Horizontal symmetry and the fourth generation

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A systematic investigation of all possible horizontal symmetries acting on four generations of quarks in a minimal left-right-symmetric model is carried out. There are only two consistent models with realistic constraints on the quark masses and mixing angles. It is shown that Z_4 is the unique symmetry group leading to these models. The fourth-generation quark masses $m_{b'}$ and $m_{t'}$ are constrained to be (A) $m_b/m_t \simeq m_{b'}/m_{t'}$, (B) $m_s/m_c \simeq (m_b+m_{b'})/(m_t+m_{t'})$. Thus, model (A) predicts $m_t, m_{b'} \leq 43$ GeV whereas model (B) has $m_{b'} \leq 58$ GeV. The two models differ in the mixing of the fourth generation into the first three. A crucial test which can distinguish the two models is a direct measurement of the mixing matrix element $|V_{ub}|$. Model (A) predicts it to be $\sim 10^{-4}$, an order of magnitude smaller than the prediction of model (B).

I. INTRODUCTION

The replication of fermion families is one of the least understood aspects of present-day particle physics. The discovery of the top quark will complete the third generation of quarks and leptons in the standard-model menu. However, there is no convincing argument that it will be the last entry. It is then natural to consider the possible existence of a fourth generation of quarks and leptons. This possibility, although not new, has been the subject of vigorous discussions lately.[1] The recent measurement of B-$\bar{B}$ mixing by the ARGUS Collaboration[2] will only strengthen the case for a fourth generation if the top quark is indeed discovered in the mass range 25–50 GeV (Ref. 3).

In this paper we shall take the possible existence of a fourth generation seriously. Within the framework of the standard model, this only proliferates the number of free parameters, since the model has all the fermion masses and mixing angles arbitrary. However, meaningful relations among the quark masses and mixing angles can be obtained by resorting to additional symmetries acting in the family space. These "horizontal symmetries" are essential if the relations are to be stable under radiative corrections. Although the number of free parameters can be reduced considerably in this approach, we still lack convincing arguments as to what the horizontal symmetry should be. Therefore, a general investigation of all possible horizontal symmetries will serve to be of great value. Such investigations have been carried out in the literature for the case of two and three generations.[4–6] Here we propose to extend them to the case of four generations.

An aesthetically pleasing and phenomenologically viable alternative to the standard model is left-right-symmetric gauge theories based on the gauge group $SU(2)_L \times SU(2)_R \times U(1)$ (Ref. 7). These theories have been shown to be very successful in obtaining natural relations between the quark masses and the mixing angles.[8] The invariance of the Lagrangian under space inversion naturally leads to Hermitian Yukawa coupling matrices, which is a considerable simplification. In this paper we shall confine ourselves to a left-right-symmetric model with a minimal Higgs sector and investigate the effect of all possible horizontal symmetries acting on the four generations of quarks. The analysis can in principle be extended to the leptonic sector as well, but lacking the experimental information on the leptonic mixing angles and the neutrino masses, we shall not pursue it here. Ecker, Grimus, and Konetschny[6] have carried out a general analysis of all horizontal symmetries within the framework of such a minimal left-right-symmetric model for the case of two and three generations of quarks. These authors show that in either case there is essentially one model which leads to phenomenologically acceptable predictions on the quark masses and mixing angles. Furthermore, the cyclic group Z_4 was shown to be the unique symmetry group that leads to these predictions.

The number of possible symmetry groups proliferates considerably while going from three to four generations. However, we have been able to show that the minimality of the Higgs sector when combined with the requirement that none of the generations decouple from each other implies that one can choose a basis in which the horizontal symmetry is essentially Abelian, thus simplifying our analysis. We show that there are only two models with realistic predictions on the quark masses and mixing angles. Remarkably, Z_4 is again the unique symmetry group which leads to these models. In both models, the masses of the top and the fourth-generation quarks (t', b') are constrained to be m_t, $m_{b'} \lesssim 50$ GeV, $m_{t'} \gtrsim 180$ GeV. The two models differ in the mixing of the fourth generation into the first three. In model (A) the fourth generation mixes preferentially with the second, whereas in model (B) it mixes with the third generation. Another test which could distinguish the two models is the mixing matrix element $|V_{ub}|$ —model (A) predicts it to be $\sim 10^{-4}$, an order of magnitude smaller than the prediction of model (B).

In the next section we describe the minimal left-right-symmetric model in some detail and begin investigating the action of possible horizontal symmetries on the four generations of quarks. There we arrive at the two realistic models (A) and (B). In Sec. III we analyze the special case of degenerate Yukawa coupling matrices and assert that they do not lead to realistic models. The phenomenology of models (A) and (B) is worked out in Sec. IV. In Sec. V we conclude. A proof that the horizontal symmetry can always be chosen to be Abelian in the minimal model is given in the Appendix.

II. HORIZONTAL SYMMETRY AND A MINIMAL LEFT-RIGHT MODEL

We shall assume the gauge group to be $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)$. The left-handed and the right-handed quarks ψ_L and ψ_R transform under the group as $(3,2,1,\frac{1}{6})$ and $(3,1,2,\frac{1}{6})$ multiplets, respectively. Fermion masses arise upon spontaneous symmetry breaking through their Yukawa coupling to a Higgs multiplet $\phi(1,2,2,0)$. Additional Higgs scalars which do not couple to the quarks are also introduced in order to break $SU(2)_R \times U(1)$ down to $U(1)_Y$ at an energy scale greater than a few TeV. The model is minimal in the sense that only one $\phi(1,2,2,0)$ field is introduced.

In addition to the Higgs field ϕ, the charge conjugate field $\tilde{\phi} = \tau_2 \phi^* \tau_2$ also couples invariantly to the fermions. The most general Yukawa coupling to the four generations of quarks is defined by

$$L_Y = \sum_{i,j=1}^{4} (\bar{\psi}_{Li} \phi \Gamma_{1ij} \psi_{Rj} + \bar{\psi}_{Li} \tilde{\phi} \Gamma_{2ij} \psi_{Rj}) + \text{H.c.} \qquad (2.1)$$

When the neutral components of the ϕ field acquire vacuum expectation values, which we parametrize as

$$\langle \phi \rangle = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} , \qquad (2.2)$$

this leads to the mass matrices

$$M_u = v\Gamma_1 + w^* \Gamma_2, \quad M_d = w\Gamma_1 + v^* \Gamma_2 , \qquad (2.3)$$

for the charge $\frac{2}{3}$ and $-\frac{1}{3}$ quarks, respectively.

By virtue of the $U(4) \times U(4)$ symmetry of the quark gauge couplings, the most general parity operation acting on the quark fields has the form[6]

$$\psi_L \to \psi_R, \quad \psi_R \to S\psi_L , \qquad (2.4)$$

where S is a 4×4 unitary matrix. Under parity, $\phi \to \phi' = \eta \phi^\dagger$ with $|\eta| = 1$. By redefining the field ϕ, we may set $\eta = 1$, in which case the invariance of (2.1) under parity implies

$$\Gamma_i^\dagger = \Gamma_i S \quad \text{or } [\Gamma_i, S] = 0, \quad i = 1,2 . \qquad (2.5)$$

In a basis where S is diagonal, Γ_1 and Γ_2 and therefore M_u and M_d will be block diagonal depending on the degeneracy of S. If none of the generations decouples from the rest, S should be completely degenerate, i.e.,

$$S = e^{i\sigma} \mathbb{1} . \qquad (2.6)$$

Absorbing the phase factor into the definition of Γ_i one can restrict oneself to Hermitian Γ_i.

Now consider an arbitrary horizontal symmetry group H (discrete or continuous) represented by unitary transformations on the quark fields as well as on the Higgs field ϕ:

$$\begin{aligned} \psi_L' &= K_L(g)\psi_L , \\ \psi_R' &= K_R(g)\psi_R , \\ \phi' &= e^{i\alpha(g)}\phi \end{aligned} \qquad (2.7)$$

for each $g \in H$. Invariance of the Yukawa coupling under this transformation implies

$$K_L^\dagger \Gamma_1 K_R = e^{-i\alpha} \Gamma_1, \quad K_L^\dagger \Gamma_2 K_R = e^{i\alpha} \Gamma_2 , \qquad (2.8)$$

or, using the Hermiticity of Γ_i,

$$[\Gamma_i^2, K_L] = [\Gamma_i^2, K_R] = 0 \quad (i = 1,2) . \qquad (2.9)$$

This relation will prove to be quite powerful in eliminating many possible horizontal symmetries.

Before we proceed, we observe that, by virtue of Eqs. (2.8) and (2.9),

$$\begin{aligned} [K_L, M_u M_u^\dagger] &= [K_L, M_d M_d^\dagger] \\ &= vw\Gamma_1\Gamma_2 K_L (e^{2i\alpha} - 1) + \text{H.c.} , \\ [K_R, M_u M_u^\dagger] &= [K_R, M_d M_d^\dagger] \\ &= vw\Gamma_1\Gamma_2 K_R (e^{-2i\alpha} - 1) + \text{H.c.} \end{aligned} \qquad (2.10)$$

If $e^{2i\alpha} = 1$, all the commutators above vanish. But then in a basis where K_L (or K_R) is diagonal either some flavors decouple or K_L (and K_R) are completely degenerate leading to no constraints on the mass matrices at all. Hence, the case $e^{2i\alpha} = 1$ is "trivial" and will not be tolerated in the subsequent discussions.

If both Γ_1^2 and Γ_2^2 are fourfold degenerate, $\Gamma_i^2 = C_i^2 \mathbb{1}$ and it follows that $[M_u M_u^\dagger, M_d M_d^\dagger] = 0$ leading to a trivial quark mixing matrix. Hence at least one of the Γ_i^2, say Γ_1^2, cannot be fourfold degenerate. We are then led to consider the cases (1) Γ_1^2 nondegenerate, (2) Γ_1^2 twofold degenerate, (3) Γ_1^2 twice twofold degenerate, and (4) Γ_1^2 threefold degenerate. Consider the case of nondegenerate Γ_1^2. Because of Eq. (2.9) there exists a basis where Γ_1, K_L, and K_R are simultaneously diagonal:

$$\begin{aligned} \Gamma_1 &= \text{diag}[g_{11}, g_{22}, g_{33}, g_{44}] , \\ K_L &= \text{diag}[e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}, e^{i\beta_4}] , \\ K_R &= \text{diag}[e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}, \gamma^{i\gamma_4}] . \end{aligned} \qquad (2.11)$$

In this basis at least three of the off-diagonal elements of Γ_2 should be nonzero in order that none of the generations decouple. Denoting Γ_{2ij} by h_{ij} we can choose them to be (a) $h_{14}h_{24}h_{34} \neq 0$ or (b) $h_{12}h_{23}h_{34} \neq 0$. Any other choice can be reduced to one of the above by a permutation of the family indices.

Combining Eq. (2.8) with (2.11) we obtain

$$g_{ii}=e^{i(\alpha-\beta_i+\gamma_i)}g_{ii}, \quad h_{ij}=e^{-i(\alpha+\beta_i-\gamma_j)}h_{ij} . \tag{2.12}$$

For case (a), $h_{14}h_{24}h_{34}\neq 0$ implies

$$\begin{aligned} &e^{i\beta_1}=e^{i\beta_2}=e^{i\beta_3}, \quad e^{i\gamma_1}=e^{i\gamma_2}=e^{i\gamma_3} , \\ &e^{i\gamma_4}=e^{i(\gamma_1+\beta_1-\beta_4)}, \quad e^{i\alpha}=e^{i(\gamma_1-\beta_4)} . \end{aligned} \tag{2.13}$$

Consequently

$$\Gamma_1=\mathrm{diag}[g_{11}e^{i(2\gamma_1-\beta_1-\beta_4)}, \ g_{22}e^{i(2\gamma_1-\beta_1-\beta_4)}, g_{33}e^{i(2\gamma_1-\beta_1-\beta_4)}, g_{44}e^{i(2\gamma_1+\beta_1-3\beta_4)}] . \tag{2.14}$$

Clearly if $e^{i\beta_1}=e^{i\beta_4}$, $\Gamma_1=0$ for nontrivial symmetry which means a trivial mixing matrix. Therefore $e^{i\beta_1}\neq e^{i\beta_4}$, in which case the second of Eq. (2.12) implies that all elements of Γ_2 besides h_{14}, h_{24}, h_{34}, and their complex conjugates are zero. For example,

$$h_{44}=e^{i(\beta_1-\beta_4)}h_{44}=0 .$$

Furthermore, from Eq. (2.14) it follows that

$$e^{i(2\gamma_1-\beta_1-\beta_4)}=1 ,$$

for a realistic mass spectrum. Hence we arrive at the following matrices for case (a):

$$\begin{aligned} &\Gamma_1=\mathrm{diag}[g_{11},g_{22},g_{33},g_{44}e^{4i\alpha}] , \\ &\Gamma_2=\begin{bmatrix} 0 & 0 & 0 & h_{14} \\ 0 & 0 & 0 & h_{24} \\ 0 & 0 & 0 & h_{34} \\ h_{14}^* & h_{24}^* & h_{34}^* & 0 \end{bmatrix} . \end{aligned} \tag{2.15}$$

If $e^{4i\alpha}\neq 1$, $g_{44}=0$ in which case

$$|\det M_u| = |\det M_d| \quad \text{or}$$

or

$$m_u m_c m_t m_{t'} = m_d m_s m_b m_{b'} .$$

Such a mass relation is inconsistent with the experimental lower bound of 22 GeV (Ref. 9) and the upper bound of about 300 GeV from the measurement of the electroweak ρ parameter[10] on the heavy-quark masses. Hence $e^{4i\alpha}=1$ is required. We shall call the matrices of Eq. (2.15) with this choice of α model (A).

The simplest symmetry group which yields the matrices of Eq. (2.15) is the group Z_4. Alternately, any symmetry group which reproduces model (A) should have Z_4 as one of its subgroups. Under Z_4 the transformation properties of the quark and the Higgs fields may, for example, be[11]

$$\begin{aligned} &e^{i\alpha}=-i , \\ &K_L=\mathrm{diag}[1,1,1,-1] , \\ &K_R=\mathrm{diag}[i,i,i,-i] . \end{aligned} \tag{2.16}$$

We shall analyze the phenomenological consequences of this model in Sec. IV.

Now consider case (b) with $h_{12}h_{23}h_{34}\neq 0$. From Eq. (2.12) it follows that

$$\begin{aligned} &e^{i\beta_1}=e^{i\beta_3}, \quad e^{i\beta_2}=e^{i\beta_4} , \\ &e^{i\gamma_1}=e^{i\gamma_3}, \quad e^{i\gamma_2}=e^{i\gamma_4}=e^{i(\gamma_1+\beta_1-\beta_2)} , \\ &e^{i\alpha}=e^{i(\gamma_1-\beta_2)} , \end{aligned} \tag{2.17}$$

and consequently

$$\Gamma_1=\mathrm{diag}[g_{11}e^{i(2\gamma_1-\beta_1-\beta_2)}, g_{22}e^{i(2\gamma_1+\beta_1-3\beta_2)}, g_{33}e^{i(2\gamma_1-\beta_1-\beta_2)}, g_{44}e^{i(2\gamma_1+\beta_1-3\beta_2)}] . \tag{2.18}$$

For $e^{i\beta_1}=e^{i\beta_2}$, Γ_1 is identically zero for a nontrivial symmetry. Hence $e^{i\beta_1}\neq e^{i\beta_2}$ in which case Eq. (2.12) implies that only h_{14} can be nonzero besides h_{12}, h_{23}, h_{34}. For example,

$$h_{13}=e^{i(\beta_2-\beta_1)}h_{13}=0$$

and so on. Furthermore, from Eq. (2.18) we have

$$e^{i(2\gamma_1-\beta_1-\beta_2)}=e^{i(2\gamma_1+\beta_1-3\beta_2)}=1 , \tag{2.19}$$

or else Γ_1 will be degenerate contrary to our assumption. Thus, we arrive at the matrices for model (B):

$$\begin{aligned} &\Gamma_1=\mathrm{diag}[g_{11},g_{22},g_{33},g_{44}] , \\ &\Gamma_2=\begin{bmatrix} 0 & h_{12} & 0 & h_{14} \\ h_{12}^* & 0 & h_{23} & 0 \\ 0 & h_{23}^* & 0 & h_{34} \\ h_{14}^* & 0 & h_{34}^* & 0 \end{bmatrix} . \end{aligned} \tag{2.20}$$

It is remarkable that the simplest symmetry group which produces these mass matrices is again Z_4 with the following assignment:[11]

$$e^{i\alpha}=i \; ,$$
$$K_L=\mathrm{diag}[i,-i,i,-i] \; , \tag{2.21}$$
$$K_R=\mathrm{diag}[1,-1,1,-1] \; .$$

Such a model has been studied recently by Mohopatra and Mohopatra.[12] In Sec. IV we shall come back to the phenomenology of this model and compare it against model (A).

III. THE CASE OF DEGENERATE $\Gamma_1{}^2$

In this section we turn to the remaining possibility that $\Gamma_1{}^2$ is degenerate. As discussed before, we have to consider three cases: $\Gamma_1{}^2$ twofold, twice twofold, or threefold degenerate. First of all, we note one simplification. In the previous section we analyzed the case of nondegenerate $\Gamma_1{}^2$, allowing for an arbitrary $\Gamma_2{}^2$. Therefore, while discussing degenerate $\Gamma_1{}^2$, we may safely exclude the possibility that $\Gamma_2{}^2$ is nondegenerate since it will not give any new model. (Note that the problem at hand has a symmetry $\Gamma_1 \leftrightarrow \Gamma_2$, $\phi \leftrightarrow \tilde{\phi}$.) Next we state a very useful lemma.

Lemma. In the minimal model, the requirement that the horizontal symmetry and the resulting mixing matrix be nontrivial and that none of the generation decouple from each other implies that there exists a basis in which K_L and K_R are simultaneously diagonal.

The proof of the lemma is somewhat tedious and is given in the Appendix. The lemma means that the horizontal symmetry group can be chosen to be Abelian. This brings in considerable simplifications in our analysis. In a basis where K_L and K_R are diagonal, if $\Gamma_1{}^2$ is twofold degenerate, Γ_1 has the form

$$\Gamma_1=\begin{bmatrix} 0 & g_{12} & 0 & 0 \\ g_{12}^* & 0 & 0 & 0 \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & g_{34}^* & g_{44} \end{bmatrix} \tag{3.1}$$

up to a permutation of the family indices. A twice twofold $\Gamma_1{}^2$ is obtained by setting $g_{33}=g_{44}=0$ in the above. Similarly, a threefold degenerate $\Gamma_1{}^2$ is possible only if

$$\Gamma_1=\mathrm{diag}[0,0,0,g_{44}] \; . \tag{3.2}$$

A fourfold degenerate $\Gamma_2{}^2$ implies $\Gamma_2=0$. These restricted forms of the matrices follow simply because the nonzero elements of the matrices are unrelated if the horizontal symmetry is Abelian.[13] The case of fourfold degenerate $\Gamma_2{}^2$ is trivially excluded from the above arguments. Similarly if $\Gamma_1{}^2$ is threefold degenerate some generation will necessarily decouple from the others because of Eqs. (3.1) and (3.2). Hence we are left with the following possibilities: (1) $\Gamma_1{}^2$ and $\Gamma_2{}^2$ twofold degenerate, (2) $\Gamma_1{}^2$ twofold degenerate and $\Gamma_2{}^2$ 2 + 2 degenerate, or (3) $\Gamma_1{}^2$ and $\Gamma_2{}^2$ 2 + 2 degenerate. Setting $g_{33}=g_{44}=0$ in Eq. (3.1) and allowing for all possible permutations of the family indices we see that in case (3) two generations decouple from the remaining two which is unacceptable. Furthermore, case (2) can be obtained as a special case of (1). Hence our task reduces to analyzing the matrices

$$\Gamma_1=\begin{bmatrix} 0 & g_{12} & 0 & 0 \\ g_{12}^* & 0 & 0 & 0 \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & g_{34}^* & g_{44} \end{bmatrix} , \quad \Gamma_2=\begin{bmatrix} 0 & 0 & 0 & h_{14} \\ 0 & h_{22} & h_{23} & 0 \\ 0 & h_{23}^* & h_{33} & 0 \\ h_{14}^* & 0 & 0 & 0 \end{bmatrix} . \tag{3.3}$$

Here we labeled the family indices so that decoupling of generations does not occur.

So far we have not used the invariance of the Lagrangian under the horizontal symmetry. Since

$$K_L=\mathrm{diag}[e^{i\beta_1},e^{i\beta_2},e^{i\beta_3},e^{i\beta_4}] \; ,$$
$$K_R=\mathrm{diag}[e^{i\gamma_1},e^{i\gamma_2},e^{i\gamma_3},e^{i\gamma_4}] \; , \tag{3.4}$$

we have, from Eq. (2.8),

$$g_{ij}=e^{i(\alpha-\beta_i+\gamma_j)}g_{ij} \; , \tag{3.5}$$
$$h_{ij}=e^{-i(\alpha+\beta_i-\gamma_j)}h_{ij} \; . \tag{3.6}$$

We immediately see that

$$g_{ij}h_{ij}^*(1-e^{2i\alpha})=0 \quad \text{(no summation)} \; . \tag{3.7}$$

For nontrivial symmetry we need $g_{ij}h_{ij}=0$. Consequently we can set $g_{33}=0$ without loss of generality in Eq. (3.3) and consider the cases $g_{44}=0$ and $g_{44}\neq 0$.

If in Eq. (3.3) $g_{12}=0$, then $g_{34}h_{14}h_{23}\neq 0$ so that no generation decouples. If $h_{14}=0$, $g_{12}g_{34}h_{23}\neq 0$. For nonzero g_{12} and h_{14}, either g_{34} or h_{23} has to be nonzero as well in order that all generations mix. We shall analyze these four possibilities one by one.

(i) $g_{12}=0$, $g_{34}h_{14}h_{23}\neq 0$. From Eqs. (3.5) and (3.6) we have

$$e^{i\beta_2}=e^{i(\gamma_1-3\alpha)}, \quad e^{i\beta_3}=e^{i(\beta_1+2\alpha)}, \quad e^{i\beta_4}=e^{i(\gamma_1-\alpha)} ,$$
$$e^{i\gamma_2}=e^{i(\beta_1+3\alpha)}, \quad e^{i\gamma_3}=e^{i(\gamma_1-2\alpha)}, \quad e^{i\gamma_4}=e^{i(\beta_1+\alpha)} . \tag{3.8}$$

Furthermore,

$$g_{24}=e^{i(\beta_1-\gamma_1+5\alpha)}g_{24}, \quad g_{33}=e^{i(\gamma_1-\beta_1-3\alpha)}g_{33} \; ,$$
$$g_{44}=e^{i(\beta_1-\gamma_1+3\alpha)}g_{44} \; , \tag{3.9}$$
$$h_{22}=e^{i(\beta_1-\gamma_1+5\alpha)}h_{22}, \quad h_{33}=e^{i(\gamma_1-\beta_1-5\alpha)}h_{33} \; .$$

Since $g_{24}=g_{33}=0$, $e^{i(\beta_1-\gamma_1+5\alpha)}\neq 1$, $e^{i(\gamma_1-\beta_1-3\alpha)}\neq 1$. This implies that $g_{44}=h_{22}=h_{33}=0$ leading to decoupling of generations.

(ii) $h_{14}=0$, $g_{12}g_{34}h_{23}\neq 0$. In this case from Eqs. (3.5) and (3.6) we have

$$e^{i\beta_2}=e^{i\beta_4}=e^{i(\gamma_1+\alpha)},\quad e^{i\beta_3}=e^{i(\beta_1-2\alpha)},$$
$$e^{i\gamma_2}=e^{i\gamma_4}=e^{i(\beta_1-\alpha)},\quad e^{i\gamma_3}=e^{i(\gamma_1+2\alpha)}. \tag{3.10}$$

Then from Eq. (3.5) we see that g_{14} cannot be kept zero consistently. Hence this case is disallowed by symmetry.

(iii) $g_{12}g_{34}h_{14}\neq 0$. Equations (3.5) and (3.6) imply

$$e^{i\beta_2}=e^{i(\gamma_1+\alpha)},\quad e^{i\beta_3}=e^{i(\beta_1+2\alpha)},\quad e^{i\beta_4}=e^{i(\gamma_1-\alpha)},$$
$$e^{i\gamma_2}=e^{i(\beta_1-\alpha)},\quad e^{i\gamma_3}=e^{i(\gamma_1-2\alpha)},\quad e^{i\gamma_4}=e^{i(\beta_1+\alpha)}, \tag{3.11}$$

and consequently

$$\Gamma_1=\begin{bmatrix} g_{11}e^{i(\gamma_1-\beta_1+\alpha)} & g_{12} & g_{13}e^{i(\gamma_1-\beta_1-\alpha)} & 0 \\ g_{12}^* & g_{22}e^{i(\beta_1-\gamma_1-\alpha)} & 0 & g_{24}e^{i(\beta_1-\gamma_1+\alpha)} \\ g_{13}^*e^{i(\gamma_1-\beta_1-\alpha)} & 0 & g_{33}e^{i(\gamma_1-\beta_1-3\alpha)} & g_{34} \\ 0 & g_{24}^*e^{i(\beta_1-\gamma_1+\alpha)} & g_{34}^* & g_{44}e^{i(\beta_1-\gamma_1+3\alpha)} \end{bmatrix},$$
$$\Gamma_2=\begin{bmatrix} h_{11}e^{i(\gamma_1-\beta_1-\alpha)} & 0 & h_{13}e^{i(\gamma_1-\beta_1-3\alpha)} & h_{14} \\ 0 & h_{22}e^{i(\beta_1-\gamma_1-3\alpha)} & h_{23}e^{-4i\alpha} & h_{24}e^{i(\beta_1-\gamma_1-\alpha)} \\ h_{13}^*e^{i(\gamma_1-\beta_1-3\alpha)} & h_{23}^*e^{-4i\alpha} & h_{33}e^{i(\gamma_1-\beta_1-5\alpha)} & 0 \\ h_{14}^* & h_{24}^*e^{i(\beta_1-\gamma_1-\alpha)} & 0 & h_{44}e^{i(\beta_1-\gamma_1+\alpha)} \end{bmatrix}, \tag{3.12}$$

where the vanishing entries result from $e^{2i\alpha}\neq 1$. Now since $h_{13}=0$, $e^{i(\gamma_1-\beta_1-3\alpha)}\neq 1$ which implies $g_{44}=0$. As $g_{24}=0$, $e^{i(\beta_1-\gamma_1+\alpha)}\neq 1$. Then if $h_{23}\neq 0$, $e^{-4i\alpha}=1$, which means $h_{22}=h_{33}=0$ leading to generation decoupling. Hence $h_{23}=0$ in which case one can consistently choose

$$\Gamma_1=\begin{bmatrix} 0 & g_{12} & 0 & 0 \\ g_{12}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{34} \\ 0 & 0 & g_{34}^* & 0 \end{bmatrix},\quad \Gamma_2=\begin{bmatrix} 0 & 0 & 0 & h_{14} \\ 0 & h_{22} & 0 & 0 \\ 0 & 0 & h_{33} & 0 \\ h_{14}^* & 0 & 0 & 0 \end{bmatrix}. \tag{3.13}$$

We shall analyze the mass spectrum of this model later in this section.

(iv) $g_{12}h_{14}h_{23}\neq 0$. Analysis similar to the above yields

$$e^{i\beta_2}=e^{i(\gamma_1+\alpha)},\quad e^{i\beta_3}=e^{i(\beta_1-2\alpha)},\quad e^{i\beta_4}=e^{i(\gamma_1-\alpha)},\quad e^{i\gamma_2}=e^{i(\beta_1-\alpha)},\quad e^{i\gamma_3}=e^{i(\gamma_1+2\alpha)},\quad e^{i\gamma_4}=e^{i(\beta_1+\alpha)}, \tag{3.14}$$

and

$$\Gamma_1=\begin{bmatrix} g_{11}e^{i(\alpha-\beta_1+\gamma_1)} & g_{12} & g_{13}e^{i(\gamma_1-\beta_1+3\alpha)} & 0 \\ g_{12}^* & g_{22}e^{i(\beta_1-\gamma_1-\alpha)} & 0 & g_{24}e^{i(\beta_1-\gamma_1+\alpha)} \\ g_{13}^*e^{i(\gamma_1-\beta_1+3\alpha)} & 0 & g_{33}e^{i(\gamma_1-\beta_1+5\alpha)} & g_{34}e^{4i\alpha} \\ 0 & g_{24}^*e^{i(\beta_1-\gamma_1+\alpha)} & g_{34}^*e^{4i\alpha} & g_{44}e^{i(\beta_1-\gamma_1+3\alpha)} \end{bmatrix},$$
$$\Gamma_2=\begin{bmatrix} h_{11}e^{i(\gamma_1-\beta_1-\alpha)} & 0 & h_{13}e^{i(\gamma_1-\beta_1+\alpha)} & h_{14} \\ 0 & h_{22}e^{i(\beta_1-\gamma_1-3\alpha)} & h_{23} & h_{24}e^{i(\beta_1-\gamma_1-\alpha)} \\ h_{13}^*e^{i(\gamma_1-\beta_1+\alpha)} & h_{23}^* & h_{33}e^{i(\gamma_1-\beta_1+3\alpha)} & 0 \\ h_{14}^* & h_{24}^*e^{i(\beta_1-\gamma_1-\alpha)} & 0 & h_{44}e^{i(\beta_1-\gamma_1+\alpha)} \end{bmatrix}. \tag{3.15}$$

Since $g_{13}=0$, $e^{i(\gamma_1-\beta_1+3\alpha)}\neq 1$ implying that $h_{22}=h_{33}=0$. As $h_{24}=0$, $e^{i(\beta_1-\gamma_1-\alpha)}\neq 1$ leading to $g_{34}g_{44}=0$. If $g_{44}=0$ the model is again with decoupled generations. For $g_{44}\neq 0$, $g_{34}=0$ a consistent model emerges, but it is a special case of the matrices Eq. (3.13) and will not be treated separately.

To summarize, we have shown after some tedious manipulations that for the case of degenerate $\Gamma_1{}^2$, there is essentially one model given by the matrices of Eq. (3.13). After a permutation of the indices 1↔3, 2↔4, the resulting mass matrices are

$$M_u = \begin{bmatrix} w^*h_{33} & vg_{34} & 0 & 0 \\ vg_{34}^* & 0 & w^*h_{14} & 0 \\ 0 & w^*h_{14}^* & 0 & vg_{12} \\ 0 & 0 & vg_{12}^* & w^*h_{22} \end{bmatrix} ,$$

$$M_d = M_u(v \leftrightarrow w) . \tag{3.16}$$

If $h_{33}=0$ this is of the Fritzsch type. It has been shown in Ref. 14 that this case leads to three mass relations

$$\frac{m_b m_{b'}}{m_t m_{t'}} \simeq \frac{m_d m_s}{m_u m_c} ,$$

$$\frac{m_{b'}-m_b}{m_{t'}-m_t} \simeq \left[\frac{m_s m_b m_{b'}}{m_c m_t m_{t'}} \right]^{1/3} \simeq \frac{m_d m_s^2}{m_u m_c^2} \frac{m_b m_{b'}}{m_t m_{t'}} , \tag{3.17}$$

which are unrealistic. The case $h_{33} \neq 0$ will be analyzed by treating the first-generation quark masses m_u and m_d as perturbations. In the limit $m_u = m_d = 0$ from the determinants of M_u and M_d, disallowing the possibility that $v = w$ (in which case $M_u = M_d$), we have

$$g_{12}g_{34} = 0, \quad h_{22}h_{33} \, | h_{14} |^2 = 0 . \tag{3.18}$$

If $g_{12} = h_{22} = 0$ or $g_{34} = h_{33} = 0$ we have the following mass relations in this limit:

$$\frac{m_{b'}-m_b}{m_{t'}-m_t} \simeq \left[\frac{m_s m_b m_{b'}}{m_c m_t m_{t'}} \right]^{1/3} \simeq \frac{m_s}{m_c} \left[\frac{m_b m_{b'}}{m_t m_{t'}} \right]^2 . \tag{3.19}$$

These relations predict a t' mass below the experimental lower bound of 22 GeV. If Eq. (3.17) is satisfied by choosing $g_{34} = h_{14} = 0$, we see that

$$\frac{m_s}{m_c} \simeq \frac{m_{t'}-m_t}{m_{b'}-m_b} \simeq \left[\frac{m_b m_{b'}}{m_t m_{t'}} \right]^{1/2} , \tag{3.20}$$

which are also not realized in nature. For $g_{12} = h_{14} = 0$, one obtains the unacceptable relations

$$\frac{m_t}{m_b} \simeq \left[\frac{m_s}{m_c} \right]^{1/3} \simeq \left[\frac{m_s m_b}{m_c m_t} \right]^{1/2} . \tag{3.21}$$

Finally, if $g_{34} = h_{22} = 0$ or $g_{12} = h_{33} = 0$, $m_c = m_t$ and $m_s = m_b$ will follow. Hence the mass matrices in Eq. (3.15) do not lead to realistic models.

IV. PHENOMENOLOGY OF MODELS (A) AND (B)

Having established that even with an arbitrary horizontal symmetry there are only two realistic models [Eqs. (2.15) and (2.20)], we now turn to the phenomenological consequences of these two models. Model (A) [see Eq. (2.15)] corresponds to the following mass matrices after a permutation of family indices:

$$M_u = \begin{bmatrix} g_{11}v & h_{14}w^* & 0 & 0 \\ h_{14}^*w^* & g_{44}v & h_{34}^*w^* & h_{24}^*w^* \\ 0 & h_{34}w^* & g_{33}v & 0 \\ 0 & h_{24}w^* & 0 & g_{22}v \end{bmatrix} ,$$

$$M_d = M_u(v \leftrightarrow w) . \tag{4.1}$$

In analyzing the predictions of the model on the quark mixing matrix we shall make the following approximations. First of all we assume *CP* invariance so that the Yukawa couplings h_{ij} and the vacuum expectation values v and w are real. In left-right-symmetric models the observed *CP* violation can be explained by the right-handed currents alone. With this assumption the mass matrices becomes Hermitian. The ratio of the vacuum expectations values is given by

$$\kappa = \frac{w}{v} = \frac{m_d + m_s + m_b + m_{b'}}{m_u + m_c + m_t + m_{t'}} . \tag{4.2}$$

If we identify the first row with the first generation, second row with the second, and so on in the matrices of Eq. (4.1) (other possible identifications will be discussed later), in the limit of neglecting the first two generation masses we have the mass relation

$$\kappa = \frac{m_b}{m_t} = \frac{m_{b'}}{m_{t'}} , \tag{4.3}$$

barring unnatural cancellations. This relation will get only small corrections when the first two generation masses are turned on. Since the mixings are known to be small and since they arise through h_{ij} in Eq. (4.1) we shall assume the h's to be small compared to the g's.

Because of Eq. (4.3), $\kappa \ll 1$ and we can safely ignore the off-diagonal elements of M_u. The up-quark masses are then

$$m_u = g_{11}v, \quad m_c = g_{44}v, \quad m_t = g_{33}v ,$$

and

$$m_{t'} = g_{22}v .$$

With these approximations, the eigenvalue equation for M_d yields

$$m_b m_{b'} \simeq m_t m_{t'} \kappa^2 - v^2(h_{14}^2 + h_{24}^2 + h_{34}^2) ,$$

$$m_s m_b m_{b'} \simeq m_c m_t m_{t'} \kappa^3 - \kappa v^2 [m_t(h_{14}^2 + h_{24}^2) + m_c h_{34}^2] , \tag{4.4}$$

$$m_d m_s m_b m_{b'} \simeq m_u m_c m_t m_{t'} \kappa^4 - \kappa^2 v^2 (m_c m_t h_{14}^2 + m_u m_t h_{24}^2 + m_u m_c h_{34}^2) .$$

The quark-mixing matrix has the approximate form (in units $v = 1$)

$$V \simeq \begin{bmatrix} 1 & \dfrac{m_d-\kappa m_u}{\kappa_{14}} & \dfrac{-h_{34}}{h_{14}}\dfrac{m_d-\kappa m_u}{\kappa m_t} & \dfrac{-h_{24}}{h_{14}}\dfrac{m_d-\kappa m_u}{\kappa m_{t'}} \\ \dfrac{h_{14}}{m_s-\kappa m_u} & 1 & \dfrac{-h_{34}}{\kappa m_t-m_s} & \dfrac{-h_{24}}{\kappa m_{t'}-m_s} \\ \dfrac{-h_{14}}{h_{34}}\dfrac{\kappa m_t-m_b}{m_b} & \dfrac{-(\kappa m_t-m_b)}{h_{34}} & 1 & \dfrac{h_{24}}{h_{34}}\dfrac{\kappa m_t-m_b}{\kappa m_{t'}-m_b} \\ \dfrac{-h_{14}}{h_{24}}\dfrac{\kappa m_{t'}-m_{b'}}{m_{b'}} & \dfrac{-(\kappa m_{t'}-m_{b'})}{h_{24}} & \dfrac{h_{34}}{h_{24}}\dfrac{\kappa m_{t'}-m_{b'}}{\kappa m_t-m_{b'}} & 1 \end{bmatrix} . \tag{4.5}$$

A few remarks are in order on the masses of the t, b', and t' quarks in the model. The approximate equality $m_b/m_t \simeq m_{b'}/m_{t'}$ when combined with the DESY PETRA lower bound of 22 GeV on the heavy-quark masses and the upper bound resulting from the measurement of the electroweak ρ parameter severely constrains m_t, $m_{b'}$, and $m_{t'}$. If we choose a conservative value of 300 GeV for the upper bound from ρ parameter,[10] we have

$$m_t,\ m_{b'} \le 43 \text{ GeV} , \tag{4.6}$$

where we used $m_b = 5.3$ GeV (Ref. 15) at the scale 1 GeV and took account of QCD corrections with $\Lambda_{\rm QCD} = 100$ MeV. A recent and more complete analysis of the neutral-current processes indicates that the upper bound on the quark masses may be considerably smaller—in the neighborhood of 200 GeV (Ref. 16). If we choose it to be 225 GeV, for example, the bound Eq. (4.6) reduces to 32 GeV. Both the top and the b' quarks should then be observable at KEK TRISTAN which is indeed an exciting possibility.

In order to get an idea of the kind of mixing matrix predicted by the model, we present two typical examples. The light-quark masses defined at the energy scale 1 GeV are chosen to be[15]

$$\begin{aligned} &m_u = 5.1 \text{ MeV}, \quad m_c = 1.35 \text{ GeV} , \\ &m_s = -175 \text{ MeV}, \quad m_b = 5.3 \text{ GeV} , \end{aligned} \tag{4.7}$$

and QCD corrections are taken into account with $\Lambda_{\rm QCD} = 100$ MeV. The mixing matrix is rather sensitive to m_d defined at 1 GeV which are chosen differently for the two cases. m_t, $m_{b'}$, and $m_{t'}$ given below are the physical masses.

Case (a). $m_d = 9.56$ MeV, $m_t = 30$ GeV, $m_{b'} = -35$ GeV, $m_{t'} = -235$ GeV:

$$V \simeq \begin{bmatrix} 0.976 & -0.220 & 3.22\times10^{-4} & 9.81\times10^{-6} \\ 0.220 & 0.975 & 0.043 & -0.014 \\ -9.79\times10^{-3} & -0.042 & 0.999 & 5.31\times10^{-5} \\ 2.98\times10^{-3} & 0.013 & 5.33\times10^{-4} & 1.00 \end{bmatrix} . \tag{4.8}$$

Case (b). $m_d = 10$ MeV, $m_t = 30$ GeV, $m_{b'} = -25$ GeV, $m_{t'} = -184$ GeV:

$$V \simeq \begin{bmatrix} 0.974 & -0.226 & 4.54\times10^{-4} & 1.19\times10^{-5} \\ 0.225 & 0.972 & 0.059 & -0.011 \\ -0.014 & -0.057 & 0.998 & 8.26\times10^{-5} \\ 2.52\times10^{-3} & 0.011 & 5.81\times10^{-4} & 1.00 \end{bmatrix} . \tag{4.9}$$

Note that in both the cases the fourth-generation mixing into the second generation is the largest, the reason for which is obvious from the form of the mass matrices Eq. (4.1). The value of the matrix element $|V_{ub}|$ is predicted to be $\sim 10^{-4}$, 2 orders of magnitude below the present experimental bound. A direct measurement of $|V_{ub}|$ can thus confirm or rule out the model.

By permuting the generation indices in the mass matrices of Eq. (4.1), it is possible to obtain three more independent models (i.e., we may, for example, identify the first row with the third generation, etc.). However, we see by analysis similar to the above that all these cases lead to unrealistic mixing matrices—either the Cabibbo angle is predicted to be too small or $|V_{ub}|$ is predicted to be greater than $|V_{cb}|$ in contradiction with experiment.

Now we turn to model (B) defined by the mass matrices

$$M_u = \begin{bmatrix} g_{11}v & h_{12}w^* & 0 & h_{14}w^* \\ h_{12}^*w^* & g_{22}v & h_{23}w^* & 0 \\ 0 & h_{23}^*w^* & g_{33}v & h_{34}w^* \\ h_{14}^*w^* & 0 & h_{34}^*w^* & g_{44}v \end{bmatrix} ,$$

$$M_d = M_u(v \leftrightarrow w) . \tag{4.10}$$

Such a model is studied in detail in Ref. 11 and we shall only summarize the main results and compare them against the predictions of model (A). With the same approximations as for model (A), model (B) yields the mass relation

$$\frac{m_s}{m_c} \simeq \frac{m_b + m_{b'}}{m_t + m_{t'}} , \tag{4.11}$$

yielding an upper bound of 58 GeV on the b'-quark mass. The third and the fourth generations mix even in the limit of zero first- and second-generation masses. This implies that $V_{tb'}$ can be even as high as 0.3 in contrast with model (A). Furthermore, $V_{cb'}$ is smaller than $V_{ub'}$ and $V_{tb'}$ in the model. Recall that model (A) had $V_{cb'}$ larger than $V_{ub'}$ and $V_{tb'}$. A crucial test which can distinguish the two models is a direct measurement of $|V_{ub}|$. Model (B) predicts it to be $\sim 10^{-3}$, an order of magnitude larger than the prediction of model (A). If the top quark is not discovered below 45 GeV or so, model (A) will be ruled out while model (B) may still stand some chance.

V. CONCLUSIONS

In this paper we have carried out a systematic investigation of all possible horizontal symmetries in left-right-symmetric gauge theories with a minimal Higgs sector and four generations of fermions. Although horizontal symmetries are quite powerful in constraining the parameters of the model, there is no convincing argument as to what the correct symmetry is. We hope that a general analysis such as the one presented here will help gain deeper insight into the problems of family replication and mixing-angle hierarchy.

We have shown that there are only two consistent models with realistic predictions on the quark masses and mixing angles in the case of four generations. Z_4 is the unique symmetry group which leads to these models. The phenomenology of the two models is studied in Sec. IV. Both the models predict $m_t, m_{b'} \lesssim 50$ GeV, and $m_{t'} \gtrsim 180$ GeV. They differ in the mixing of the fourth generation into the first three. In one of the models, the fourth generation mixes preferentially with the second, whereas in the other it mixes with the third generation. A direct measurement of $|V_{ub}|$ can distinguish the two models. Accelerator experiments planned for the near future will tell us which one of the two, if either, is the correct description of nature.

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APPENDIX

In this appendix we prove the lemma stated in Sec. III, viz., in order that none of the generations decouple and that the symmetry be nontrivial, there must exist a basis where K_L and K_R are simultaneously diagonal in the minimal model.

The invariance of the Lagrangian under the horizontal symmetry implies

$$K_L^\dagger \Gamma_1 K_R = e^{-i\alpha}\Gamma_1, \quad K_L^\dagger \Gamma_2 K_R = e^{i\alpha}\Gamma_2 \tag{A1}$$

or

$$[\Gamma_i{}^2, K_L] = [\Gamma_i{}^2, K_R] = 0 \quad (i = 1,2) . \tag{A2}$$

Because of Eq. (A2), in a basis where $\Gamma_1{}^2$ is diagonal K_L and K_R will be block diagonal depending on the degeneracy of $\Gamma_1{}^2$. Clearly for a nondegenerate $\Gamma_1{}^2$ there exists a basis where K_L and K_R are diagonal.

Consider the case of a twofold degenerate $\Gamma_1{}^2$. In a basis where $\Gamma_1{}^2$ is diagonal K_L and K_R will have the form

$$K_L = \begin{bmatrix} k & 0 & \\ & e^{i\beta_3} & 0 \\ 0 & 0 & e^{i\beta_4} \end{bmatrix} ,$$

$$K_R = \begin{bmatrix} \tilde{k} & 0 & \\ & e^{i\gamma_3} & 0 \\ 0 & 0 & e^{i\gamma_4} \end{bmatrix} , \tag{A3}$$

where k and $\tilde{k}$ are 2×2 unitary matrices. We make a basis transformation so that K_L is diagonal without altering the form of K_R:

$$K_L = \text{diag}[e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}, e^{i\beta_4}] . \tag{A4}$$

Γ_1 has the general form

$$\Gamma_1 = \begin{bmatrix} G & 0 & \\ & g_{33} & 0 \\ 0 & 0 & g_{44} \end{bmatrix} , \tag{A5}$$

with 2×2 Hermitian G.

If $e^{i\beta_1} = e^{i\beta_2}$, K_R can be diagonalized. Hence consider $e^{i\beta_1} \neq e^{i\beta_2}$. From Eq. (A1) we have

$$h_{13} = e^{i(\gamma_3 - \beta_1 - \alpha)} h_{13}, \quad h_{14} = e^{i(\gamma_4 - \beta_1 - \alpha)} h_{14} ,$$
$$h_{23} = e^{i(\gamma_3 - \beta_2 - \alpha)} h_{23}, \quad h_{24} = e^{i(\gamma_4 - \beta_2 - \alpha)} h_{23} , \tag{A6}$$

and

$$
\begin{aligned}
&h^*_{13}\tilde{k}_{11}+h^*_{23}\tilde{k}_{21}=h^*_{13}e^{i(\alpha+\beta_3)}\ ,\\
&h^*_{14}\tilde{k}_{11}+h^*_{24}\tilde{k}_{21}=h^*_{14}e^{i(\alpha+\beta_4)}\ ,\\
&h^*_{13}\tilde{k}_{12}+h^*_{23}\tilde{k}_{22}=h^*_{23}e^{i(\alpha+\beta_3)}\ ,\\
&h^*_{14}\tilde{k}_{12}+h^*_{24}\tilde{k}_{22}=h^*_{24}e^{i(\alpha+\beta_4)}\ .
\end{aligned}
\tag{A7}
$$

From Eq. (A6) it follows that $h_{13}h_{23}=h_{14}h_{24}=0$ since $e^{i\beta_1}\neq e^{i\beta_2}$. If $h_{13}=h_{14}=0$, $h_{23}\tilde{k}_{21}=h_{24}\tilde{k}_{21}=0$ from Eq. (A7). Choosing $h_{23}=h_{24}=0$ will result in a block diagonal Γ_2 resulting in decoupling of families. Hence $\tilde{k}_{21}=0$ in which case K_R is diagonal. Now let $h_{13}=h_{24}=0$. Then $h^*_{23}\tilde{k}_{21}=h^*_{14}\tilde{k}_{12}=0$. Again the only consistent choice is $\tilde{k}_{21}=\tilde{k}_{12}=0$ for no decoupling. Similarly $h_{23}=h_{14}=0$ and $h_{23}=h_{24}=0$ also give either decoupling of families or $\tilde{k}_{12}=\tilde{k}_{21}=0$. This proves that K_L and K_R can be simultaneously diagonalized for the case of a twofold degenerate $\Gamma_1{}^2$.

Now consider the case of a twice twofold degenerate $\Gamma_1{}^2$. In this case we can find a basis where

$$
K_L=\begin{bmatrix} e^{i\beta_1} & 0 & 0 & 0\\ 0 & e^{i\beta_2} & 0 & 0\\ 0 & 0 & k_{33} & k_{34}\\ 0 & 0 & k_{43} & k_{44}\end{bmatrix},
$$

$$
K_R=\begin{bmatrix} \tilde{k}_{11} & \tilde{k}_{12} & 0 & 0\\ \tilde{k}_{21} & \tilde{k}_{22} & 0 & 0\\ 0 & 0 & e^{i\gamma_3} & 0\\ 0 & 0 & 0 & e^{i\gamma_4}\end{bmatrix},
\tag{A8}
$$

$$
\Gamma_1=\begin{bmatrix} G_1 & 0\\ 0 & G_2\end{bmatrix},
$$

with 2×2 Hermitian G_1 and G_2. Then Eq. (A1) implies

$$
\begin{aligned}
&h_{14}e^{i\alpha}=e^{i(\gamma_4-\beta_1)}h_{14},\quad h_{24}e^{i\alpha}=e^{i(\gamma_4-\beta_2)}h_{24}\ ,\\
&h_{13}e^{i\alpha}=e^{i(\gamma_3-\beta_1)}h_{13},\quad h_{23}e^{i\alpha}=e^{i(\gamma_3-\beta_2)}h_{23}\ .
\end{aligned}
\tag{A9}
$$

Note that at least one among h_{14}, h_{24}, h_{13}, and h_{23} has to be nonzero for generation decoupling not to occur.

Let $e^{i\beta_1}=e^{i\beta_2}$ in which case K_R can be diagonalized. If $e^{i\gamma_3}=e^{i\gamma_4}$, K_L can also be diagonalized proving the lemma. For $e^{i\gamma_3}\neq e^{i\gamma_4}$, from Eq. (A9) we have

$$
h_{14}h_{24}\neq 0,\quad h_{13}=h_{23}=0\ .
\tag{A10}
$$

Using

$$
h^*_{13}e^{i\alpha}=e^{i\gamma_1}(k^*_{33}h^*_{13}+k^*_{43}h^*_{14})\ ,
\tag{A11}
$$

we see that $k_{43}=0$ and hence K_L is also diagonal. Similarly if $e^{i\beta_1}\neq e^{i\beta_2}$, $e^{i\gamma_3}=e^{i\gamma_4}$, K_L and K_R are simultaneously diagonalizable. We are left with the case $e^{i\beta_1}\neq e^{i\beta_2}$, $e^{i\gamma_3}\neq e^{i\gamma_4}$ in which case, from Eq. (A9),

$$
h_{14}\neq 0,\quad h_{13}=h_{23}=h_{24}=0\ ,
\tag{A12}
$$

without loss of generality. From Eq. (A1) we then have

$$
k_{11}\tilde{k}^*_{43}=k_{12}\tilde{k}^*_{43}=k_{12}\tilde{k}^*_{44}=0\ .
\tag{A13}
$$

The only solution is $k_{12}=\tilde{k}_{43}=0$ in which case K_L and K_R are diagonal. This proves the lemma for the case of twice twofold degenerate $\Gamma_1{}^2$.

Finally, consider the case of threefold degenerate $\Gamma_1{}^2$. (As discussed in the text, a fourfold degenerate $\Gamma_1{}^2$ leads to trivial mixing matrix and is ruled out.) By virtue of Eq. (A2) one can find a basis where

$$
\begin{aligned}
&K_L=\mathrm{diag}[e^{i\beta_1},\ e^{i\beta_2},\ e^{i\beta_3},\ e^{i\beta_4}]\ ,\\
&K_R=\begin{bmatrix}\tilde{k} & 0\\ 0 & e^{i\gamma_4}\end{bmatrix},\quad \Gamma_1=\begin{bmatrix} G & 0\\ 0 & g_{44}\end{bmatrix}\ ,
\end{aligned}
\tag{A14}
$$

with 3×3 unitary $\tilde{k}$ and Hermitian G. Clearly if $e^{i\beta_1}=e^{i\beta_2}=e^{i\beta_3}$, K_R can also be diagonalized. Equation (A1) results in

$$
\begin{aligned}
&h_{14}e^{i\alpha}=h_{14}e^{i(\gamma_4-\beta_1)}\ ,\\
&h_{24}e^{i\alpha}=h_{24}e^{i(\gamma_4-\beta_2)}\ ,\\
&h_{34}e^{i\alpha}=h_{34}e^{i(\gamma_4-\beta_3)}\ .
\end{aligned}
\tag{A15}
$$

For all families to mix at least one of h_{14}, h_{24}, or h_{34} must be nonzero. Consider $e^{i\beta_1}\neq e^{i\beta_2}=e^{i\beta_3}$. Then by a unitary transformation in the 2-3 plane $\tilde{k}$ can be brought to

$$
\tilde{k}=\begin{bmatrix}\tilde{k}_{11} & \tilde{k}_{12} & \tilde{k}_{13}\\ \tilde{k}_{21} & \tilde{k}_{22} & \tilde{k}_{23}\\ 0 & \tilde{k}_{32} & \tilde{k}_{33}\end{bmatrix}.
\tag{A16}
$$

Furthermore, from Eq. (A15) either $h_{14}=0$, $h_{24}h_{34}\neq 0$ or $h_{14}\neq 0$, $h_{24}=h_{34}=0$. Consider $h_{14}=0$. Using

$$
h^*_{14}e^{i\alpha}=e^{-i\beta_4}(h^*_{14}\tilde{k}_{11}+h^*_{24}\tilde{k}_{21}+h^*_{34}\tilde{k}_{31})\ ,
\tag{A17}
$$

we see that $\tilde{k}_{21}=0$. Since $e^{i\beta_2}=e^{i\beta_3}$, K_R can be diagonalized in this case. Similarly if $h_{14}\neq 0$, $h_{24}=h_{34}=0$, with

$$
\begin{aligned}
&h^*_{34}e^{i\alpha}=e^{-i\beta_4}(h^*_{14}\tilde{k}_{13}+h^*_{23}\tilde{k}_{23}+h^*_{34}\tilde{k}_{33})\ ,\\
&h^*_{24}e^{i\alpha}=e^{-i\beta_4}(h^*_{14}\tilde{k}_{12}+h^*_{24}\tilde{k}_{22}+h^*_{34}\tilde{k}_{32})\ ,
\end{aligned}
\tag{A18}
$$

it follows that $\tilde{k}_{12}=\tilde{k}_{13}=0$. Again K_R can be diagonalized.

If $e^{i\beta_1}\neq e^{i\beta_2}\neq e^{i\beta_3}$, we have from Eq. (A15) $h_{14}\neq 0$, $h_{24}=h_{34}=0$. Then using Eq. (A18) we see that $\tilde{k}_{12}=\tilde{k}_{13}=0$, or

$$
\tilde{k}=\begin{bmatrix} e^{i\gamma_1} & 0 & 0\\ 0 & \tilde{k}_{22} & \tilde{k}_{23}\\ 0 & \tilde{k}_{32} & \tilde{k}_{33}\end{bmatrix}.
\tag{A19}
$$

With this simplified form of K_R, we have

$$h^*_{12}e^{i\alpha}=h^*_{12}e^{i(\gamma_1-\beta_2)}, \quad g^*_{12}e^{-i\alpha}=g^*_{12}e^{i(\gamma_1-\beta_2)},$$
$$h^*_{13}e^{i\alpha}=h^*_{13}e^{i(\gamma_1-\beta_3)}, \quad g^*_{13}e^{-i\alpha}=g^*_{13}e^{i(\gamma_1-\beta_3)}. \tag{A20}$$

Clearly $h_{12}h_{13}=0$, $g_{12}g_{13}=0$ from above. Furthermore, one among h_{12}, h_{13}, g_{12}, and g_{13} must be nonzero, or else the first and fourth families will decouple from the second and third. Consider $h_{12}\neq 0$. Then from

$$h_{12}e^{i\alpha}=e^{-i\beta_1}(h_{12}\tilde{k}_{22}+h_{13}\tilde{k}_{32}), \tag{A21}$$

it follows that $|\tilde{k}_{22}|=1$, or K_R is diagonal. Similarly $g_{12}\neq 0$ yields the same result. This completes the proof of the lemma.

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