{\mathscr{g}}_{kp}+X^{k(a)(0)}(\bar{\mathscr{g}}_{kp}\hat{\Gamma}_q{}^{(0)}{}_{(b)}-\bar{\mathscr{g}}_{kq}\hat{\Gamma}_p{}^{(0)}{}_{(b)})+X^k{}_{(b)}{}^{(0)}(-\bar{\mathscr{g}}_{kp}\hat{\Gamma}_q{}^{(a)}{}_{(0)}+\bar{\mathscr{g}}_{kq}\hat{\Gamma}_p{}^{(a)}{}_{(0)})
\end{aligned} \tag{A15}$$

The $\hat{D}_0$ and $\hat{D}_k$ derivatives are natural covariant operators for SO(3)-object-valued spatial tensor fields on the surfaces of a slicing. In the Hamiltonian formulations of SO(3,1) gauge theories of gravity, however, most of 3-geometric quantities are generated by corresponding SO(3,1)-object-valued spacetime tensor fields. Let $f^{(\alpha)}{}_{(\beta)}$ be a field of SO(3,1) tensors [i.e., a function with values in SO(3,1) tensors]; then $\hat{f}^{(\alpha)}{}_{(\beta)}$ is a family of functions with values in SO(3) objects. We define the dagger derivatives

$$^{\dagger}\hat{D}_0\hat{f}^{(\alpha)}{}_{(\beta)}=\bar{\partial}_0\hat{f}^{(\alpha)}{}_{(\beta)}+\hat{\Gamma}_0{}^{(\alpha)}{}_{(\tau)}\hat{f}^{(\tau)}{}_{(\beta)}-\hat{\Gamma}_0{}^{(\sigma)}{}_{(\beta)}\hat{f}^{(\alpha)}{}_{(\sigma)} , \tag{A16}$$

$$^{\dagger}\hat{D}_k\hat{f}^{(\alpha)}{}_{(\beta)}=\bar{\partial}_k\hat{f}^{(\alpha)}{}_{(\beta)}+\hat{\Gamma}_k{}^{(\alpha)}{}_{(\tau)}\hat{f}^{(\tau)}{}_{(\beta)}-\hat{\Gamma}_k{}^{(\sigma)}{}_{(\beta)}\hat{f}^{(\alpha)}{}_{(\sigma)} .$$

We observe that

$$\begin{aligned}
^{\dagger}\hat{D}_\lambda\hat{f}^{(\alpha)}{}_{(\beta)}=&\ \hat{D}_\lambda\hat{f}^{(\alpha)}{}_{(\beta)}+\delta^{(\alpha)}{}_{(0)}\hat{\Gamma}_\lambda{}^{(0)}{}_{(c)}\hat{f}^{(c)}{}_{(\beta)}\\
&+\delta^{(\alpha)}{}_{(a)}\hat{\Gamma}_\lambda{}^{(a)}{}_{(0)}\hat{f}^{(0)}{}_{(\beta)}-\delta^{(0)}{}_{(\beta)}\hat{\Gamma}_\lambda{}^{(d)}{}_{(0)}\hat{f}^{(\alpha)}{}_{(d)}\\
&-\delta^{(b)}{}_{(\beta)}\hat{\Gamma}_\lambda{}^{(0)}{}_{(b)}\hat{f}^{(\alpha)}{}_{(0)} .
\end{aligned} \tag{A17}$$

For SO(3)-tensor quantities with holonomic and anholonomic indices the dagger derivatives are defined by means of relations (A17). That is,

$$^{\dagger}\hat{D}_\lambda\hat{f}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{\nu}=\hat{D}_\lambda\hat{f}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{\nu} + \text{the correction terms in (A17)} . \tag{A18}$$

Therefore the objects with purely holonomic indices the dagger derivatives $^{\dagger}\hat{D}_\lambda$ coincide with the $\hat{D}_\lambda$ derivatives, e.g.,

$$^{\dagger}\hat{D}_0\bar{g}_{ij}=\hat{D}_0\bar{g}_{ij}, \quad {}^{\dagger}\hat{D}_k\bar{\mathscr{g}}_{ij}=\hat{D}_k\bar{\mathscr{g}}_{ij} . \tag{A19}$$

We would like to emphasize that the dagger derivatives may be applied only to such SO(3) objects that are generated by some SO(3,1) objects. That is to say we are not able to compute $^{\dagger}\hat{D}_\lambda\hat{f}^{(a)}$ if $\hat{f}^{(0)}$ is unknown.

For the connection coefficients we define

$$\begin{aligned}
&^{\dagger}\hat{D}_0\hat{\Gamma}_s{}^{(\alpha)}{}_{(\beta)}=\hat{R}^{(\alpha)}{}_{(\beta)0s} ,\\
&^{\dagger}\hat{D}_k\hat{\Gamma}_s{}^{(\alpha)}{}_{(\beta)}=\hat{R}^{(\alpha)}{}_{(\beta)ks} .
\end{aligned} \tag{A20}$$

Let $(\hat{F}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{\nu})$ be the system of SO(3) quantities on slices generated by an SO(3,1)-tensor-valued spacetime tensor density of weight r. Then we have the following commutation relations for the dagger derivatives:

$$\begin{aligned}
[{}^{\dagger}\hat{D}_i,{}^{\dagger}\hat{D}_j]\hat{F}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{\nu}=&\ {}^{\dagger}\hat{D}_i\hat{\Gamma}_j{}^{(\alpha)}{}_{(\tau)}\hat{F}^{(\tau)}{}_{(\beta)}{}^{\mu}{}_{\nu}\\
&-{}^{\dagger}\hat{D}_i\hat{\Gamma}_j{}^{(\sigma)}{}_{(\beta)}\hat{F}^{(\alpha)}{}_{(\sigma)}{}^{\mu}{}_{\nu}\\
&+\delta^{\mu}{}_{m}\,{}^{R}\bar{R}^{m}{}_{pij}\hat{F}^{(\alpha)}{}_{(\beta)}{}^{p}{}_{\nu}\\
&-\delta^{n}{}_{\nu}\,{}^{R}\bar{R}^{q}{}_{nij}\hat{F}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{q} ,
\end{aligned} \tag{A21}$$

where $^{R}\bar{R}^{m}{}_{pij}$ is the curvature tensor of the Riemannian connection $\bar{\gamma}_k{}^p{}_q$ on σ;

$$\begin{aligned}
[{}^{\dagger}\hat{D}_0,{}^{\dagger}\hat{D}_j]\hat{F}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{\nu}=&\ \partial_j\ln N\,{}^{\dagger}\hat{D}_0\hat{F}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{\nu}+{}^{\dagger}\hat{D}_0\hat{\Gamma}_j{}^{(\alpha)}{}_{(\tau)}\hat{F}^{(\tau)}{}_{(\beta)}{}^{\mu}{}_{\nu}-{}^{\dagger}\hat{D}_0\hat{\Gamma}_j{}^{(\sigma)}{}_{(\beta)}\hat{F}^{(\alpha)}{}_{(\sigma)}{}^{\mu}{}_{\nu}\\
&-r\lambda_j{}^p{}_p\hat{F}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{\nu}+\delta^{\mu}{}_{m}\lambda_j{}^m{}_p\hat{F}^{(\alpha)}{}_{(\beta)}{}^{p}{}_{\nu}-\delta^{n}{}_{\nu}\lambda_j{}^q{}_n\hat{F}^{(\alpha)}{}_{(\beta)}{}^{\mu}{}_{q} ,
\end{aligned} \tag{A22}$$

where $\lambda_j{}^q{}_p$ are the Christoffel-type tensors:

$$\begin{aligned}
\lambda_j{}^q{}_p=\tfrac{1}{2}\bar{g}^{qu}[&(\hat{D}_j+\partial_j\ln N)\hat{D}_0\bar{g}_{up}+(\hat{D}_p+\partial_p\ln N)\hat{D}_0\bar{g}_{uj}\\
&-(\hat{D}_u+\partial_u\ln N)\hat{D}_0\bar{g}_{jp}] .
\end{aligned} \tag{A23}$$

A similar commutation relation can be obtained for $\hat{D}_\lambda$ operators [cf. (A13) and (A14)].

The Bianchi identities in dagger derivatives

We have

$$\begin{aligned}
&^{\dagger}\hat{D}_{[k}\hat{R}^{(\alpha)(\beta)}{}_{pq]}=0 ,\\
&^{\dagger}\hat{D}_0\hat{R}^{(\alpha)(\beta)}{}_{pq}+({}^{\dagger}\hat{D}_p+\partial_p\ln N)\hat{R}^{(\alpha)(\beta)}{}_{q0}\\
&\qquad+({}^{\dagger}\hat{D}_q+\partial_q\ln N)\hat{R}^{(\alpha)(\beta)}{}_{0p}=0 .
\end{aligned} \tag{A24}$$

SO(3,1)-covariant Lie derivatives were described profoundly in Refs. 64 and 12. Their (3 + 1) decompositions define SO(3)-covariant Lie derivatives on three-dimensional slices as well as SO(3)-covariant time derivatives $\hat{D}_0$. If in corresponding formulas for SO(3)-covariant Lie derivatives we replace the $\hat{D}_k$ operators with ${}^{\dagger}\hat{D}_k$ operators then we get dagger Lie derivatives on slices. In particular, if $\hat{F}^{(A)B}$ is a spatial tensor density of weight r with values in SO(3) tensors then for a vector field u^k on σ we have

$$ {}^{\dagger 3}_{\hat{\Gamma}}L_u \hat{F}^{(A)B} = u^{k\,\dagger}\hat{D}_k \hat{F}^{(A)B} + r\hat{D}_k u^k \hat{F}^{(A)B} - \hat{D}_p u^q \bar{h}_q{}^{pB}{}_J \hat{F}^{(A)J} \quad [\text{cf. (A10a)}] . $$

In our consideration we also use the dagger Lie derivative ${}^{\dagger 3}_{\zeta}L$ with respect to an auxiliary connection $\zeta_\lambda{}^{(\alpha)(\beta)}$.

Yang-Mills covariant derivatives

For the strength of a Yang-Mills field the covariant derivative is given by the formula

$$ D_\lambda f^I{}_{\mu\nu} = \partial_\lambda f^I{}_{\mu\nu} - \gamma_\lambda{}^\tau{}_\mu f^I{}_{\tau\nu} - \gamma_\lambda{}^\tau{}_\nu f^I{}_{\mu\tau} + c^I{}_{UV} A^U{}_\lambda f^V{}_{\mu\nu} . \tag{A25} $$

For the corresponding density we have

$$ D_\lambda F^{I\lambda\nu} = \partial_\lambda F^{I\lambda\nu} + c^I{}_{UV} A^U{}_\lambda F^{V\lambda\nu} . \tag{A25'} $$

For the Yang-Mills connection we have

$$ f^I{}_{\mu\nu} = D_\mu A^I{}_\nu \doteq \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu + c^I{}_{UV} A^U{}_\mu A^V{}_\nu . \tag{A26} $$

In the (3 + 1) decomposition we have

$$ \begin{aligned} \hat{D}_0 \hat{f}^I{}_{0n} &= \bar{\partial}_0 \hat{f}^I{}_{0n} - \sigma^q{}_n \hat{f}^I{}_{0q} + c^I{}_{UV} \hat{A}^U{}_0 \hat{f}^V{}_{0n} , \\ \hat{D}_0 \hat{f}^I{}_{mn} &= \bar{\partial}_0 \hat{f}^I{}_{mn} - \sigma^p{}_m \hat{f}^I{}_{pn} - \sigma^q{}_n \hat{f}^I{}_{mq} \\ &\quad + c^I{}_{UV} \hat{A}^U{}_0 \hat{f}^V{}_{mn} , \\ \hat{D}_r \hat{f}^I{}_{0n} &= \bar{\partial}_r \hat{f}^I{}_{0n} - \bar{\gamma}_r{}^p{}_n \hat{f}^I{}_{0p} + c^I{}_{UV} \hat{A}^U{}_0 \hat{f}^V{}_{0n} , \\ \hat{D}_r \hat{f}^I{}_{mn} &= \bar{\partial}_r \hat{f}^I{}_{mn} - \bar{\gamma}_r{}^p{}_m \hat{f}^I{}_{pn} - \bar{\gamma}_r{}^q{}_n \hat{f}^I{}_{mq} \\ &\quad + c^I{}_{UV} \hat{A}^U{}_r \hat{f}^V{}_{mn} , \end{aligned} \tag{A27} $$

and some formulas for densities

$$ \begin{aligned} \hat{D}_0 \hat{F}^{I0n} &= \bar{\partial}_0 \hat{F}^{I0n} - \sigma^p{}_p \hat{F}^{I0n} + \sigma^n{}_q \hat{F}^{I0q} + c^I{}_{UV} \hat{A}^U{}_0 \hat{F}^{V0n} , \\ \hat{D}_m \hat{F}^{Imn} &= \bar{\partial}_m \hat{F}^{Imn} + c^I{}_{UV} \hat{A}^U{}_m \hat{F}^{Vmn} , \\ \hat{D}_m \hat{F}^{Im0} &= \bar{\partial}_m \hat{F}^{Im0} + c^I{}_{UV} \hat{A}^U{}_m \hat{F}^{Vm0} . \end{aligned} \tag{A28} $$

The 3-covariant derivatives for the YM connection coefficients are given by the formulas

$$ \begin{aligned} \hat{D}_0 \hat{A}^I{}_m &= \hat{f}^I{}_{0m} = \bar{\partial}_0 \hat{A}^I{}_m - \bar{\partial}_m \hat{A}^I{}_0 + c^I{}_{UV} \hat{A}^U{}_0 \hat{A}^V{}_m \\ &\quad - \partial_m \ln N \hat{A}^I{}_0 - \sigma^r{}_m \hat{A}^I{}_r , \\ \hat{D}_k \hat{A}^I{}_m &= \hat{f}^I{}_{km} = \bar{\partial}_k \hat{A}^I{}_m - \bar{\partial}_m \hat{A}^I{}_k + c^I{}_{UV} \hat{A}^U{}_k \hat{A}^V{}_m . \end{aligned} \tag{A29} $$

We have the commutation formula

$$ \hat{D}_0 \hat{f}^I{}_{mn} = (\hat{D}_m + \partial_m \ln N) \hat{f}^I{}_{0n} - (\hat{D}_n + \partial \ln N) \hat{f}^I{}_{0m} . \tag{A30} $$

Let us observe that for the Yang-Mills fields the caret operation coincides with the bar operation. The covariant operators $\hat{D}_\lambda$ for the Yang-Mills fields correspond to the dagger covariant derivatives ${}^{\dagger}\hat{D}_\lambda$ in gravity.

Canonical variables for a scalar field

For a matter Lagrangian $L_{\text{mat}} = L_{\text{mat}}(e^{(\alpha)}{}_\lambda, f^{(\alpha)}, D_\lambda f^{(\alpha)})$ the matter momenta and current are

$$ \begin{aligned} p^\lambda{}_{(\alpha)} &= \partial L_{\text{mat}} / \partial (D_\lambda f^{(\alpha)}) , \\ J_{(\alpha)} &= \partial L_{\text{mat}} / \partial f^{(\alpha)} . \end{aligned} \tag{A31} $$

The EL equations read

$$ -D_\lambda p^\lambda{}_{(\alpha)} + J_{(\alpha)} = 0 . \tag{A32} $$

For the Lagrangian (6.14) we have, in the (3 + 1) decomposition,

$$ \begin{aligned} \hat{p}^0{}_0 &= 2\hat{e}\,{}^{\dagger}\hat{D}_0 \hat{f}^{(0)} + \hat{e}\,\hat{e}^p{}_{(c)}\,{}^{\dagger}\hat{D}_p \hat{f}^{(c)} , \\ \hat{p}^0{}_{(a)} &= -\hat{e}\,{}^{\dagger}\hat{D}_0 \hat{f}_{(a)} , \end{aligned} \tag{A33} $$

$$ \begin{aligned} \hat{p}^k{}_{(0)} &= \hat{e}\bar{g}^{ks\,\dagger}\hat{D}_s \hat{f}_{(0)} , \\ \hat{p}^k{}_{(a)} &= \hat{e}\bar{g}^{ks\,\dagger}\hat{D}_s \hat{f}_{(a)} + \hat{e}\,\hat{e}^k{}_{(a)} ({}^{\dagger}\hat{D}_0 \hat{f}^{(0)} + \hat{e}^s{}_{(b)}\,{}^{\dagger}\hat{D}_s \hat{f}^{(b)}) . \end{aligned} \tag{A34} $$

For the right-hand side of (6.17) we have

$$ \begin{aligned} \hat{e}(\hat{T}^{ks} + \bar{g}^{ks} \hat{T}^0{}_{(0)}) &= \bar{g}^{ks} [-\hat{p}^0{}_{(c)} \hat{p}^{0(c)} + \hat{e}^2 \bar{g}^{rz\,\dagger}\hat{D}_r \hat{f}_{(c)}\,{}^{\dagger}\hat{D}_z \hat{f}^{(c)} + \tfrac{1}{2}\hat{e}^2 (\hat{e}^p{}_{(a)}\,{}^{\dagger}\hat{D}_p \hat{f}^{(a)})^2 \\ &\quad + \tfrac{1}{2}\hat{e}\,\hat{p}^0{}_{(0)} (\hat{e}^p{}_{(a)}\,{}^{\dagger}\hat{D}_p \hat{f}^{(a)}) + m^2 \hat{e}^2 \hat{f}^{(\alpha)} \hat{f}_{(\alpha)}] \\ &\quad - \hat{e}^2 \bar{g}^{kp} \bar{g}^{sq} ({}^{\dagger}\hat{D}_p \hat{f}_{(\alpha)}\,{}^{\dagger}\hat{D}_q \hat{f}^{(\alpha)}) + \tfrac{1}{2}\hat{e}\,\hat{e}^k{}_{(c)} (\hat{p}^0{}_{(0)} + \hat{e}\,\hat{e}^p{}_{(u)}\,{}^{\dagger}\hat{D}_p \hat{f}^{(u)}) \bar{g}^{sr\,\dagger}\hat{D}_r \hat{f}^{(c)} . \end{aligned} \tag{A35} $$

By virtue of (A35) the skew-symmetric part of (6.17) reads

$$ \bar{g}^{sr\,\dagger}\hat{D}_r \hat{f}^{(c)} \hat{e}^k{}_{(c)} - \bar{g}^{kr\,\dagger}\hat{D}_r \hat{f}^{(c)} \hat{e}^s{}_{(c)} = 0 . \tag{A36} $$

(This is the symmetry condition $T^{ks} = T^{sk}$.)

Now we make use of the following lemma that generalizes the polar decomposition of a given nonsingular matrix in its symmetric and unitary parts.[82]

Lemma. For a given metric $\bar{g}_{ij}$ on σ and a nonsingular tensor $Z_{r(c)}$ there exists a nonsingular matrix $X^{(c)}{}_k$ such that (i) $X^{(a)}{}_i X^{(b)}{}_j \delta_{(a)(b)} = \bar{g}_{ij}$ and (ii) $Z_{r(c)} X^{(c)}{}_k$ is a symmetric matrix.

The matrix elements of X are algebraic functions of $Z_{r(c)}$ and $\bar{g}_{ij}$. If we set $Z_{r(c)} = \hat{D}_r \hat{f}_{(c)}$ and $X^{(c)}{}_r = \hat{e}^{(c)}{}_r$ then we determine the triad components in terms of the components of the three-metric and symplectic matter field variables. Moreover, Eqs. (A36) are satisfied. Now six equations that constitute the symmetric part of (6.17) are algebraic relations for $\bar{g}^{ij}$. These are highly nonlinear equations but they, in principle, enable us to determine the three-metric in terms of the gravitational and matter symplectic variables. The time-conservation conditions for (6.20) read

$$(\hat{e}^{\dagger}\hat{D}_k\hat{f}^{(c)}+\hat{e}^{(c)}{}_k\hat{p}^{0}{}_{(0)})(\hat{\Gamma}_0{}^{(0)}{}_{(c)}-\hat{e}^{s}{}_{(c)}\partial_s\ln N)=\text{expression of symplectic variables and their spatial derivatives} . \quad \text{(A37)}$$

If the 3×3 matrix $(\hat{e}^{\dagger}\hat{D}_k\hat{f}^{(c)}+\hat{e}^{(c)}{}_k\hat{p}^{0}{}_{(0)})$ is invertible then we can determine $\hat{\Gamma}_0{}^{(0)}{}_{(c)}$ in terms of symplectic variables and the quantities $\partial_s\ln N$.

APPENDIX B

If (x^0,x^k) are local coordinates consistent with a chosen slicing of spacetime and the surfaces of the slicing are defined by relations $x^0=\text{const}$ then the components of the unit vector field orthonormal to the slicing are

$$\mathcal{N}=(1/N,-N^k/N) .$$

If $e_{(\alpha)}=e^{\mu}{}_{(\alpha)}\partial/\partial x^{\mu}$ then the angle coefficients $n_{(\alpha)}=e_{(\alpha)}\mathcal{N}$ satisfy relations (5.15).

The gravitational momenta $M_{(a)}$ are defined by the formula

$$M_{(a)}=2(-(\hat{E}2)^{0}{}_{(c)(0)}\{\delta^{(c)}{}_{(a)}-n^{(c)}n_{(a)}[n^{(0)}(1+n^{(0)})]^{-1}\}-n^{(c)}(1+n^{(0)})^{-1}(\hat{E}2)^{0}{}_{(c)(a)}) . \quad \text{(B1)}$$

It is remarkable that for the SKY theory $M_{(a)}$ are functions of $X^k{}_{(\alpha)(\beta)}$, $\hat{\Gamma}_k{}^{(\alpha)(\beta)}$, their spatial derivatives and $n^{(a)}$. For a general gravitational Lagrangian this is not true.

If B_{abcd} is a tensor on a three-dimensional manifold satisfying

$$B_{abcd}=B_{cdab} ,$$

$$B_{abcd}=-B_{bacd}=B_{badc}$$

then

$$\tfrac{1}{4}B_{abcd}\epsilon^{abu}\epsilon^{cdv}=\bar{g}(\tfrac{1}{2}\bar{g}^{uv}B_{abcd}\bar{g}^{ac}\bar{g}^{bd}-B_{abcd}\bar{g}^{bd}\bar{g}^{au}\bar{g}^{cv}) . \quad \text{(B2)}$$

In our case $B_{abcd}=\hat{R}_{(\alpha)(\beta)ab}\hat{R}^{(\alpha)(\beta)}{}_{cd}$ [cf. (3.5)].

APPENDIX C

We discuss the space of symmetric tensors on a three-dimensional manifold. We have the following component representation:

$$g=\bar{g}_{ab}dx^a\otimes dx^b . \quad \text{(C1)}$$

We take the following basis $(E^{\{ij\}})_{i\le j}$ in the space of symmetric tensors:

$$dx^i\otimes dx^j \quad \text{for } i=j ,$$
$$(dx^i\otimes dx^j+dx^j\otimes dx^i) \quad \text{for } i<j . \quad \text{(C2)}$$

Indices in the six-dimensional space spanned by the basis (C2) are pairs of numbers $\{ij\}$ with the natural order

$$\{11\},\ \{22\},\ \{33\},\ \{23\},\ \{13\},\ \{12\} .$$

If we transform local coordinates on the manifold $\sigma, x^k\to x^{k'}=x^{k'}(x^s)$ then

$$\bar{g}_{ab}\to\bar{g}_{a'b'}=X^{-1c}{}_{a'}X^{-1d}{}_{b'}\bar{g}_{cd} \quad \text{where } X^{a'}{}_c=\partial x^{a'}/\partial x^c ,$$
$$dx^a\otimes dx^b\to dx^{a'}\otimes dx^{b'}=X^{a'}{}_iX^{b'}{}_jdx^i\otimes dx^j ; \quad \text{(C3)}$$

or in matrix form

$$E^{\{a'b'\}}=(X\otimes X)^{\{a'b'\}}{}_{\{ij\}}E^{\{ij\}} . \quad \text{(C4)}$$

In the explicit form

$$(X\otimes X)^{\{a'b'\}}{}_{\{ij\}}=\begin{cases} X^{a'}{}_iX^{b'}{}_j & \text{for } a'=b' \text{ and } i\le j ,\\ 2X^{a'}{}_iX^{b'}{}_j & \text{for } a'<b' \text{ and } i=j ,\\ X^{a'}{}_iX^{b'}{}_j+X^{a'}{}_jX^{b'}{}_i & \text{for } a'<b' \text{ and } i<j . \end{cases} \quad \text{(C5)}$$

It can be proved that

$$\det[X\otimes X]_{6\times 6}=(\det[X]_{3\times 3})^4 \quad \text{(C6)}$$

(see Ref. 82 , pp. 257–259). Let A be an operator from the space of two-covariant tensor densities of weight -1 to the space of two-contravariant tensor densities of weight $+2$:

$$A(\mathscr{g})=h ,$$

where

$$\begin{aligned}&\mathscr{g}=\bar{g}_{ab}(dx^a\otimes dx^b)\otimes \wedge\, ^3\partial/\partial x\ ,\\&h=\bar{h}^{ab}(\partial/\partial x^a\otimes\partial/\partial x^b)\otimes\wedge\, ^d dx\otimes\wedge\, ^3dx\ ,\\&A\,(dx^i\otimes dx^j\otimes\wedge\, ^3\partial/\partial x)=A^{abij}(\partial/\partial x^a\otimes\partial/\partial x^b)\otimes\wedge\, ^3dx\otimes\wedge\, ^3dx\ ,\\&\bar{h}^{ab}=A^{abij}\bar{g}_{ij}\ ,\end{aligned}\tag{C7}$$

where

$$\wedge\, ^3dx=dx^1\wedge dx^2\wedge dx^3,\ \ \wedge\, ^3\partial/\partial x=\partial/\partial x^1\wedge\partial/\partial x^2\wedge\partial/\partial x^3\ .$$

In the bases $E^{\{ij\}}$ and $F_{\{ab\}}$ we write

$$A\,(E^{\{ij\}})=\sum_{a\le b}M^{\{ab\}\{ij\}}F_{\{ab\}}\ .\tag{C8}$$

Here

$$\begin{aligned}&E^{\{ij\}}=\begin{cases}(dx^i\otimes dx^j)\otimes\wedge\, ^3\partial/\partial x & \text{for } i=j\ ,\\(dx^i\otimes dx^j+dx^j\otimes dx^i)\otimes\wedge\, ^3\partial/\partial x & \text{for } i<j\ ,\end{cases}\\&F^{\{ab\}}\begin{cases}(\partial/\partial x^a\otimes\partial/\partial x^b)\otimes\wedge\, ^3dx\otimes\wedge\, ^3dx & \text{for } a=b\ ,\\(\partial/\partial x^a\otimes\partial/\partial x^b+\partial/\partial x^b\otimes\partial/\partial x^a)\otimes\wedge\, ^3dx\otimes\wedge\, ^3dx & \text{for } a<b\ .\end{cases}\end{aligned}\tag{C9}$$

In these bases the elements of the matrix $[M^{\{ab\}\{ij\}}]$ read

$$M^{\{ab\}\{ij\}}=\begin{cases}A^{abij} & \text{for } a\le b \text{ and } i=j\ ,\\2A^{abij} & \text{for } a\le b \text{ and } i<j\ .\end{cases}\tag{C10}$$

Lemma 1. If $A^{abcd}=-A^{cdab}$ then

$$\det[M^{\{ab\}\{cd\}}]_{6\times 6}=[1/(8\times 36)][\epsilon_{\{a_1b_1\}\{a_2b_2\}\{a_3b_3\}\{a_4b_4\}\{a_5b_5\}\{a_6b_6\}}A^{\{a_1b_1\}\{a_2b_2\}}A^{\{a_3b_3\}\{a_4b_4\}}A^{\{a_5b_5\}\{a_6b_6\}}]^2\ .\tag{C11}$$

Here $\epsilon_{\{\ \}\{\ \}\{\ \}\{\ \}\{\ \}\{\ \}}$ is the Levi-Civita skew-symmetric symbol in the six-dimensional space and the summation in (C11) is over all indices $\{\ \}$ in the six-dimensional space.

If the transformation (C3) is performed then

$$\begin{aligned}A^{\{ab\}\{cd\}}&\to(\det X)^{-3}X^{a'}{}_iX^{b'}{}_jX^{c'}{}_mX^{d'}{}_n A^{ijmn}\\&=(\det X)^{-3}\sum_{i\le j,m\le n}(X\otimes X)^{\{a'b'\}}{}_{\{ij\}}(X\otimes X)^{\{c'd'\}}{}_{\{mn\}}A^{\{ij\}\{mn\}}\ .\end{aligned}$$

Therefore

$$\det[M^{\{a'b'\}\{c'd'\}}]_{6\times 6}=(\det X)^{-10}\det[M^{\{ab\}\{cd\}}]\ .$$

The determinant is a scalar density of weight 10.

If A^{kspq} is a tensor density of weight 3 having the symmetry properties

$$A^{kspq}=A^{skpq}=A^{ksqp}\ ,$$

$$A^{kspq}=-A^{pqks}$$

then

$$A^{kspq}=\epsilon^{kzq}W_z{}^{ps}+\epsilon^{szq}W_z{}^{pk}+\epsilon^{kzp}W_z{}^{qs}+\epsilon^{szp}W_z{}^{qk}\ ,\tag{C12}$$

where $W_z{}^{ps}$ is a tensor density of weight 2 having the symmetry properties

$$W_z{}^{zs}=0,\quad W_z{}^{ps}=W_z{}^{sp}\ .\tag{C13}$$

It is clear that both the A^{kspq} and $W_z{}^{ps}$ have 15 independent components. The inverse formula reads

$$W_j{}^{qs}=-\tfrac{1}{4}A^{kspq}\epsilon_{kpj}\ .\tag{C12a}$$

Lemma 2.

$$\begin{aligned}\tfrac{1}{48}\epsilon_{\{a_1b_1\}\{a_2b_2\}\{a_3b_3\}\{a_4b_4\}\{a_5b_5\}\{a_6b_6\}}&A^{\{a_1b_1\}\{a_2b_2\}}A^{\{a_3b_3\}\{a_4b_4\}}A^{\{a_5b_5\}\{a_6b_6\}}\\&=\tfrac{8}{3}\epsilon_{a_1a_2a_3}W_{c_3}{}^{c_1a_1}W_{c_1}{}^{c_2a_2}W_{c_2}{}^{c_3a_3}\\&=\tfrac{1}{24}\epsilon_{a_1a_2a_3}\epsilon_{b_1d_1c_2}\epsilon_{b_2d_2c_3}\epsilon_{b_3d_3c_1}A^{a_1b_1c_1d_1}A^{a_2b_2c_2d_2}A^{a_3b_3c_3d_3}\ .\end{aligned}\tag{C14}$$

APPENDIX D

We prove that the time-maintanence condition of nth order for Eq. (3.19),

$$\hat{D}_0{}^{(n)}(A^{kspq}\bar{\mathscr{g}}_{pq})=0\ , \tag{D1}$$

gives rise to a consistent system of linear equations for $\hat{D}_0{}^{(n)}\bar{\mathscr{g}}_{pq}$. In Sec. III we proved this fact for $n=1$ and 2. Now we use the inductive method. Equation (D1) may be rewritten as

$$A^{kspq}\hat{D}_0{}^{(n)}\bar{\mathscr{g}}_{pq}=-\hat{D}_0{}^{(n)}A^{kspq}\bar{\mathscr{g}}_{pq}-\sum_{i=1}^{n-1}\begin{bmatrix}n\\ i\end{bmatrix}\hat{D}_0{}^{(i)}A^{kspq}\hat{D}_0{}^{(n-i)}\bar{\mathscr{g}}_{pq} \tag{D1$'$}$$

or in the operator form

$$A\hat{D}_0{}^{(n)}\mathscr{g}=b\ . \tag{D2}$$

The matrix A is singular; therefore the Eq. (D2) has solutions only for b orthogonal to the kernel of the adjoint operator A^*. But $A^*=-A$ and the kernel of A^* coincides with the one-dimensional space spanned by $\bar{\mathscr{g}}_{ks}$. Hence the consistency condition for (D1) reads

$$-\bar{\mathscr{g}}_{ks}\left[\hat{D}_0{}^{(n)}A^{kspq}\bar{\mathscr{g}}_{pq}+\sum_{i=1}^{n-1}\begin{bmatrix}n\\ i\end{bmatrix}\hat{D}_0{}^{(i)}A^{kspq}\hat{D}_0{}^{(n-i)}\bar{\mathscr{g}}_{pq}\right]=0\ . \tag{D3}$$

Skew symmetry of A^{kspq} [cf. (3.23)] annihilates the first term in (D3). We have

$$\begin{aligned}-\bar{\mathscr{g}}_{ks}\sum_{i=1}^{n}\begin{bmatrix}n\\ i\end{bmatrix}\hat{D}_0{}^{(i)}A^{kspq}\hat{D}_0{}^{(n-i)}\bar{\mathscr{g}}_{pq}=&-\sum_{i=1}^{n-1}\begin{bmatrix}n\\ i\end{bmatrix}\hat{D}_0{}^{(i)}(A^{kspq}\bar{\mathscr{g}}_{ks})\hat{D}_0{}^{(n-i)}\bar{\mathscr{g}}_{pq}\\ &+\sum_{i=1}^{n-1}\begin{bmatrix}n\\ i\end{bmatrix}\sum_{j=0}^{i-1}\begin{bmatrix}i\\ j\end{bmatrix}\hat{D}_0{}^{(j)}A^{kspq}\hat{D}_0{}^{(i-j)}\bar{\mathscr{g}}_{ks}\hat{D}_0{}^{(n-i)}\bar{\mathscr{g}}_{pq}\ .\end{aligned} \tag{D4}$$

From the inductive assumption

$$\hat{D}_0{}^{(i)}(A^{kspq}\bar{\mathscr{g}}_{ks})=0,\quad i=1,\ldots,n-1\ . \tag{D5}$$

Hence the left-hand side of (D4) equals

$$\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}n![j!(n-i)!(i-j)!]^{-1}\hat{D}_0{}^{(j)}A^{kspq}\hat{D}_0{}^{(i-j)}\bar{\mathscr{g}}_{ks}\hat{D}_0{}^{(n-i)}\bar{\mathscr{g}}_{pq}\ . \tag{D6}$$

If we set $r=n-i$, $r=1,\ldots,n-j-1$ then we get

$$\sum_{j=0}^{n-2}\sum_{r=1}^{n-j-1}n![j!r!(n-j-r)!]^{-1}\hat{D}_0{}^{(j)}A^{kspq}\hat{D}_0{}^{(n-j-r)}\bar{\mathscr{g}}_{ks}\hat{D}_0{}^{(r)}\bar{\mathscr{g}}_{pq}\ . \tag{D6$'$}$$

This expression is invariant with respect to the transformation $r\to n-j-r$ and simultaneously it is skew symmetric with respect to the transformation $k,s\to p,q$. Therefore it vanishes. The consistency condition (D3) is proven.

APPENDIX E

The time maintenance of constraints (3.4) and (3.8) are given below. If the dynamical equations (3.9) and (3.10) are satisfied then

$$\begin{aligned}{}^{\dagger}\hat{D}_0(\epsilon_{pij}X^i{}_{(\alpha)(\beta)}Y^{j(\alpha)(\beta)})=&-(\tfrac{1}{2}{}^{\dagger}\hat{D}_p+\partial_p\ln N)(X^i{}_{(\alpha)(\beta)}X^{j(\alpha)(\beta)}+Y^i{}_{(\alpha)(\beta)}Y^{j(\alpha)(\beta)})\bar{\mathscr{g}}_{ij}\\ &+({}^{\dagger}\hat{D}_j+\partial_j\ln N)(X^{z(\alpha)(\beta)}X^j{}_{(\alpha)(\beta)}+Y^z{}_{(\alpha)(\beta)}Y^{j(\alpha)(\beta)})\bar{\mathscr{g}}_{zp}-{}^{\dagger}\hat{D}_jX^j{}_{(\alpha)(\beta)}X^{z(\alpha)(\beta)}\bar{\mathscr{g}}_{zp}\ ,\end{aligned} \tag{E1}$$

$$^{\dagger}\hat{D}_0({}^{\dagger}\hat{D}_kX^k{}_{(\alpha)(\beta)})=0\ . \tag{E2}$$

The time derivatives of the left-hand sides of constraints (3.4) and (3.8) are linear combinations of the left-hand sides of constraints (3.4), (3.5), and (3.8) as well as spatial derivatives of these quantities.

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