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# Relativistic formulation of the radiative transitions of charmonium and $b$-quarkonium

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Bander, Silverman, Klima, and Maor recently developed a relativistic formalism for quark-antiquark bound states. It was used to calculate the spectra and leptonic decay widths of charmonium, $b$-quarkonium, $D$ mesons, $F$ mesons, $B$ mesons, and mesons composed of light quarks. The form of the interactions between quark and antiquark was highly motivated by QCD. We extend this relativistic treatment to a relativistic formulation of radiative decays that automatically takes into account the effects of recoil, uses boosted final-state meson wave functions, includes relativistic bound-state wave functions involving relativistic mixtures of terms, and treats the valence quarks as four-component spinors. We then calculate radiative transition rates of the charmonium and $b$-quarkonium systems in this formalism and find that transitions which can be compared with experiment are in good agreement. Predictions are also made for decays that are not yet measured.

## I. INTRODUCTION

In this paper, we extend a relativistic formalism[1–5] with QCD interactions for quark-antiquark systems previously used for spectra[1] to calculate relativistically the radiative decay rate of such systems. By carrying out the calculation relativistically, we automatically include full recoil effects with boosted final states, relativistic wave functions that treat both the quark and antiquark as four-component spinors to give relativistic currents, and a natural mixing of $L$-$S$ terms for a given $J$ and parity. We then apply this formalism to calculate the radiative decay rates of charmonium and $b$-quarkonium and find them in agreement with experiment.

It has been known that the relativistic effects are important in finding the radiative decay rates of the charmonium system.[6–10] The various relativistic approximations of the order $v^2/c^2$ were found inadequate, especially in calculating the magnetic dipole and hindered magnetic transitions. The relativistic treatment by a Dirac-type equation gives results that go beyond the $v^2/c^2$ expansion and avoids some of the difficulties of the singularities of the perturbative terms.

The relativistic treatment of the bound-state equation for the quark-antiquark bound state that was used in Ref. 1 was based on taking the wave function or matrix element of the quark field equation between the bound state and the antiquark state and expanding in intermediate states with the quantum numbers of the antiquark. The sum over all intermediate states is approximated with only one on-mass-shell antiquark. The resulting equation, after partial-wave analysis, is a single-variable integral equation in the one off-shell quark momentum. It is simpler than the Bethe-Salpeter equation, which is a double-variable integral equation in both off-mass-shell momenta. This type of equation was formulated by Greenberg[2] in the $N$-quantum approximation and applied to the deuteron problem by Gross.[3] It has been applied to the problem of deeply bound composites by Bander, Chiu, Shaw, and Silverman.[4] It was applied to higher-order calculations in QED by Lepage.[5]

This bound-state equation reduces to the Breit-Fermi equation to first order in $v^2/c^2$; it is gauge invariant for the on-mass-shell antiquark and, in the limit that the antiquark is much heavier than the quark, it reduces to the Dirac equation for the quark field.

The quark-antiquark interaction that was used in this equation and the range of parameters considered were highly motivated by QCD. At short distances it has the form of an asymptotically free vector-gluon exchange and its only parameter is $\alpha_{\mathrm{QCD}}$ or the effective $\Lambda_R$ for the bound state. At large distances the interaction is an effective scalar exchange which is the linear confining potential, and its slope is found by the fit to the charmonium energy levels. The effect of multiquark states in decreasing the valence-quark–antiquark channel at high momentum is taken into account with a cutoff. In the nonrelativistic limit the interactions reduce to the Richardson[11] potential.

Starting with the electromagnetic-current matrix element of the quark current, we formulate the radiative decay rate in terms of the eigenfunctions of the relativistic bound-state equation. This includes the effect of the recoil of the final mesonic state, and to calculate this we boosted the rest-frame eigenfunctions of the final meson. This boost also rotates the spin states of the antiquark in the final state. We do the calculations numerically without resorting to a long-wavelength limit as in the dipole approximation.

The relativistic formulation automatically includes recoil or, equivalently, a sum over all multipoles, which is important in the strongly coupled QCD quarkonium. The $e^{i\mathbf{p}_\gamma\cdot\mathbf{r}}$ photon wave function, when evaluated for quarkonium transitions, gives a phase $p_\gamma r \simeq p_\gamma/\langle p_q\rangle$, where $\langle p_q \rangle$ is the average quark momentum. For the larger energy transitions, where $p_\gamma \simeq 600$ MeV, and for charmonium $\langle p_q \rangle \sim 500$ MeV or $b$-quarkonium $\langle p_q \rangle \sim 1000$ MeV, the phase cannot be ignored, which is to say the multipole expansion is not useful (the dipole approximation, of course, even drops the exponential). Keeping the entire exponential leads to a momentum $\delta$ function with $\mathbf{p}_f + \mathbf{p}_\gamma = \mathbf{p}_i$ and shows up as evaluating the final state with a boosted momentum ($\mathbf{p}_f = -\mathbf{p}_\gamma$ if $\mathbf{p}_i = 0$) and is called the recoil effect. This is in contrast with atomic spectra where $p_\gamma / \langle p_e \rangle \simeq \alpha$ or for nuclear spectra where $p_\gamma R \simeq$(few MeV)/(200 MeV) and the dipole or lowest-order multipole is appropriate.

To study the effect of the inclusion of asymptotic freedom in the interaction on the decay rates, we calculated the decay rates with a single vector-gluon exchange that did not include the asymptotic-freedom effects. This interaction in the nonrelativistic limit reduces to the Cornell potential.[12]

We have applied this formalism to the charmonium and $b$-quarkonium systems and found good agreement with experimental decay rates. Our calculations both with the asymptotic freedom included or left out of the vector interaction agree with the measured decay rates of the $b$-quarkonium system within the experimental errors. For the charmonium system, our calculated values for the electric-dipole-moment transitions are generally in good agreement with experiment. For the allowed and hindered magnetic dipole transitions, our calculations also agree with experiment. We observe that the decay rates calculated with the interaction that includes the effects of asymptotic freedom are, in general, closer to the experimental values than those calculated without it. Comparison with a nonrelativistic calculation of the decay rates with the potentials that arise from the nonrelativistic limit of the relativistic interactions clearly shows that the relativistic effects are very important in the radiative decay rates, especially for the case of charmonium.

We also include calculations of radiative transitions to and from the ${}^1P_1$ level and from the ${}^3D_2$ and ${}^1D_2$ levels in charmonium. While the $D$-wave levels are above the $D\bar{D}$ threshold, the $D\bar{D}$ system has $J^{PC} = 0^{++}, 1^{--}, 2^{++}, 3^{--}, \ldots,$ while charmonium $D$ waves have $J^{PC}$: ${}^3D_3$ $(3^{--})$, ${}^3D_2$ $(2^{--})$, ${}^3D_1$ $(1^{--})$, ${}^1D_2$ $(2^{-+})$. This leaves the ${}^3D_2$ and ${}^1D_2$ unable to decay to $D\bar{D}$ and possibly narrow enough to see their electromagnetic decays, especially if created as resonances in precision low-energy $p\bar{p}$ colliders.

In Sec. II the bound-state equation and the interactions and cutoffs are briefly described. In Sec. III, the relativistic formalism for the radiative decay rate is developed; and in Sec. IV we present the results of our calculation of the radiative decay rates of the charmonium and $b$-quarkonium systems. In Appendix A we show that the quark and antiquark currents give the same contributions. In Appendix B we find the formula to Lorentz transform the moving final-state meson wave function to one at rest. In Appendix C we show that the electromagnetic current is conserved for the valence-quark formulation. In Appendix D we take the nonrelativistic limit of our formalism to establish the connection with the electric and magnetic dipoles and the hindered magnetic nonrelativistic results. In Appendix E, we relate the matrix form of our equations to the spin wave-function form.

## II. FORMULATION OF RELATIVISTIC EQUATIONS FOR QUARK-ANTIQUARK SYSTEMS

In this section we are going to give a brief summary of the bound-state equation and interactions. They were reported and explained in detail in Ref. 1.

### A. Derivation of the bound-state equation

The relativistic bound-state equation that is used is based on the equation for a quark field $\psi(x)$ of mass $m_1$ coupled to a gauge potential $A_\mu$ and an effective QCD scalar potential $S(x)$:

$$(i\partial\!\!\!/ - m_1)\psi(x) = [gA\!\!\!/(x) + S(x)]\psi(x) \ . \tag{2.1}$$

The bound-state equation is obtained from the matrix element of Eq. (2.1) between the meson bound state of four-momentum $B$ and an antiquark state of mass $m_2$, momentum $\mathbf{p}$, and spin $\sigma$:

$$(i\partial\!\!\!/ - m_1)\langle \mathbf{p},\sigma \mid \psi(x) \mid B \rangle = \sum_n \langle \mathbf{p},\sigma \mid gA\!\!\!/(x) + S(x) \mid n \rangle \times \langle n \mid \psi(x) \mid B \rangle \ . \tag{2.2}$$

A complete set of states has been inserted into the right-hand side of Eq. (2.2). Up to this point, the equations are exact. The approximation that is going to be used is to take only the lowest order or single antiquark states in the sum. The justification for this valence-quark approximation is that, at least at large distances, the valence-quark model appears to work quite well; in this model the mesons are made up of only a quark and an antiquark. The effect of the inclusion of multiquark states or states composed of quarks and gluons will be discussed at the end of this section. In this approximation, Eq. (2.2) becomes an integral equation for the matrix elements of the quark field between the bound state and an antiquark state, with eigenvalues being the bound-state mass. By definition,

$$\Psi_\alpha(\mathbf{p},\sigma) = (2\pi)^3 \left[ \frac{2\omega(B)\omega}{m_2} \right]^{1/2} \langle \mathbf{p},\sigma \mid \psi_\alpha(0) \mid B \rangle \ , \tag{2.3}$$

where $\omega(B) = (B^2 + M^2)^{1/2}$, $\omega = (\mathbf{p}^2 + m_2{}^2)^{1/2}$, $m_2$ is the antiquark mass and $M$ is the unknown bound-state mass. The interactions for this problem are obtained from the matrix element of the fields or their source currents in the antiquark states. The bound-state equation becomes[1]

$$(\not{B}-\not{p}-m_1)\Psi(\mathbf{p},\sigma)=\sum_{\sigma'}\int\frac{d^3p'm_2}{(2\pi)^3\omega'}[V_V((p-p')^2)\gamma_\mu\Psi(\mathbf{p}',\sigma')\bar{v}(p',\sigma')\gamma^\mu v(p,\sigma)$$
$$+V_s((p'-p)^2)\Psi(\mathbf{p}',\sigma')\bar{v}(p',\sigma')v(p,\sigma)]\ , \tag{2.4}$$

where $\omega'=(\mathbf{p}'^2+m_2{}^2)^{1/2}$. The details of the interactions $V_V((p-p')^2)$ and $V_S((p-p')^2)$ will be discussed later.

### B. Angular-momentum decomposition

Equation (2.4) is solved by first performing an angular-momentum decomposition. A $4\times4$-component wave function $\Phi(p)$ arises naturally from the above equation:

$$\Phi_{\alpha\beta}(p)=\sum_\sigma \Psi_\alpha(\mathbf{p},\sigma)\bar{v}_\beta(p,\sigma)\ . \tag{2.5}$$

The wave equation for $\Phi$ is

$$(\not{B}-\not{p}-m_1)\Phi(p)=\int\frac{d^3p'}{(2\pi)^3 2\omega'}[V_V((p-p')^2)\gamma_\mu\Phi(p')\gamma^\mu+V_S((p-p')^2)\Phi(p')](\not{p}-m_2)\ . \tag{2.6}$$

As in the case of the Dirac equation, the large and small components are introduced; in this situation we have four $2\times2$ submatrices:

$$\Phi(p)=\begin{bmatrix}\tilde{G}_u(p) & \tilde{G}_d(p)\\ \tilde{F}_u(p) & \tilde{F}_d(p)\end{bmatrix}\ . \tag{2.7}$$

$\tilde{G}$ and $\tilde{F}$ denote the upper and lower components for the quark, and $u$ and $d$ the upper and lower components for the antiquark. In the nonrelativistic limit, $\tilde{G}_d$ is the large component, $\tilde{G}_u$ and $\tilde{F}_d$ are of order $v/c$, and $\tilde{F}_u$ is of order $(v/c)^2$. Since the antiquark is on the mass shell, its wave function satisfies the free Dirac equation, as can be seen from Eq. (2.5):

$$\Phi(p)(\not{p}+m_2)=0\ . \tag{2.8}$$

This results in the following relations, leaving $\tilde{G}_d$ and $\tilde{F}_d$ as independent:

$$\tilde{G}_u(p)=-\tilde{G}_d(p)\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega+m_2}\right],\quad \tilde{F}_u(p)=-\tilde{F}_d(p)\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega+m_2}\right]\ . \tag{2.9}$$

The equations for the two independent components are

$$\begin{bmatrix}(M-\omega-m_1)\tilde{G}_d(p)+\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{F}_d(p)\\(M-\omega+m_1)\tilde{F}_d(p)+\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{G}_d(p)\end{bmatrix}$$
$$=\int\frac{d^3p'}{(2\pi)^3 2\omega'}\left[V_V((p-p')^2)\begin{bmatrix}(\omega'+m_2)(\tilde{G}_d+\boldsymbol{\sigma}\cdot\tilde{F}_u\boldsymbol{\sigma})-\tilde{G}_u\boldsymbol{\sigma}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{F}_d+i\mathbf{p}\cdot(\boldsymbol{\sigma}\times\tilde{F}_d\boldsymbol{\sigma})\\(\omega'+m_2)(\tilde{F}_d+\boldsymbol{\sigma}\cdot\tilde{G}_u\boldsymbol{\sigma})-\tilde{F}_u\boldsymbol{\sigma}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{G}_d+i\mathbf{p}\cdot(\boldsymbol{\sigma}\times\tilde{G}_d\boldsymbol{\sigma})\end{bmatrix}\right.$$
$$\left.+V_S((p-p')^2)\begin{bmatrix}-(\omega+m_2)\tilde{G}_d-\tilde{G}_u\boldsymbol{\sigma}\cdot\mathbf{p}\\(\omega+m_2)\tilde{F}_d+\tilde{F}_u\boldsymbol{\sigma}\cdot\mathbf{p}\end{bmatrix}\right]\ , \tag{2.10}$$

where the components of $\Phi$ on the right-hand side are evaluated at $p'$. It is easier to perform the angular-momentum decomposition by going to the direct-product state-vector representation for the quark and antiquark spins instead of the present matrix form where rows are the quark 2 represents and columns are the antiquark $\bar{2}$ representation. We do this by performing a $\sigma_y$ rotation on the antiquark $\bar{2}$ representation, taking it to the 2 representation. We then define $|G_u\rangle$, $|G_d\rangle$, $|F_u\rangle$, and $|F_d\rangle$ to be the direct-product state-vector representations which are isomorphic to $\tilde{G}_u\sigma_y$, $\tilde{G}_d\sigma_y$, $\tilde{F}_u\sigma_y$, and $\tilde{F}_d\sigma_y$, respectively. All Pauli matrices on the left act on the quark spin and are now called $\boldsymbol{\sigma}_1$. Using $\sigma_y\boldsymbol{\sigma}\sigma_y=-\boldsymbol{\sigma}_T$ for Pauli matrices on the right brings them to the left where they are called $\boldsymbol{\sigma}_2$ and act on the antiquark spin. We now obtain

$$\begin{bmatrix}(M-\omega-m_1)|G_d\rangle+\boldsymbol{\sigma}_1\cdot\mathbf{p}|F_d\rangle\\(M-\omega+m_1)|F_d\rangle+\boldsymbol{\sigma}_1\cdot\mathbf{p}|G_d\rangle\end{bmatrix}$$
$$=\int\frac{d^3p'}{(2\pi)^3 2\omega'}\left[V_V((p-p')^2)\begin{bmatrix}(\omega'+m_2)(|G_d\rangle-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|F_u\rangle)+\boldsymbol{\sigma}_2\cdot\mathbf{p}|G_u\rangle-[\mathbf{p}\cdot\boldsymbol{\sigma}_1+i\mathbf{p}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]|F_d\rangle\\(\omega'+m_2)(|F_d\rangle-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|G_u\rangle)+\boldsymbol{\sigma}_2\cdot\mathbf{p}|F_u\rangle-[\mathbf{p}\cdot\boldsymbol{\sigma}_1+i\mathbf{p}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]|G_d\rangle\end{bmatrix}\right.$$
$$\left.+V_S((p-p')^2)\begin{bmatrix}\boldsymbol{\sigma}_2\cdot\mathbf{p}|G_u\rangle-(\omega+m_2)|G_d\rangle\\-\boldsymbol{\sigma}_2\cdot\mathbf{p}|F_u\rangle+(\omega+m_2)|F_d\rangle\end{bmatrix}\right]\ . \tag{2.11}$$

It can be shown that the nonrelativistic limit of these equations gives an equation for $|G_d\rangle$, which to order $v^2/c^2$ coincides with the Darwin-Breit Hamiltonian. Also, in the limit $m_2 \gg m_1$, it reduces to the Dirac equation for the quark in the potential of a static antiquark. In order to do the angular-momentum decomposition, the states $|G_d\rangle$ and $|F_d\rangle$, of total angular momentum $j$ and $\hat{\mathbf{z}}$-component $m_j$, are expanded in terms of the allowed $L$ and $S$, which are not good quantum numbers relativistically:

$$\begin{aligned}|G_d(\mathbf{p})\rangle &= g_-(p)\,|j,m_j;L=j-1,S=1\rangle + g_+(p)\,|j,m_j;L=j+1,S=1\rangle\ ,\\ |F_d(\mathbf{p})\rangle &= f_0(p)\,|j,m_j;L=j,S=0\rangle + f_1(p)\,|j,m_j;L=j,S=1\rangle\ ,\end{aligned} \tag{2.12}$$

for the natural-parity states $[P=(-1)^j]$; and for the unnatural-parity states $[P=(-1)^{j+1}]$ we have

$$\begin{aligned}|G_d(\mathbf{p})\rangle &= g_0(p)\,|j,m_j;L=j,S=0\rangle + g_1(p)\,|j,m_j;L=j,S=1\rangle\ ,\\ |F_d(\mathbf{p})\rangle &= f_-(p)\,|j,m_j;L=j-1,S=1\rangle + f_+(p)\,|j,m_j;L=j+1,S=1\rangle\ .\end{aligned} \tag{2.13}$$

The angular-momentum decomposition of the interactions are

$$V_{V,S}((p-p')^2) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')V_{V,S}^l(p,p')\ , \tag{2.14}$$

$$K_{V,S}^l(p,p') = \frac{p'^2}{(2\pi)^3 2\omega'} V_{V,S}^l(p,p')\ .$$

Substituting these in Eq. (2.11), and using the space and spin properties of Eqs. (2.12) and (2.13), the integral equation for the angular-momentum components of the wave functions reduces to the form of a single integral over the magnitude of the momentum

$$T(p)\begin{bmatrix} g_-(p)\\ g_+(p)\\ f_0(p)\\ f_1(p)\end{bmatrix} = \int dp' K(p,p')\begin{bmatrix} g_-(p)\\ g_+(p)\\ f_0(p)\\ f_1(p)\end{bmatrix}\ , \tag{2.15}$$

and the matrices $T(p)$ and $K(p,p')$ are given in Ref. 1.

### C. Relativistic interactions and the cutoff

The QCD interaction between quark and antiquark used in the integral equation consists of a vector-gluon exchange with asymptotic-freedom effects included that dominate at short distances and an effective scalar exchange which is the linear confining potential that dominates at large distances. This form of the interaction is motivated by QCD and the requirements of a relativistic equation.

The asymptotic-freedom-corrected vector-gluon exchange is taken for three flavors as

$$V_V(q^2) = -4\pi \left[\frac{16\pi}{27}\right] \left[\frac{1}{-q^2\ln(1-q^2/\Lambda_R{}^2)} - \frac{\Lambda_R{}^2}{(q^2)^2}\right]\ , \tag{2.16}$$

where the pole in the logarithmic term has been displaced to the unphysical point $q^2=\Lambda_R{}^2$, following Richardson,[11] and the $1/(q^2)^2$ linear part has been removed from the vector part since it would lead to a Klein paradox.

In the nonrelativistic limit, i.e., $q^2 \to -\mathbf{q}^2$, it reduces to the Richardson potential with its linear in $r$ part subtracted out. The linear potential is restored as a scalar potential. The relativistic form of the linear part is not known. However, since it is important only at smaller $\mathbf{q}^2$, where the retardation in converting to $-q^2$ is only a $v^2/c^2$ correction, the difference does not have a significant effect on the calculation. We take the nonrelativistic form of the scalar exchange to be a linear potential with the slope $\kappa_S$, which for numerical purposes levels off at a distance $b_0$ or at a height $\kappa b_0$:

$$V_S(r) = \kappa_S r\theta(b_0 - r) + \kappa_S b_0 \theta(r-b_0)\ . \tag{2.17}$$

$b_0$ is chosen large enough so that there are no numerical differences between this potential and an infinitely rising one. We take the Fourier transform of this and then change $|\mathbf{q}^2| \to |-q^2|$, and include $\omega/m$ with the $\delta$ function for Lorentz invariance[1] to obtain our relativistic scalar interaction

$$\begin{aligned}V_S(\kappa, q=(-q^2)^{1/2}) = (2\pi)^3\kappa_S b_0 \frac{\omega}{m}\delta^3(\mathbf{q}) &+ \frac{4\pi}{q^4}\kappa_S[b_0 q\sin(b_0 q)\\ &+ 2\cos(b_0 q) - 2]\ .\end{aligned} \tag{2.18}$$

In practice we chose the value of this ramp height to be 3 GeV, so that the results are insensitive to it.

In the sum over states in Eq. (2.2), only the antiquark states were considered. To include the effect of other quantum states in damping out the two-particle intermediate states at high momentum, we introduce a large-momentum cutoff function, $S_\epsilon(\Lambda_0, p')$, which multiplies the kernel in the integral equation. This function approaches zero for $p \gg \Lambda_\epsilon$ and approaches 1 for $\Lambda_\epsilon \to \infty$. The form of the cutoff function that we use is

$$S_\epsilon(\Lambda_0, p) = \left[\frac{\Lambda_\epsilon{}^2}{\Lambda_\epsilon{}^2 + p^2}\right]^{1+\epsilon} \quad \text{with } \Lambda_\epsilon = \Lambda_0\sqrt{1+\epsilon}\ .$$

Expanding in $p^2/\Lambda_\epsilon{}^2$, this becomes

$$S_\epsilon(\Lambda_0,p)\simeq\left[1+(1+\epsilon)\frac{p^2}{\Lambda_\epsilon^2}\right]^{-1}=\left[1+\frac{p^2}{\Lambda_0^2}\right]^{-1},$$

so that the predominant corrections to the wave function at low momentum is independent of $\epsilon$.

For $\epsilon=0$ we find that the leptonic decay width is slowly divergent, but is regulated for $\epsilon>0$. In our calculations, we have used both $\epsilon=0.01$, which is essentially the $S_1(\Lambda,p)$ case in Ref. 1, and $\epsilon=0.25$. We saw very little change between these two cases in the spectra and radiative decay widths.

## III. DERIVATION OF THE RADIATIVE DECAY RATE

Having found the eigenvalues and eigenfunctions of the bound-state equation [Eq. (2.15)], such as $f_0(p)$, $f_1(p)$, $g_+(p)$, and $g_-(p)$ for the natural-parity states, we can now find an expression for the radiative decay rate in terms of these functions. The decay rate is defined as

$$\Gamma=\frac{(2\pi)^4}{2j+1}\int\frac{\delta^4(B_2'-B_1+k)}{2k}\times\sum_{m_j,m_j',\lambda}|M_\mu\epsilon^\mu(k,\lambda)|^2d^3k\,d^3B_2',\quad(3.1)$$

where $j$ is the total angular momentum of the initial meson, $m_j$ and $m_j'$ are the $\hat{\mathbf{z}}$ components of the angular momentum of the initial and final mesons, $\lambda$ is the photon polarization, and $\mathbf{k}$ and $\mathbf{B}_2'$ are the decay photon and the final meson momenta, respectively.

The electromagnetic-current matrix element $M_\mu$ is

$$M_\mu=eQ\langle B_2'm_j'\,|:\bar\psi(0)\gamma_\mu\psi(0):|\,B_1m_j\rangle\,,\quad(3.2)$$

where $eQ$ is the electric charge of the constituent quark and $\langle B_2',m_j'|$ and $|B_1,m_j\rangle$ are the final and initial bound states. To find the decay rate we have to integrate over the photon momentum. Instead, we use the physical argument that after we have summed over the initial and final polarizations of the mesons, the decay probability does not depend on the direction of the photon, and, hence, the integration over the direction of the photon momentum gives $4\pi$. The decay rate becomes

$$\Gamma=\frac{4\pi k\,(2\pi)^4}{2j+1}\frac{M_1^2+M_2^2}{4M_1^2}\sum_{m_j,m_j',\lambda}|M_\mu\epsilon^\mu(k,\lambda)|^2,\quad(3.3)$$

where $k=(M_1^2-M_2^2)/2M_1$ is the energy of the photon, with $M_1$ and $M_2$ the masses of the initial and final mesons, respectively. To evaluate the current we insert a complete set of antiquark states in Eq. (3.2) and use the definition (2.3) to obtain

$$M_\mu=2eQ\sum_\sigma\int\frac{d^3p}{(2\pi)^3}\langle B_2'm_j'\,|\,\bar\psi(0)\,|\,\mathbf{p},\sigma\rangle\times\gamma_\mu\Psi_{B_1m_j}(\mathbf{p},\sigma)\left[\frac{m}{2\omega(B_1)\omega}\right]^{1/2},\quad(3.4)$$

where the factor of 2 arises from equal contributions of both quark and antiquark electromagnetic currents due to the correlated motion in the two-body bound state (see Appendix A).

Now we want to take into consideration the effect of the recoil of the final meson state. We take the direction of the photon momentum $\mathbf{k}$ to be the $z$ direction, and the final meson momentum is $\mathbf{B}_2'=-\mathbf{k}$. Let $\Lambda$ be the Lorentz transformation that brings the moving final meson to rest, i.e.,

$$\Lambda B_2'=B_2=(M_2,0)\,.\quad(3.5)$$

We then define $U(\Lambda)$ to be the corresponding transformation on the Hilbert space of the mesonic states[13]

$$\langle B_2',m_j'\,|\,U^{-1}(\Lambda)=\langle B_2,m_j'\,|\left[\frac{M_2}{\omega(B_2')}\right]^{1/2},\quad(3.6)$$

where $m_j'$ does not change since the boost is along the $z$ axis.

We use the Lorentz transformation of the antiquark states[13]

$$U[\Lambda]\,|\,\mathbf{p},\sigma\rangle=\left[\frac{\omega(\Lambda p)}{\omega(p)}\right]^{1/2}\sum_{\sigma'}\bar D_{\sigma\sigma'}\,|\,\Lambda\mathbf{p},\sigma'\rangle\,,\quad(3.7)$$

where $\Lambda\mathbf{p}$ is the antiquark momentum boosted by the Lorentz transformation $\Lambda$, which is a boost with the velocity of the final meson in the negative $z$ direction. The Lorentz transformations $S(\Lambda)$ for the spinor fields

$$U(\Lambda)\bar\psi(x)U^{-1}(\Lambda)=\bar\psi(\Lambda x)S(\Lambda)\quad(3.8)$$

is given by[14]

$$S(\Lambda)=\begin{bmatrix}a & \sigma\cdot\mathbf{b}\\ \sigma\cdot\mathbf{b} & a\end{bmatrix},\quad(3.9)$$

where

$$a=\left[\frac{\omega(B_2')+M_2}{2M_2}\right]^{1/2}=\frac{M_1+M_2}{2\sqrt{M_1M_2}}\,,$$
$$\mathbf{b}=\frac{a\mathbf{B}_2'}{\omega(B_2')+M_2}=-b\hat{\mathbf{z}},\quad b=\frac{M_1-M_2}{2\sqrt{M_1M_2}}\,.\quad(3.10)$$

Inserting $U^{-1}(\Lambda)U(\Lambda)$ on both sides of $\bar\psi(0)$ in Eq. (3.4) and using Eqs. (3.6)–(3.8), we obtain

$$M_\mu=2eQ\sum_\sigma\int d^3p\left\langle B_2,m_j'\left|\bar\psi(0)S(\Lambda)\gamma_\mu\sum_{\sigma'}\bar D_{\sigma\sigma'}(\Lambda)\right|\Lambda\mathbf{p},\sigma'\right\rangle\Psi_{B_1,m_j}(\mathbf{p},\sigma)\left[\frac{mM_2\omega(\Lambda p)}{(2\pi)^6 2M_1\omega\omega(B_2')}\right]^{1/2}.\quad(3.11)$$

$\bar D_{\sigma\sigma'}(\Lambda)$ is found (Appendix B) to be

$$\bar D_{\sigma\sigma'}(\Lambda)=-\bar v(p,\sigma)S^{-1}(\Lambda)v(\Lambda p,\sigma')\,.\quad(3.12)$$

By using Eq. (3.12) in Eq. (3.11), we find

$$M_\mu=\frac{-2Qem}{(2\pi)^6[M_1\omega(B_2')]^{1/2}}\sum_{\sigma,\sigma'}\int\frac{d^3p}{2\omega}[\bar{\Psi}_{B_2m_j'}(\Lambda p,\sigma')S(\Lambda)\gamma_\mu\Psi_{B_1m_j}(p,\sigma)]\bar{v}(p,\sigma)S^{-1}(\Lambda)v(\Lambda p,\sigma') . \quad (3.13)$$

Performing the $\sigma,\sigma'$ sums using the definition of $(\Phi_{B_1m_j})_{\alpha\beta}$ in Eq.(2.5), we find

$$M_\mu=\frac{-2meQ}{(2\pi)^6[M_1\omega(B_2')]^{1/2}}\int\frac{d^3p}{2\omega}\mathrm{Tr}[\gamma_0\Phi'^{\dagger}_{B_2m_j'}(\Lambda p)\gamma_0S(\Lambda)\gamma_\mu\Phi_{B_1m_j}(p)S^{-1}(\Lambda)] . \quad (3.14)$$

The matrix $\Phi_{B_1m_j}(p)$ is the eigenfunction matrix of the bound-state equation for the initial meson state, and $\Phi'_{B_2m_j'}(\Lambda p)$ is the eigenfunction for the final meson state with the boosted momentum.

Three points should be mentioned.

(a) It can be shown that the above current (3.14) is conserved (Appendix C).

(b) The decay rate is more general than the lowest-order multipole (i.e., electric dipole or magnetic dipole) or static limit approximation. Equation (3.14) also takes into account the effect of the recoil of the final meson and the relativistic mixing of terms.

(c) It can be shown that in the nonrelativistic limit (3.14) coincides exactly with the nonrelativistic formula for radiative decay (Appendix D).

To find the decay rate we substitute the relation for the current, Eq. (3.14), into the formula for the decay rate, Eq. (3.3). It can be shown that the decay rate for emitting a positive-helicity photon is equal to that for emitting a negative-helicity photon when summed over spins; therefore, the total decay rate would be twice that for a negative-helicity photon:

$$\Gamma=\frac{16\alpha Q^2k}{2j+1}\sum_{m_j,m_j'}\left[\int\frac{d^3p\, m}{(2\pi)^3 2\omega M_1}\mathrm{Tr}[\mathbf{J}]\cdot\hat{\mathbf{e}}^*_-\right]^2 , \quad (3.15)$$

where $\mathbf{J}$ is defined as

$$\mathbf{J}=\gamma_0\Phi'^{\dagger}_{B_2m_j'}(\Lambda p)\gamma_0S(\Lambda)\boldsymbol{\gamma}\Phi_{B_1m_j}(p)S^{-1}(\Lambda)$$

and (3.16)

$$\hat{\mathbf{e}}_+=\frac{-1}{\sqrt{2}}(\hat{\mathbf{e}}_x+i\hat{\mathbf{e}}_y),\quad \hat{\mathbf{e}}_-=\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_x-i\hat{\mathbf{e}}_y) .$$

Before taking the trace we substitute for $\Phi_{B_1m_j}(p)$ and $\Phi'_{B_2m_j'}(\Lambda p)$ using Eq.(2.7), and we use Eq. (2.9) to substitute for $\tilde{F}_u$ and $\tilde{G}_u$ in terms of $\tilde{F}_d$ and $\tilde{G}_d$. Noting that $\hat{\mathbf{e}}^*_-=-\hat{\mathbf{e}}_+$, we have

$$\begin{aligned}\mathrm{Tr}[\mathbf{J}]\cdot\hat{\mathbf{e}}_+=&\frac{a}{\sqrt{2}}\mathrm{Tr}[(C+i\mathbf{D}\cdot\boldsymbol{\sigma})(\tilde{F}'^{\dagger}_d\sigma_+\tilde{G}_d+\tilde{G}'^{\dagger}_d\sigma_+\tilde{F}_d)]\\&-\frac{b}{\sqrt{2}}\mathrm{Tr}[(C+i\mathbf{D}\cdot\boldsymbol{\sigma})\\&\times(\tilde{G}'^{\dagger}_d\sigma_+\tilde{G}_d+\tilde{F}'^{\dagger}_d\sigma_+\tilde{F}_d)] ,\end{aligned} \quad (3.17)$$

where $C$ and $\mathbf{D}$ are defined as

$$\begin{aligned}C&=a\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}'+\mathbf{b}\cdot\boldsymbol{\kappa}'-\boldsymbol{\kappa}\cdot\mathbf{b}-a ,\\ \mathbf{D}&=a\boldsymbol{\kappa}\times\boldsymbol{\kappa}'+\mathbf{b}\times\boldsymbol{\kappa}'-\boldsymbol{\kappa}\times\mathbf{b} ,\end{aligned} \quad (3.18)$$

$$\boldsymbol{\kappa}=\frac{-\mathbf{p}}{\omega+m_2},\quad \boldsymbol{\kappa}'=\frac{-\Lambda\mathbf{p}}{\omega(\Lambda p)+m_2} , \quad (3.19)$$

and

$$\sigma_\pm=\sigma_1\pm i\sigma_2 .$$

$\tilde{F}_d(p)$ and $\tilde{G}_d(p)$ refer to the initial meson, and $\tilde{F}'^{\dagger}_d(\Lambda p)$ and $\tilde{G}'^{\dagger}_d(\Lambda p)$ refer to the final meson.

We then transform the $2\times2$ matrices $\tilde{G}$ and $\tilde{F}$ to their direct-product state representation, see Appendix E, and take the simplified notation $|G\rangle\equiv|G_d(\mathbf{p})\rangle$ and $|F\rangle\equiv|F_d(\mathbf{p})\rangle$ for the initial states and $|G'\rangle$ and $|F'\rangle$ for the final states. This gives, for the trace

$$\begin{aligned}\mathrm{Tr}[\mathbf{J}]\cdot\hat{\mathbf{e}}_+=&a(\langle F'|W_+|G\rangle+\langle G'|W_+|F\rangle)\\&-b(\langle G'|W_+|G\rangle+\langle F'|W_+|F\rangle) ,\end{aligned} \quad (3.20)$$

where

$$W_+=\frac{1}{\sqrt{2}}(C-i\mathbf{D}\cdot\boldsymbol{\sigma}_2)\sigma_{1+} . \quad (3.21)$$

Noting that the vector $\mathbf{b}$ is in the negative $\hat{\mathbf{z}}$ direction and $\mathbf{b}\times\boldsymbol{\kappa}$, $\mathbf{b}\times\boldsymbol{\kappa}'$, and $\boldsymbol{\kappa}'\times\boldsymbol{\kappa}$ are in the $\hat{\boldsymbol{\phi}}$ direction (that of the azimuthal angle $\phi$ of the antiquark momentum), we obtain

$$\mathbf{D}=\frac{iD}{\sqrt{2}}(\hat{\mathbf{e}}_-e^{i\phi}+\hat{\mathbf{e}}_+e^{-i\phi}) , \quad (3.22)$$

where $D$ is the magnitude of $\mathbf{D}$. Defining spin-related operators as

$$\begin{aligned}S_+&=\tfrac{1}{2}(\sigma_{1+}+\sigma_{2+}),\quad T_+=\tfrac{1}{2}(\sigma_{1+}-\sigma_{2+}) ,\\ S_-&=\tfrac{1}{2}(\sigma_{1-}+\sigma_{2-}),\quad T_-=\tfrac{1}{2}(\sigma_{1-}-\sigma_{2-}) ,\end{aligned} \quad (3.23)$$

$W_+$ can be written in terms of $S_+$, $S_-$, $T_+$, and $T_-$:

$$W_+=\frac{1}{\sqrt{2}}\left\{C+\frac{D}{2}[(S_--T_-)e^{i\phi}-(S_+-T_+)e^{-i\phi}]\right\}(S_++T_+) . \quad (3.24)$$

In Eq. (3.20) we substitute for the eigenstates $|F\rangle$, $|G\rangle$, $\langle G'|$, and $\langle F'|$ their angular-momentum forms, Eqs. (2.12) or Eqs. (2.13), depending on whether it is a natural- or unnatural-parity eigenstate.

We begin with the case of a radiative decay from a natural-parity meson state to another natural-parity meson, and later we will show how this can be generalized. Each total angular-momentum eigenstate in Eq. (3.20) is expanded in terms of spin and orbital-angular-momentum basis states. Then Eq. (2.12) becomes (see Appendix E)

$$|G\rangle = g_-(p)\sum_{m_l+m_s=m_j} C^{j,m_j}_{j-1,m_l;1,m_s} Y^{m_l}_{j-1}(\theta,\phi)\,|S=1,m_s\rangle + g_+(p)\sum_{m_l+m_s=m_j} C^{j,m_j}_{j+1,m_l;1,m_s} Y^{m_l}_{j+1}(\theta,\phi)\,|S=1,m_s\rangle \quad (3.25)$$

and

$$|F\rangle = f_0(p)Y^{m_j}_j(\theta,\phi)\,|S=0,0\rangle + f_1(p)\sum_{m_l+m_s=m_j} C^{j,m_j}_{j,m_l;1,m_s} Y^{m_l}_j(\theta,\phi)\,|S=1,m_s\rangle\ , \quad (3.26)$$

where the $C$'s are Clebsch-Gordan coefficients. For the angular-momentum decomposition of the final states, we have

$$\langle G'| = \sum_{m_l'+m_s'=m_j'} [g'_+(\Lambda p)C^{j',m_j'}_{j'+1,m_l';1,m_s'} Y^{m_l'*}_{j'+1}(\theta_\Lambda,\phi)\langle S'=1,m_s'| + g'_-(\Lambda p)C^{j',m_j'}_{j'-1,m_s';1,m_l'} Y^{m_l'*}_{j'-1}(\theta_\Lambda,\phi)\langle S'=1,m_s'|\,]\ , \quad (3.27)$$

where the spherical harmonics are complex conjugated and depend on the direction of the boosted antiquark momentum $\Lambda\mathbf{p}$, and $f'$ and $g'$ are the eigenfunctions of the final-state meson evaluated at the magnitude of the boosted antiquark momentum $|\Lambda\mathbf{p}|$. There is a similar equation for $\langle F'|$.

Substituting Eqs. (3.25)–(3.27) and Eqs. (3.24) into Eq. (3.20), we have operators ($S$'s and $T$'s) acting on the spin basis states. The effect of these operators are

$$\begin{aligned}
&S_+\,|1,m_s\rangle = \sqrt{2}\,|1,m_s+1\rangle\ ,\\
&S_-\,|1,m_s\rangle = \sqrt{2}\,|1,m_s-1\rangle\ ,\\
&T_+\,|0,0\rangle = -\sqrt{2}\,|1,1\rangle\ ,\\
&T_-\,|0,0\rangle = \sqrt{2}\,|1,-1\rangle\ ,\\
&T_+\,|1,m_s\rangle = \sqrt{2}\,|0,0\rangle\delta_{m_s,-1}\ ,\\
&T_-\,|1,m_s\rangle = -\sqrt{2}\,|0,0\rangle\delta_{m_s,1}\ .
\end{aligned} \quad (3.28)$$

Using Eq. (3.28) and the orthogonality of the spin states, we find relationships between $m_s$ and $m_s'$ of the initial and final mesons. When integrating over $\phi$ we obtain another relationship between $m_l$ and $m_l'$, but since $m_l+m_s=m_j$ and $m_l'+m_s'=m_j'$, we finally obtain a relation between $m_j$ and $m_j'$, which is

$$m_j' = m_j + 1\ . \quad (3.29)$$

This is clear physically, since we are considering a negative-helicity photon being emitted in the $z$ direction.

Since the boost is along the negative $z$ direction, the azimuthal angle of $\Lambda\mathbf{p}$ is equal to $\phi$, the azimuthal angle of $\mathbf{p}$. But the angle $\theta$, which is the angle between $\mathbf{p}$ and the $z$ axis, is not equal to the angle $\theta_\Lambda$ between $\Lambda\mathbf{p}$ and the $z$ axis, and, therefore, we cannot simply perform the integration over the angle $\theta$ by using the orthogonality relations of the spherical harmonics, as is done in the nonrelativistic formalism.

We find that the angle to the $z$ axis of $\Lambda\mathbf{p}$ as a function of $\theta$ and $p$ is

$$\theta_\Lambda = \arctan\left[\frac{p\sin\theta}{\gamma p\cos\theta+\gamma\beta\omega}\right]\ , \quad (3.30)$$

where $\beta$ is the magnitude of the velocity of the boost along the negative $z$ direction. The magnitude of the boosted antiquark momentum, in terms of $\theta$ and $p$, is

$$|\Lambda\mathbf{p}| = [p^2\sin^2\theta+\gamma^2(p\cos\theta+\beta\omega)^2]^{1/2}\ , \quad (3.31)$$

where $\beta = k/(M_2{}^2+k^2)^{1/2} \equiv V/c$, and $V$ is the velocity of the recoil and boost.

In terms of the magnitudes of $\kappa$ and $\kappa'$, Eq. (3.19), and angles $\theta$ and $\theta_\Lambda$, we have for $C$ and $D$ in Eq. (3.18), and Eq. (3.22),

$$\begin{aligned}
C &= a\kappa\kappa'\cos(\theta_\Lambda-\theta)+b\kappa'\cos(\theta_\Lambda)-b\kappa\cos\theta-a\ ,\\
D &= a\kappa\kappa'\sin(\theta_\Lambda-\theta)+b\kappa\sin\theta+b\kappa'\sin(\theta_\Lambda)\ .
\end{aligned} \quad (3.32)$$

After the above operations are carried out in Eq. (3.20), the $\theta$ and $p=|\mathbf{p}|$ integrations in Eq. (3.15) are carried out numerically.

From the expanded equations for the case of the radiative decay of a natural-parity meson to another natural-parity meson, we find that the relation for the decay of a natural-parity meson to an unnatural-parity meson, or vice versa, is obtained by the interchange of $a$ and $b$, and, also, for the meson that is the unnatural-parity state, by the interchange $g_+\to f_+$, $g_-\to f_-$, $f_0\to g_0$, and $f_1\to g_1$, as can be seen from Eq. (3.20).

In order to illustrate the importance of the recoil in quarkonia decays we expand Eqs. (3.30) and (3.31) to lowest order in $V/c$ and in quark velocity $v/c=p/\omega$ and find

$$\theta_\Lambda \simeq \arctan\left[\frac{\sin\theta}{\cos\theta+V/v}\right]\ , \quad (3.33)$$

$$|\Lambda \mathbf{p}| \simeq p(1+2\cos\theta V/v + V^2/v^2)^{1/2} . \quad (3.34)$$

In large-energy decays we find for the velocity ratio

$$V/v \simeq \frac{k}{2m} \Big/ \frac{p}{m} = \tfrac{1}{2}k/p .$$

For typical quark momentum $p \simeq 500$ MeV and transitions where $k \simeq 600$ MeV (as in the $\psi' \to \eta_c + \gamma$ hindered magnetic transition), we find we cannot neglect the recoil since the recoil momentum is of the same order as the internal momentum. This effect is of order 1, even though $V/c \simeq 0.2$. The non-normality of the angularly shifted spherical harmonics with the unshifted ones from (3.33) and the momentum shift in (3.34), separates the wave functions in momentum space and decreases the overlap and the transition rates.

As an illustration of the importance of the recoil, we give here the analytic result for the electric dipole moment for a $1P \to 1S$ transition to a circularly polarized photon in a three-dimensional harmonic oscillator with $V(r) = \frac{1}{2}kr^2$. In terms of the average velocity squared of the particle in the oscillator, $\langle v^2 \rangle = \frac{3}{2}(k/m^3)^{1/2}$, and the recoil velocity $V$ depending on the total mass, the electric dipole matrix element is

TABLE I. (a) Experimental and theoretical transition rates of charmonium (in keV). (b) Photon energies of charmonium radiative transitions (in MeV).

(a)

| No. | Decay mode | Experiment | Relativistic asymptotic freedom + linear | Relativistic Coulomb + linear | Nonrelativistic adjusted by $k_{\rm rel}$ |
|---|---|---|---|---|---|
| | | | Electric dipole transitions | | |
| (1) | $\psi' \to \chi_2 + \gamma$ | $17 \pm 4$ | 24 | 29 | 66 |
| (2) | $\psi' \to \chi_1 + \gamma$ | $19 \pm 4$ | 23 | 25 | 94 |
| (3) | $\psi' \to \chi_0 + \gamma$ | $20 \pm 4$ | 12 | 10 | 73 |
| (4) | $\chi_2 \to \psi + \gamma$ | $429^{+266}_{-163}$ | 210 | 221 | 463 |
| (5) | $\chi_1 \to \psi + \gamma$ | $< 390$ | 100 | 134 | 296 |
| (6) | $\chi_0 \to \psi + \gamma$ | $97 \pm 38$ | 56 | 66 | 160 |
| | | | Magnetic dipole transitions | | |
| (1) | $\psi' \to \eta_c' + \gamma$ | $2.3 \to 0.43$ | 0.60* | 0.53* | 1.44 |
| (2) | $\psi \to \eta_c + \gamma$ | $0.92 \pm 0.45$ | 0.91 | 1.06 | 2.98 |
| | | | Hindered magnetic transitions | | |
| (1) | $\psi' \to \eta_c + \gamma$ | $0.6 \pm 0.2$ | 1.0 | 1.3 | 1.8 |
| (2) | $\eta_c' \to \psi + \gamma$ | | 8.3 | 10.0 | 1.3 |

(b)

| No. | Decay mode | Experiment | Relativistic asymptotic freedom + linear $k_{\rm rel}$ | Relativistic Coulomb + linear | Nonrelativistic |
|---|---|---|---|---|---|
| | | | Electric dipole transitions | | |
| (1) | $\psi' \to \chi_2 + \gamma$ | 128 | 153 | 160 | 161 |
| (2) | $\psi' \to \chi_1 + \gamma$ | 172 | 206 | 197 | 161 |
| (3) | $\psi' \to \chi_0 + \gamma$ | 261 | 275 | 272 | 161 |
| (4) | $\chi_2 \to \psi + \gamma$ | 429 | 402 | 399 | 397 |
| (5) | $\chi_1 \to \psi + \gamma$ | 389 | 353 | 364 | 397 |
| (6) | $\chi_0 \to \psi + \gamma$ | 303 | 286 | 291 | 397 |
| | | | Magnetic dipole transitions | | |
| (1) | $\psi' \to \eta_c' + \gamma$ | 94 | 39 | 50 | 0 |
| (2) | $\psi \to \eta_c + \gamma$ | 114 | 99 | 107 | 0 |
| | | | Hindered magnetic transitions | | |
| (1) | $\psi' \to \eta_c + \gamma$ | 638 | 622 | 633 | 544 |
| (2) | $\eta_c' \to \psi + \gamma$ | 459 | 505 | 498 | 544 |

$$M=\int \frac{d^3p}{(2\pi)^3}\phi^*_{1S}(\Lambda \mathbf{p})\frac{\mathbf{p}\cdot\epsilon_+}{m}\phi_{1P}^{m_l=-1}(\mathbf{p}) \ ,$$

$$M=-i\left[\frac{\langle v^2\rangle}{3}\right]^{1/2}\exp\left[-\frac{3}{8}\frac{V^2}{\langle v^2\rangle}\right] .$$

This shows the rapid falloff in the overlap of the wave functions depending on the ratio of the recoil velocity to the average velocity in the wave function.

## IV. RESULTS AND CONCLUSIONS

The decay rates of different charmonium and $b$-quarkonium radiative decays have been calculated by using the formalism that was developed in previous sections. Parameters of the potentials and cutoffs for the calculation of the wave functions for the decays were taken from previous spectral fits.[1] These parameters include the slope of the linearly rising scalar potential, which was found to be $\kappa_S=0.15$ GeV$^2$, and $\Lambda_R=0.4$ GeV. The large momentum cutoffs $\Lambda_0$ for each system were also previously found by fitting to spectra and to the leptonic decay widths. It was found that the best value of $\Lambda_0$ for the charmonium system is $\Lambda_0=4$ GeV and for $b$-quarkonium is $\Lambda_0=7$ GeV. No attempt was made to vary any of these parameters to obtain a better fit for the radiative decay rates.

For the linear potential we use the purely scalar form since the previous calculation[1] gave the best fits for spectra and leptonic widths with this form. With the new $S_\epsilon(\Lambda,p)$ cutoffs the fit to the spectra with $\epsilon=0.25$ are essentially the same as the $S1$ fit ($\epsilon=0$) of Ref. 1. We used $\epsilon=0.01$ in the calculation of the decay rates.

The results of our relativistic calculation of the radiative decay rates of the charmonium system are presented in Table I(a) for measured electric dipole, magnetic dipole, and hindered magnetic dipole transitions, under the column labeled "relativistic asymptotic freedom + linear" and called $\Gamma_{\rm rel}^{\rm AFL}$. We label the transitions by their corresponding dominant nonrelativistic designation although in our relativistic calculation the exact inclusion of recoil includes all higher-order terms in the equivalent position space $kr$ expansion as well as the mixing of $L,S$ terms. The corresponding energies of the transitions with the same interactions are shown in Table I(b) and called $k_{\rm rel}$. The experimental values for the decay rates and the photon energies are taken from the 1986 Particle Data Group compilation.[15] For the charmonium system we can see that our relativistic results with the asymptotic-freedom coupling are in good agreement with the experimental values of the $E1$ and $M1$ transitions. In the spectrum calculations we obtain a low value for the $\psi'-\eta_c'$ mass splitting, which gives a $k_{\rm rel}$ of 37 MeV instead of the experimental value of 95 MeV. We correct the relativistic decay rate by the ratio of the experimental value to calculated value of $k^3$ as is exact in the nonrelativistic limit for magnetic dipole transitions. This gives the value of 0.60 keV for the rate of $\psi'\to\eta_c'+\gamma$, in agreement with the experimental range.

The hindered magnetic transition $\psi'\to\eta_c+\gamma$ is of particular importance for the study of relativistic effects. If the system is treated nonrelativistically, since the wave functions of $\psi'$ and $\eta_c$ are orthogonal, the magnetic di-

TABLE II. Predictions of $^1P_1$, $^1D_2$, and $^3D_2$ related decay rates (in keV) and photon energies (in MeV) of charmonium.

| | | $\Gamma^{\rm AF}$ | $k^{\rm AF}$ | $\Gamma^{\rm CS}$ | $k^{\rm CS}$ |
|---|---|---|---|---|---|
| | Electric dipole transitions | | | | |
| (1) | $^3D_2\to\chi_2+\gamma$ | 31 | 239 | 21 | 246 |
| (2) | $^3D_2\to\chi_1\to\gamma$ | 82 | 292 | 46 | 281 |
| (3) | $^1D_2\to{}^1P_1+\gamma$ | 108 | 277 | 56 | 275 |
| (4) | $\eta_c'\to{}^1P_1+\gamma$ | 31 | 155 | 26 | 135 |
| (5) | $^1P_1\to\eta_c+\gamma$ | 220 | 456 | 248 | 475 |
| | Electric quadrupole transitions | | | | |
| (1) | $^3D_2\to\eta_c+\gamma$ | 2.5 | 606 | 6.2 | 708 |
| (2) | $^3D_2\to\eta_c'+\gamma$ | 0.02 | 125 | 0.06 | 138 |
| | Magnetic dipole transitions | | | | |
| (1) | $^1P_1\to\chi_0+\gamma$ | 0.1 | 94 | 0.1 | 93 |
| (2) | $\chi_2\to{}^1P_1+\gamma$ | 0.06 | 39 | 0.01 | 24 |
| | Hindered magnetic dipole transitions | | | | |
| (1) | $^1D_2\to\psi'+\gamma$ | 0.006 | 91 | 0.007 | 97 |
| (2) | $^1D_2\to\psi+\gamma$ | 1.5 | 616 | 7.7 | 625 |
| | Magnetic quadrupole transitions | | | | |
| (1) | $^3D_2\to\chi_0+\gamma$ | 0.04 | 362 | 0.002 | 356 |

pole moment transition rate would be zero. The effect of recoil is also important in this transition since the emitted photon momentum is 620 MeV, giving a significant correction when quark momentum are less than or of the order of this momentum. Our relativistic calculation results in $\Gamma_{\rm rel}^{\rm AFL}=1.0$ keV compared to the experimental value of $\Gamma_{\rm expt}=0.6\pm0.2$ keV.

For the charmonium system we have calculated theoretical predictions for the radiative transition rates involving the possibly newly found[16] $^1P_1$ state, and the yet unobserved $^3D_2$ and $^3D_1$ states, which, although above charmed threshold, cannot decay to $D$-$\bar{D}$ states. These predictions are tabulated in Table II.

Our relativistic results for the electric dipole transitions for the $b$-quarkonium system are presented in the columns labeled "relativistic asymptotic freedom + linear" in Table III(a) with the corresponding transition energies in Table III(b). In the cases of the five transitions with experimentally known rates, our calculations agree with all of them within the one-sigma experimental errors. In Table IV we give predictions for significant transitions involving the yet-to-be-discovered

TABLE III. (a) Experimental and theoretical electric dipole transition rates in $b$-quarkonium (in keV). (b) Experimental and theoretical photon energies in electric dipole transitions of $b$-quarkonium (in MeV).

(a)

| No. | Decay mode | Experiment | Relativistic asymptotic freedom + linear | Relativistic Coulomb + linear | Nonrelativistic adjusted by $k_{\rm rel}{}^3$ |
|---|---|---|---|---|---|
| (1) | $\Upsilon'\to\chi_{b2}+\gamma$ | $1.98\pm0.5$ | 1.81 | 0.91 | 2.74 |
| (2) | $\Upsilon'\to\chi_{b1}+\gamma$ | $2.01\pm0.55$ | 1.64 | 1.05 | 2.88 |
| (3) | $\Upsilon'\to\chi_{b0}+\gamma$ | $1.29\pm0.42$ | 0.87 | 0.67 | 1.8 |
| (4) | $\chi_{b2}\to\Upsilon+\gamma$ | | 28.3 | 33.6 | 37.0 |
| (5) | $\chi_{b1}\to\Upsilon+\gamma$ | | 22.5 | 27.5 | 31.6 |
| (6) | $\chi_{b0}\to\Upsilon+\gamma$ | | 19.0 | 21.1 | 24.7 |
| (7) | $\Upsilon''\to\chi'_{b2}+\gamma$ | $1.52\pm1.3$ | 2.54 | 1.46 | 2.87 |
| (8) | $\Upsilon''\to\chi'_{b1}+\gamma$ | $1.89\pm1.6$ | 2.17 | 1.65 | 2.83 |
| (9) | $\Upsilon''\to\chi'_{b0}+\gamma$ | | 1.06 | 1.04 | 1.61 |
| (10) | $\chi'_{b2}\to\Upsilon'+\gamma$ | | 13.0 | 15.2 | 17.3 |
| (11) | $\chi'_{b1}\to\Upsilon'+\gamma$ | | 11.1 | 13.5 | 13.6 |
| (12) | $\chi'_{b0}\to\Upsilon'+\gamma$ | | 9.7 | 10.7 | 10.1 |
| (13) | $\chi'_{b2}\to\Upsilon+\gamma$ | | 7.7 | 13.0 | 10.5 |
| (14) | $\chi'_{b1}\to\Upsilon+\gamma$ | | 4.13 | 7.25 | 9.8 |
| (15) | $\chi'_{b0}\to\Upsilon+\gamma$ | | 1.8 | 2.3 | 9.1 |
| (16) | $\Upsilon''\to\chi_{b2}+\gamma$ | | 0.13 | 0.65 | 0.03 |
| (17) | $\Upsilon''\to\chi_{b1}+\gamma$ | | | 0.10 | 0.02 |
| (18) | $\Upsilon''\to\chi_{b0}+\gamma$ | | 0.042 | 0.016 | 0.008 |

(b)

| No. | Decay mode | Experiment | Relativistic asymptotic freedom + linear | Relativistic Coulomb + linear | Nonrelativistic |
|---|---|---|---|---|---|
| (1) | $\Upsilon'\to\chi_{b2}+\gamma$ | 109 | 118 | 95 | 120 |
| (2) | $\Upsilon'\to\chi_{b2}+\gamma$ | 131 | 142 | 120 | 120 |
| (3) | $\Upsilon'\to\chi_{b0}+\gamma$ | 162 | 175 | 161 | 120 |
| (4) | $\chi_{b2}\to\Upsilon+\gamma$ | 443 | 431 | 476 | 430 |
| (5) | $\chi_{b1}\to\Upsilon+\gamma$ | 422 | 408 | 452 | 430 |
| (6) | $\chi_{b0}\to\Upsilon+\gamma$ | 392 | 376 | 412 | 430 |
| (7) | $\Upsilon''\to\chi'_{b2}+\gamma$ | 84 | 86 | 72 | 95 |
| (8) | $\Upsilon''\to\chi'_{b1}+\gamma$ | 101 | 101 | 91 | 95 |
| (9) | $\Upsilon''\to\chi'_{b0}+\gamma$ | | 121 | 119 | 95 |
| (10) | $\chi'_{b2}\to\Upsilon'+\gamma$ | 245 | 230 | 250 | 233 |
| (11) | $\chi'_{b1}\to\Upsilon'+\gamma$ | 228 | 215 | 234 | 233 |
| (12) | $\chi'_{b0}\to\Upsilon'+\gamma$ | | 195 | 206 | 233 |
| (13) | $\chi'_{b2}\to\Upsilon+\gamma$ | 777 | 762 | 805 | 766 |
| (14) | $\chi'_{b1}\to\Upsilon+\gamma$ | 763 | 747 | 788 | 766 |
| (15) | $\chi'_{b0}\to\Upsilon+\gamma$ | | 728 | 761 | 766 |
| (16) | $\Upsilon''\to\chi_{b2}+\gamma$ | 432 | 427 | 415 | 443 |
| (17) | $\Upsilon''\to\chi_{b1}+\gamma$ | 452 | 452 | 439 | 443 |
| (18) | $\Upsilon''\to\chi_{b0}+\gamma$ | 484 | 483 | 479 | 443 |

TABLE IV. Predictions of $1\,^1P_1$, $2\,^1P_1$, $2\,^3D_1$, $1\,^3D_2$, and $1\,^1D_2$ related decays and largest magnetic decays of $b$-quarkonium. Decays rates in keV and photon energies in MeV.

| | Decay mode | $\Gamma^{AF}$ | $k^{AF}$ | $\Gamma^{CS}$ | $k^{CS}$ | $\Gamma^{NR}_{adj}$ | $k^{NR}$ |
|---|---|---|---|---|---|---|---|
| | | Electric dipole transitions | | | | | |
| (1) | $\chi_2' \to 1\,^3D_1+\gamma$ | 0.03 | 124 | 0.03 | 122 | 0.08 | 104 |
| (2) | $\chi_1' \to 1\,^3D_1+\gamma$ | 0.48 | 107 | 0.41 | 101 | 0.68 | 104 |
| (3) | $\chi_0' \to 1\,^3D_1+\gamma$ | 1.09 | 85 | 0.59 | 68 | 0.48 | 104 |
| (4) | $1\,^1P_1 \to \eta_b+\gamma$ | 31.6 | 477 | 38.1 | 567 | 75.6 | 430 |
| (5) | $\eta_b' \to 1\,^1P_1+\gamma$ | 2.5 | 111 | 0.69 | 72 | 4.1 | 120 |
| (6) | $\eta_b'' \to 1\,^1P_1+\gamma$ | 0.44 | 421 | 2.11 | 396 | 0.41 | 413 |
| (7) | $2^1P_1 \to \eta_b'+\gamma$ | 14.0 | 242 | 15.7 | 278 | 29.0 | 233 |
| (8) | $2\,^1P_1 \to \eta_b+\gamma$ | 9.6 | 811 | 16.8 | 896 | 18.6 | 766 |
| (9) | $\eta_b'' \to 2\,^1P_1+\gamma$ | 3.7 | 73 | 1.6 | 49 | 4.6 | 95 |
| (10) | $1\,^3D_2 \to \chi_2+\gamma$ | 4.5 | 235 | 4.7 | 235 | 6.0 | 250 |
| (11) | $1\,^3D_2 \to \chi_1+\gamma$ | 17.6 | 261 | 16.9 | 260 | 24.5 | 250 |
| (12) | $1\,^3D_1 \to \chi_2+\gamma$ | 0.46 | 223 | 0.50 | 226 | 0.34 | 250 |
| (13) | $1\,^3D_1 \to \chi_1+\gamma$ | 8.5 | 250 | 8.7 | 252 | 7.2 | 250 |
| (14) | $1\,^3D_1 \to \chi_0+\gamma$ | 14.7 | 284 | 15.4 | 297 | 14.0 | 250 |
| (15) | $\chi_2' \to 1\,^3D_2+\gamma$ | 0.3 | 113 | 0.3 | 113 | 0.86 | 104 |
| (16) | $1\,^1D_2 \to 1\,^1P_1+\gamma$ | 22.5 | 254 | 21.3 | 250 | 28.9 | 250 |
| | | Hindered magnetic dipole transitions | | | | | |
| (1) | $\eta_b' \to \Upsilon+\gamma$ | 0.12 | 525 | 0.3 | 533 | 0.02 | 545 |
| (2) | $\eta_b'' \to \Upsilon+\gamma$ | 0.18 | 823 | 0.29 | 847 | 0.03 | 854 |
| (3) | $1\,^1D_2 \to \Upsilon+\gamma$ | 0.05 | 662 | 0.007 | 708 | | 0 |

$2\,^1P_1$, $3\,^1P_1$, $3\,^3D_1$, $3\,^3D_2$, and $3\,^1D_2$ states of $b$-quarkonium.

In order to test whether or not the asymptotic-freedom modification of the gluon-exchange vector interaction by the logarithmic term in Eq. (2.16) shows up significantly in the electromagnetic transitions, we have also calculated the transition rates with a vector interaction that is a single perturbative exchange propagator: $4\pi\alpha_{\text{eff}}/q^2$. The nonrelativistic limit of this is the Cornell potential,[8] i.e., a Coulomb plus linear potential $V(r)=-\alpha_{\text{eff}}/r+K_S r$. For this we use the same parameters that were used in Ref. 12, namely, $K_S=0.183$ GeV$^2$ and $\alpha_{\text{eff}}=0.52$. The decay rates and the photon energies for this case are given in the "relativistic Coulomb + linear" columns of the Tables and denoted by $\Gamma^{CL}_{\text{rel}}$.

For the $b$-quarkonium system, after correcting for differences in rates due to energy differences appearing in $k^3$ factors, our calculated values for the asymptotic freedom and Coulomb cases both agree with the known experimental rates. Thus, even for the heaviest system with the presently known decays, we cannot verify the presence of the asymptotic-freedom $q^2$-dependent coupling. We have made predictions for decay rates that are not yet experimentally measured. In one of these cases, $\Upsilon''\to\chi_{b1}+\gamma$, we obtain a large difference, namely, $\Gamma^{AFL}_{\text{rel}}=0.00013$ keV, while $\Gamma^{CL}_{\text{rel}}=0.10$ keV. We tested the sensitivity of this rate to variations in $\kappa_S$ and $\epsilon$ and found that it is very sensitive to these changes. For this reason we do not provide a prediction for this transition. The $\Upsilon''\to\chi_{b0}+\gamma$ and $\Upsilon''\to\chi_{b2}+\gamma$ decay rates are not sensitive, and still give large differences. Unfortunately, the cases with large differences may be unmeasurably small.

A nonrelativistic Schrödinger equation calculation for the bound states and radiative transitions was used to compare the decay rates with the results of our relativistic calculations. In this Schrödinger equation the poten-

TABLE V. Total widths of $\chi$ and $\chi_b$ levels from measured branching ratios and calculated widths of radiative decays.

| Decay | Expt. branching ratio | Calc. $\Gamma_\gamma$ | Total width $\Gamma=\Gamma_\gamma/B$ | Expt. total width |
|---|---|---|---|---|
| $\chi_0\to\psi+\gamma$ | $(0.7\pm0.2)\%$ | 56 keV | $8^{+3}_{-2}$ MeV | $13.5\pm5$ MeV[a] |
| $\chi_1\to\psi+\gamma$ | $(25.8\pm2.5)\%$ | 100 keV | 0.38 MeV | $<1.3$ MeV[b] |
| $\chi_2\to\psi+\gamma$ | $(14.8\pm1.7)\%$ | 210 keV | 1.4 MeV | $2.6^{+1.4}_{-1.0}$ MeV[b] |
| $\chi_{b0}\to\Upsilon+\gamma$ | $<6\%$ | 19 keV | $>320$ keV | |
| $\chi_{b1}\to\Upsilon+\gamma$ | $(35\pm8)\%$ | 23 keV | 66 keV | |
| $\chi_{b2}\to\Upsilon+\gamma$ | $(22\pm4)\%$ | 28 keV | 130 keV | |

[a]Crystal Ball Collaboration.
[b]R704 Collaboration, CERN ISR.

tial that we have used was the nonrelativistic limit of the interactions that were used in our relativistic calculation, which becomes the Richardson potential, with the same $\kappa_S$ and $\Lambda_R$ parameters, but with a cutoff no longer required. This gives the zeroth-order wave functions to use for calculating electric and magnetic dipole matrix elements. In zeroth order the spin-spin and the spin-orbit interaction terms are left out and the electric dipole moment is independent of the total $J$ or spin-orbit and spin-spin energy splittings. The electric and magnetic dipole decay rates are proportional to $k^3$. The next step in perturbation theory is to use the first-order or perturbed energies with the zeroth-order wave functions. We do this by using the relativistic energies in the photon energy cubed terms with the lowest-order wave

TABLE VI. (a) Predictions for $3\,^3P_J$ ($\chi_J''$) related decays of $b$-quarkonium. (b) Predictions for $3\,^1P_1$, $1\,^3F_2$, and $2\,^3D_J$. Decay rates in keV and photon energies (in MeV).

(a)

| | $\Gamma_{\rm rel}^{\rm AF}$ | $k_{\rm rel}^{\rm AF}$ | $\Gamma^{\rm CS}$ | $k^{\rm CS}$ |
|---|---|---|---|---|
| $\chi_2'' \to \Upsilon'' + \gamma$ | 11.4 | 159 | 13.9 | 178 |
| $\chi_1'' \to \Upsilon'' + \gamma$ | 9.4 | 146 | 11.5 | 160 |
| $\chi_0'' \to \Upsilon'' + \gamma$ | 7.8 | 131 | 8.5 | 137 |
| $\chi_2'' \to \Upsilon' + \gamma$ | 4.4 | 460 | 6.5 | 487 |
| $\chi_1'' \to \Upsilon' + \gamma$ | 2.4 | 447 | 3.5 | 470 |
| $\chi_0'' \to \Upsilon' + \gamma$ | 1.0 | 433 | 1.0 | 448 |
| $\chi_2'' \to \Upsilon + \gamma$ | 4.6 | 982 | 8.7 | 1031 |
| $\chi_1'' \to \Upsilon + \gamma$ | 1.7 | 970 | 3.1 | 1014 |
| $\chi_0'' \to \Upsilon + \gamma$ | 0.33 | 956 | 0.29 | 993 |
| $\chi_2'' \to 2\,^3D_2 + \gamma$ | 0.52 | 69 | 0.54 | 63 |
| $\chi_1'' \to 2\,^3D_2 + \gamma$ | 1.7 | 56 | 1.4 | 45 |
| $\chi_2'' \to 2\,^3D_1 + \gamma$ | 0.05 | 79 | 0.05 | 73 |
| $\chi_1'' \to 2\,^3D_1 + \gamma$ | 0.74 | 66 | 0.63 | 55 |
| $\chi_0'' \to 2\,^3D_1 + \gamma$ | 1.8 | 51 | 1.0 | 32 |

(b)

| | $\Gamma_{\rm rel}^{\rm AF}$ | $k_{\rm rel}^{\rm AF}$ | $\Gamma^{\rm CS}$ | $k^{\rm CS}$ |
|---|---|---|---|---|
| $2\,^3D_1 \to 1\,^3F_2 + \gamma$ | 1.2 | 86 | 1.5 | 94 |
| $2\,^3D_2 \to 1\,^3F_2 + \gamma$ | 0.17 | 96 | 0.19 | 104 |
| $1\,^3F_2 \to 1\,^3D_1 + \gamma$ | 16.7 | 193 | 16.3 | 192 |
| $1\,^3F_2 \to 1\,^3D_2 + \gamma$ | 2.7 | 182 | 2.7 | 183 |
| $3\,^1P_1 \to \eta_b + \gamma$ | 4.9 | 1028 | 10.6 | 1113 |
| $3\,^1P_1 \to \eta_b' + \gamma$ | 4.4 | 472 | 6.9 | 509 |
| $3\,^1P_1 \to \eta_b'' + \gamma$ | 10.9 | 163 | 13.3 | 189 |
| $2\,^3D_1 \to \chi_2' + \gamma$ | 0.35 | 154 | 0.41 | 164 |
| $2\,^3D_1 \to \chi_1' + \gamma$ | 6.1 | 172 | 6.8 | 185 |
| $2\,^3D_1 \to \chi_0' + \gamma$ | 10.1 | 194 | 11.9 | 217 |
| $2\,^3D_1 \to \chi_2 + \gamma$ | 0.38 | 494 | 0.05 | 504 |
| $2\,^3D_1 \to \chi_1 + \gamma$ | 1.4 | 520 | 1.7 | 529 |
| $2\,^3D_1 \to \chi_0 + \gamma$ | 3.9 | 554 | 5.7 | 573 |
| $2\,^3D_2 \to \chi_2' + \gamma$ | 3.4 | 165 | 3.8 | 173 |
| $2\,^3D_2 \to \chi_1' + \gamma$ | 12.5 | 182 | 13.7 | 194 |
| $2\,^3D_2 \to \chi_2 + \gamma$ | 0.59 | 504 | 0.71 | 514 |
| $2\,^3D_2 \to \chi_1 + \gamma$ | 3.9 | 529 | 4.5 | 538 |
| $3\,^1P_1 \to 2\,^1D_2 + \gamma$ | 2.5 | 58 | 2.4 | 51 |
| $2\,^1D_2 \to 2\,^1P_1 + \gamma$ | 16.1 | 178 | 17.3 | 187 |
| $2\,^1P_1 \to 1\,^1D_2 + \gamma$ | 1.7 | 99 | 1.7 | 100 |
| $3\,^1P_1 \to 1\,^1D_2 + \gamma$ | 0.03 | 332 | 0.09 | 335 |
| $2\,^1D_2 \to 1\,^1P_1 + \gamma$ | 4.7 | 523 | 5.0 | 529 |

functions in order to give the best comparison with the relativistic decay rates. The nonrelativistic decay rates so corrected are shown in Tables I(a), III(a), and IV and are labeled with "nonrelativistic adjusted."[11] The nonrelativistic energies shown are the zeroth-order ones. After scaling by $(k_{\rm rel}/k_{\rm nonrel})^3$ factors, we are then actually comparing the "effective" relativistic matrix element with the nonrelativistic electric dipole moment. For the hindered magnetic transitions the decay rate is proportional to $k^7$ and the nonrelativistic rates are scaled up by the factor $(k_{\rm rel}/k_{\rm nonrel})^7$ in Tables I(a) and IV.

In the charmonium transitions, Table I(a), we see that the relativistic rates are reduced from the nonrelativistic ones by factors of from 2 to 6, in agreement with the experimental rates, where available. In $b$-quarkonium, Table III(a), the relativistic rates are reduced up to a factor of 2 from the nonrelativistic ones. The exceptions are the highly suppressed electric dipole ones numbered 16–18 in Table III(a), which depend on large wave-function interferences for transitions between more widely separated principal quantum numbers. Similar suppressions are found in the hydrogen-atom transition rates for the same decays.[17]

Since the formation of $\chi$ states from $\psi \to \chi + \gamma$ is detected from the associated $\gamma$ production, the latter $\chi \to \psi + \gamma$ decay allows the experiments to determine the $\chi$ radiative decay branching ratio rather than the total width. We can use our calculated rate of this width $\Gamma_\gamma$ to find the total widths of the $\chi$'s by $\Gamma = \Gamma_\gamma / B$. These are shown for the $\chi$ and $\chi_b$ states in Table V. The total widths for $\chi$ states are in agreement with the larger $\chi_0$ width measured in the crystal ball and with the smaller $\chi_2$ width measured from formation in $p$-$\bar{p}$ colliding rings at the CERN ISR.

In Table VI we present predictions for decays from the higher levels of $b$-quarkonium. We have omitted decays with very small branding ratios.

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## APPENDIX A

Here we show that quark and antiquark currents of the same flavor give identical contributions to radiative decay.

In the valence approximation, the quarkonium states are just superpositions of $b^\dagger d^\dagger |0\rangle$ states. From the normal ordering,

$$\begin{aligned} M_\mu &= \langle B_2' m_j' \,|\, {:}\bar{\psi}(x)\gamma_\mu \psi(x){:} \,|\, B_1 m_j \rangle \\ &= \langle B_2' m_j' \,|\, \bar{\psi}^{(-)}(x)\gamma_\mu \psi^{(+)}(x) \,|\, B_1 m_j \rangle \\ &\quad - \langle B_2' m_j' [\gamma_\mu \psi^{(-)}(x)]_\alpha \bar{\psi}^{(+)}(x)_\alpha \,|\, B_1 m_j \rangle \ . \end{aligned} \tag{A1}$$

We insert appropriate complete sets of single-quark (labeled by subscript $q$) or antiquark states, and add in the innocuous remaining parts of the fields:

$$M_\mu = \int d^3p \sum_\sigma \langle B_2' m_j' \mid \bar{\psi}(x) \mid \mathbf{p}\sigma \rangle \langle \mathbf{p}\sigma \mid \gamma_\mu \psi(x) \mid B_1 m_j \rangle - \int d^3p \sum_\sigma \langle B_2' m_j' \mid (\gamma_\mu \psi(x))_\alpha \mid \mathbf{p}\sigma \rangle_q \langle \mathbf{p}\sigma \mid {}_q\bar{\psi}(x)_\alpha \mid B_1 m_j \rangle$$
$$= \int d^3p \sum_\sigma \langle B_2' m_j' \mid \bar{\psi}(x) \mid \mathbf{p}\sigma \rangle \langle \mathbf{p}\sigma \mid \gamma_\mu \psi(x) \mid B_1 m_j \rangle$$
$$- \int d^3p \sum_\sigma \langle B_2' m_j' \mid C^{-1} C [\gamma_\mu \psi(x)]_\alpha C^{-1} C \mid \mathbf{p}\sigma \rangle_q \langle \mathbf{p}\sigma \mid {}_q C^{-1} C \bar{\psi}(x)_\alpha C^{-1} C \mid B_1 m_j \rangle \ , \tag{A2}$$

where C is the charge-conjugation operator.

Its effects on different states are

$$\langle B_2' m_j' \mid C^{-1} = \eta_c' \langle B_2' m_j' \mid , \quad C \mid B_1 m_j \rangle = \eta_c \mid B_1 m_j \rangle, \quad \langle \mathbf{p}\sigma \mid {}_q C^{-1} = \langle \mathbf{p}\sigma \mid , \quad C \mid \mathbf{p}\sigma \rangle_q = \mid \mathbf{p}\sigma \rangle \ . \tag{A3}$$

Since the charge-conjugation quantum number of the photon is $-1$, in any radiative decay we must have $\eta_c \eta_c' = -1$.

For the current we now have

$$M_\mu = \int d^3p \sum_\sigma \langle B_2' m_j' \mid \bar{\psi}(x) \mid \mathbf{p}\sigma \rangle \langle \mathbf{p}\sigma \mid \gamma_\mu \psi(x) \mid B_1 m_j \rangle$$
$$+ \int d^3p \sum_\sigma \langle B_2' m_j' \mid C\psi_\beta(x) C^{-1} \mid \mathbf{p}\sigma \rangle (\gamma_\mu)_{\alpha\beta} \langle \mathbf{p}\sigma \mid C\bar{\psi}_\alpha(x) C^{-1} \mid B_1 m_j \rangle \ . \tag{A4}$$

For a fermion field we have

$$C\psi_\beta(x) C^{-1} = C_{\beta\tau} \bar{\psi}_\tau(x) \quad \text{and} \quad C\bar{\psi}_\alpha(x) C^{-1} = -\psi_\xi(x) C_{\xi\alpha}{}^{-1} \ , \tag{A5}$$

and the current becomes

$$M_\mu = \int d^3p \sum_\sigma \langle B_2' m_j' \mid \bar{\psi}(x) \mid \mathbf{p}\sigma \rangle \langle \mathbf{p}\sigma \mid \gamma_\mu(x) \mid B_1 m_j \rangle$$
$$- \int d^3p \sum_\sigma \langle B_2' m_j' \mid \bar{\psi}_\tau(x) \mid \mathbf{p}\sigma \rangle C_{\beta\tau} (\gamma_\mu)_{\alpha\beta} C_{\xi\alpha}{}^{-1} \langle \mathbf{p}\sigma \mid \psi_\xi(x) \mid B_1 m_j \rangle \ . \tag{A6}$$

Since $C_{\alpha\beta} = -(C^{-1})_{\alpha\beta}$ and

$$C_{\beta\tau}(\gamma_\mu)_{\alpha\beta} C_{\xi\alpha}{}^{-1} = C_{\xi\alpha}(\gamma_\mu)_{\alpha\beta} C_{\beta\tau}{}^{-1} = -(\gamma_\mu)_{\tau\xi} \tag{A7}$$

in the equation for the current, we find that both terms now give the same contribution; hence,

$$M_\mu = 2 \int d^3p \sum_\sigma \langle B_2' m_j' \mid \bar{\psi}(x) \mid \mathbf{p}\sigma \rangle \gamma_\mu \langle \mathbf{p}\sigma \mid \psi(x) \mid B_1 m_j \rangle . \tag{A8}$$

## APPENDIX B

In this appendix we carry out the Lorentz boost of the antiquark state and derive a formula for the matrix $\bar{D}_{\sigma\sigma'}$. From

$$U(\Lambda)\psi_r(x) U^{-1}(\Lambda) = S_{rs}{}^{-1}(\Lambda)\psi_s(\Lambda x) \ , \tag{B1}$$

one finds

$$U(\Lambda)\bar{\psi}(x) U^{-1}(\Lambda) = \bar{\psi}(\Lambda x) S(\Lambda) \ . \tag{B2}$$

We then take an antifermion matrix element and use the above Lorentz-transformation formula on the field $\bar{\psi}(x)$:

$$\langle 0 \mid \bar{\psi}(x) \mid \mathbf{p},\sigma \rangle = \langle 0 \mid U^{-1}(\Lambda) \bar{\psi}(\Lambda x) S(\Lambda) U(\Lambda) \mid \mathbf{p},\sigma \rangle \ . \tag{B3}$$

Now we use the formula for $U(\Lambda) \mid p,\sigma \rangle$:

$$\langle 0 \mid \bar{\psi}(x) \mid \mathbf{p},\sigma \rangle = \langle 0 \mid \bar{\psi}(\Lambda x) S(\Lambda) \sum_{\sigma'} \mid \Lambda \mathbf{p},\sigma' \rangle \times \bar{D}_{\sigma'\sigma} \left[ \frac{\omega(\Lambda p)}{\omega(p)} \right]^{1/2} . \tag{B4}$$

From the free field expansion, the left-hand side becomes

$$\langle 0 \mid \bar{\psi}(x) \mid \mathbf{p},\sigma \rangle = \bar{v}(p,\sigma) e^{-ipx} \left[ \frac{m}{\omega(p)(2\pi)^3} \right]^{1/2} . \tag{B5}$$

Evaluating the right-hand side of Eq. (B4),

$$\langle 0 \mid \bar{\psi}(\Lambda x) S(\Lambda) \mid \Lambda \mathbf{p},\sigma' \rangle = \left[ \frac{m}{\omega(\Lambda p)(2\pi)^3} \right]^{1/2} \times \bar{v}(\Lambda p,\sigma') e^{-i\Lambda p \cdot \Lambda x} S(\Lambda) . \tag{B6}$$

Substituting Eq. (B5) and Eq. (B6) in Eq. (B4) gives

$$\left[ \frac{\omega(\Lambda p)}{\omega(p)} \right]^{1/2} \bar{v}(p,\sigma) = \sum_{\sigma'} \bar{D}_{\sigma'\sigma} \bar{v}(\Lambda p,\sigma') S(\Lambda) \ . \tag{B7}$$

Multiplying both sides by $S^{-1}(\Lambda)$ and then by $v(\Lambda p,\sigma'')$ and using the relation

$$\bar{v}(\Lambda p,\sigma')v(\Lambda p,\sigma'')=-\delta_{\sigma'\sigma''}\ ,$$

we find the desired relation for $\bar{D}_{\sigma\sigma'}$:

$$\bar{D}_{\sigma\sigma'}(\Lambda)=-\bar{v}(p,\sigma)S^{-1}(\Lambda)v(\Lambda p,\sigma')\ . \tag{B8}$$

## APPENDIX C

In this appendix we show that the electromagnetic current of the quarkonium bound state is conserved in the valence-quark treatment that was used in the bound-state equations. From Appendix A and Eq. (2.3), the electromagnetic current of a same-flavor quark-antiquark state for a transition allowed by charge conjugation is

$$M^{\mu}=Q\langle B_2'\,|\,:\bar{\psi}(0)\gamma^{\mu}\psi(0):\,|\,B_1\rangle=2Q\int d^3p\sum_{\sigma}\langle B_2'\,|\,\bar{\psi}(0)\,|\,\mathbf{p}\sigma\rangle\gamma^{\mu}\langle\mathbf{p}\sigma\,|\,\psi(0)\,|\,B_1\rangle$$

$$=2Q\frac{m}{(2\pi)^6[4\omega(B_2')\omega(B_1)]^{1/2}}\int\frac{d^3p}{\omega}\bar{\Psi}_{B_2'}(p,\sigma)\gamma^{\mu}\Psi_{B_1}(p,\sigma)\ . \tag{C1}$$

Taking the divergence of the current, $q_{\mu}M^{\mu}$, and adding and subtracting $\not{B}_1-\not{p}-m$ gives

$$q_{\mu}M^{\mu}=\frac{2Qm}{(2\pi)^2[4\omega(B_1)\omega(B_2')]^{1/2}}\int\frac{d^3p}{\omega}\sum_{\sigma}[\bar{\Psi}_{B_2'}(p,\sigma)(\not{q}+\not{p}-\not{B}_1+m)\Psi_{B_1}(p,\sigma)+\Psi_{B_2'}(p,\sigma)(\not{B}_1-\not{p}-m)\Psi_{B_1}(p,\sigma)]\ . \tag{C2}$$

In the first term we use $B_2'=B_1-q$ to obtain $(q+p-B_1+m)=-(B_2'-p-m)$.

To find the effect of $(\not{B}_2'-\not{p}-m)$ on $\bar{\Psi}_{B_2'}(p,\sigma)$ we use the bound-state equation (2.4), and, by a few algebraic steps, we find the following equation for $\bar{\Psi}_{B_2'}(p,\sigma)$:

$$\bar{\Psi}_{B_2'}(p,\sigma)(\not{B}-\not{p}-m)=\sum_{\sigma'}\int\frac{d^3p'm}{(2\pi)^3\omega'}[V_V((p-p')^2)\bar{v}(p,\sigma)\gamma^{\mu}v(p',\sigma')\bar{\Psi}_{B_2'}(p',\sigma')\gamma_{\mu}$$
$$+V_S((p-p')^2)\bar{v}(p,\sigma)v(p',\sigma')\bar{\Psi}_{B_2'}(p',\sigma')]\ . \tag{C3}$$

Using Eq. (C3) in the first term of Eq. (C2) and the bound-state equation (2.4) in the second term gives

$$q_{\mu}M^{\mu}=\frac{Qm^2}{(2\pi)^9[\omega(B_2')\omega(B_1)]^{1/2}}$$
$$\times\sum_{\sigma,\sigma'}\int\frac{d^3p\,d^3p'}{\omega\omega'}\{-V_V((p-p')^2)[\bar{v}(p,\sigma)\gamma^{\mu}v(p',\sigma')\bar{\Psi}_{B_2'}(p',\sigma')\gamma_{\mu}\Psi_{B_1}(p,\sigma)]$$
$$-V_S((p-p')^2)[\bar{v}(p,\sigma)v(p',\sigma')\bar{\Psi}_{B_2'}(p',\sigma')\Psi_{B_1}(p,\sigma)]$$
$$+V_V((p-p')^2)[\bar{\Psi}_{B_2'}(p,\sigma)\gamma_{\mu}\Psi_{B_1}(p',\sigma')\bar{v}(p',\sigma')\gamma^{\mu}v(p,\sigma)]$$
$$+V_S((p-p')^2)[\bar{\Psi}_{B_2'}(p,\sigma)\Psi_{B_1}(p',\sigma')\bar{v}(p',\sigma')v(p,\sigma)]\}\ . \tag{C4}$$

We note now that upon interchanging $p\leftrightarrow p'$ and $\sigma\leftrightarrow\sigma'$ in the third and fourth terms of the above equations they cancel the first and second terms, respectively. Thus, the current is conserved.

## APPENDIX D

In order to see how the relativistic formalism reduces to the nonrelativistic results for radiative transition matrix elements, we begin with the second of the integral equations in Eq. (2.11). In this limit, the right-hand side potential terms are of order of the kinetic energy or of order $v/c$ of the terms on the left-hand side and can be dropped. The left-hand side now gives

$$|F\rangle\simeq-\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}}{2m}|G\rangle\ . \tag{D1}$$

We use the result of the trace of the current for a generalized final-state photon with $\mathbf{W}=(C-i\mathbf{D}\cdot\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_1$ and $\mathbf{b}=-b\hat{\mathbf{z}}$:

$$\mathrm{Tr}[\mathbf{J}\cdot\boldsymbol{\epsilon}]=a\langle G'\,|\,\mathbf{W}\cdot\boldsymbol{\epsilon}^*\,|\,F\rangle+a\langle F'\,|\,\mathbf{W}\cdot\boldsymbol{\epsilon}^*\,|\,G\rangle +i\langle G'\,|\,(\mathbf{W}\times\mathbf{b})\cdot\boldsymbol{\epsilon}\,|\,G\rangle +i\langle F'\,|\,(\mathbf{W}\times\mathbf{b})\cdot\boldsymbol{\epsilon}^*\,|\,F\rangle\ . \tag{D2}$$

In the nonrelativistic limit,

$$a\simeq 1,\quad C\simeq -1,\quad \mathbf{D}=O(v^2/c^2)\ ,\qquad \mathbf{b}=-\frac{ak\hat{\mathbf{z}}}{\omega(B_2')+M_2}\simeq-\frac{\mathbf{k}}{4m},\quad \Lambda\mathbf{p}\simeq\mathbf{p}+\frac{\mathbf{k}}{2}\ . \tag{D3}$$

Using Eqs. (D3) and (D1) in Eq. (D2) yields

$$\mathrm{Tr}[\mathbf{J}\cdot\boldsymbol{\epsilon}^*]\simeq\left\langle G'\left|\sigma_1\cdot\boldsymbol{\epsilon}^*\frac{\sigma_1\cdot\mathbf{p}}{m}\right|G\right\rangle +\left\langle G'\left|\sigma_1\cdot\frac{\Lambda\mathbf{p}}{2m}\sigma\cdot\boldsymbol{\epsilon}^*\right|G\right\rangle +i\left\langle G'\left|\sigma_1\cdot\frac{\mathbf{k}\times\boldsymbol{\epsilon}^*}{4m}\right|G\right\rangle\ . \tag{D4}$$

Using the nonrelativistic recoil limit $\Lambda\mathbf{p}\simeq\mathbf{p}+\mathbf{k}/2$ and $\mathbf{k}\cdot\boldsymbol{\epsilon}=0$, we obtain the nonrelativistic result where the first term is the electric dipole and the second is the magnetic dipole contribution of the valence quark:

$$\mathrm{Tr}[\mathbf{J}\cdot\boldsymbol{\epsilon}^*]\simeq\left\langle G'\left|\frac{\mathbf{p}}{m}\cdot\boldsymbol{\epsilon}^*+i\frac{\sigma_1}{2m}\cdot(\mathbf{k}\times\boldsymbol{\epsilon}^*)\right|G\right\rangle\ . \tag{D5}$$

At this point, the final state $|G'\rangle$ is still evaluated at the recoil momentum $\Lambda\mathbf{p}=\mathbf{p}+\mathbf{k}/2$. This allows us to calculate the decay rate for hindered magnetic transitions such as $\psi'\to\eta_c+\gamma$, where the momentum wave functions would be orthogonal if we neglected recoil. To obtain the lowest-order multipole contribution to the hindered magnetic transitions we expand in $|\mathbf{k}|/|\mathbf{p}|$ (analogous to the $kr$ expansion in position space) and perform an angular integration, obtaining, for $S$-wave states,

$$\int d^3p\, g_f\left(\left|\mathbf{p}+\frac{\mathbf{k}}{2}\right|\right)g_i(p) =4\pi\int_0^\infty p^2dp\left[g_f(p)+\tfrac{1}{12}g_f'(p)\frac{k^2}{p}+\tfrac{1}{24}g_f''(p)k^2\right]\times g_i(p)\ . \tag{D6}$$

For the electric dipole rate from Eq. (D5) in the norecoil limit $\Lambda\mathbf{p}\simeq\mathbf{p}$, we can follow a rotational-group analysis analogous to that done in position space to obtain the result in momentum space:

$$\Gamma=\tfrac{16}{3}\alpha Q_q{}^2k(2j'+1)\begin{Bmatrix}l' & j' & s\\ j & l & 1\end{Bmatrix}^2\left|\left\langle l'\left\|\frac{p}{m}\right\|l\right\rangle\right|^2\ . \tag{D7}$$

The reduced matrix element is found from the $m_l=m_{l'}=0$ matrix element

$$\left\langle l',0\left|\frac{p_z}{m}\right|l,0\right\rangle=i(-1)^{l_{\max}}\left\langle l'\left\|\frac{p}{m}\right\|l\right\rangle\begin{pmatrix}l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}\ . \tag{D8}$$

The connection to the position-space dipole moment is

$$\left\langle l'\left\|\frac{p}{m}\right\|l\right\rangle=ik\langle l'\|r\|l\rangle\ . \tag{D9}$$

Upon replacing Eq. (D9) in Eq. (D7) we find the position-space formula for electric dipole decays.[18]

## APPENDIX E

Here we convert the $2\times\bar{2}$ form of the $\tilde{G}$ and $\tilde{F}$ matrices, take the trace, and end up with matrix elements in the direct-product form of two spin-$\frac{1}{2}$ particles. In the trace we insert $\sigma_y$'s to convert to $G=\tilde{G}\sigma_y$ and the $2\times 2$ representation. A typical term becomes

$$\begin{aligned}T&=\mathrm{Tr}[(C+i\mathbf{D}\cdot\boldsymbol{\sigma})\tilde{G}'^{\dagger}\boldsymbol{\sigma}\tilde{G}]\\&=\mathrm{Tr}[\sigma_y\sigma_y(C+i\mathbf{D}\cdot\boldsymbol{\sigma})\sigma_y\sigma_y\tilde{G}'^{\dagger}\boldsymbol{\sigma}\tilde{G}]\\&=\mathrm{Tr}[(C-i\mathbf{D}\cdot\boldsymbol{\sigma}^T)(\tilde{G}\sigma_y)^{\dagger}\boldsymbol{\sigma}(\tilde{G}\sigma_y)]\\&=\mathrm{Tr}[(C-i\mathbf{D}\cdot\boldsymbol{\sigma}^T)G'^{\dagger}\boldsymbol{\sigma}G]\ ,\end{aligned} \tag{E1}$$

where we have used $\sigma_y\boldsymbol{\sigma}\sigma_y=-\boldsymbol{\sigma}^T$.

We partial wave expand

$$\tilde{G}(\mathbf{p})=\sum_{LS}g_{LS}(p)\sum_{m_Lm_S}C^{jm_j}_{Lm_L,Sm_S}Y_L^{m_L}(\hat{\mathbf{p}})P_{Sm_S}\ . \tag{E2}$$

Here $P_{Sm_S}$ is a $2\times 2$ projection operator for total spin $S$ and component $m_S$,

$$P_{Sm_S}=\sum_{m_1m_2}C^{Sm_S}_{m_1m_2}|m_1\rangle\langle m_2|\ , \tag{E3}$$

constructed from spin-$\frac{1}{2}$ basis vectors $|m_1\rangle$ and $|m_2\rangle$.

The spin structure of $T$ is then

$$\begin{aligned}T'=\mathrm{Tr}[(C-i\mathbf{D}\cdot\boldsymbol{\sigma}^T)(|m_1'\rangle\langle m_2'|)^{\dagger}\boldsymbol{\sigma}|m_1\rangle\langle m_2|]&=\langle m_2|(C-i\mathbf{D}\cdot\boldsymbol{\sigma}^T)|m_2'\rangle\langle m_1'|\boldsymbol{\sigma}|m_1\rangle\\&=\langle m_2'|(C-i\mathbf{D}\cdot\boldsymbol{\sigma})|m_2\rangle\langle m_1'|\boldsymbol{\sigma}|m_1\rangle\ .\end{aligned}$$

We move the order of the spin vectors by defining $\sigma_1$ as those that act on the vectors $|m_1\rangle$ and $\sigma_2$ as those that act on $|m_2\rangle$:

$$T'=\langle m_1|\langle m_2'|(C-i\mathbf{D}\cdot\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_1|m_1\rangle|m_2\rangle\ . \tag{E4}$$

This recombines with the Clebsch-Gordan coefficients to give direct-product state vectors:

$$|Sm_S\rangle=\sum_{m_1m_2}C^{Sm_S}_{m_1m_2}|m_1\rangle|m_2\rangle \tag{E5}$$

and

$$| G \rangle = \sum_{LS} g_{LS}(p) \sum_{m_l m_s} C^{jm_j}_{Lm_L, Sm_S} Y_L^{m_L}(\hat{\mathbf{p}}) \, | Sm_S \rangle \; . \qquad \text{(E6)}$$

The result for the trace then becomes

$$T = \langle G' \, | \, (C - i\mathbf{D}\cdot\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_1 \, | \, G \rangle \; . \qquad \text{(E7)}$$


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