Energy loss in general relativity

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Implicit assumptions regarding continuity in energy-loss calculations in general relativity are examined. The Arnowitt-Deser-Misner energy integral is treated in a new manner as a universal vehicle for energy loss. Two explicit examples are given: the electric dipole radiation flux is computed using *general relativity* as well as the gravitational-radiation flux from a linear mass quadrupole oscillator. In this approach, the latter is seen as a *nonlinear* problem in the sense that the lower-order metric serves as a source for the required order metric as computed within the wave front. Logarithmic terms which threaten to induce divergences, as has been found in other works, are averted by functions of integration which are required to sustain the gauge conditions and finally yield the usual fluxes.

I. INTRODUCTION

The role of energy has been one of the central issues in general relativity. Much effort has been devoted to the study of its positivity, its localizability, and the measure of its flux in radiative systems. In the early years, the analogy with electromagnetism was used by Einstein[1] and others to express the gravitational energy flux in terms of a pseudotensor. A variety of such complexes were subsequently developed but many researchers regarded these with suspicion because of their noncovariant nature. Einstein defended his pseudotensor on the basis of the Lorentz covariance of the total four-momentum derived from it, and Eddington[2] added further support by demonstrating agreement between the pseudotensorial energy flux and the radiation damping of a radiating source.

Gravitational energy loss was generally regarded as being on a more secure footing after Bondi[3] associated mass loss with a news function using a well-adapted system of coordinates with retarded time. The work of Bondi can also be regarded as support for the pseudotensor because Madore[4] (and one of the present authors[5] later independently for a particular case) showed the equivalence of the news function to the pseudotensor.

In more recent investigations, interest has focused upon the relationship between the Bondi mass which diminishes in the presence of "news" and the Arnowitt-Deser-Misner (ADM) mass[6] which is a constant for a system when calculated with certain specific asymptotic properties.[7,8] In this paper we develop the connections between the different quantities associated with energy in general relativity in a new manner. We will see the limitations of the pseudotensor and by identification, the news function, as a vehicle to express loss. We will see how the ADM mass can be treated in a different manner to compute energy loss.

This treatment is used to calculate the energy loss from two well-known prototypical systems: an electromagnetic radiation energy loss from a linear charge dipole oscillator and a gravitational-radiation energy loss from a linear mass quadrupole oscillator. These examples are highly instructive both for their similarities and differences which will be developed in detail. In both cases, the method entails the determination of the metric to the required order within the expanding wave front. In the electric dipole example, it is a linearized gravitational problem in that the source is the electromagnetic Maxwell stress-energy tensor of the electromagnetic radiation which is the source of energy loss. However, in the mass quadrupole example where it is gravitational radiation which is carrying the energy, the problem is nonlinear for this method with *gravitational* stress energy as the source of the metric to the required order.

In spite of these differences, there is a strong analogy, indeed similarity, between the two calculations. Both involve logarithmic terms which threaten to induce divergences and both are saved by functions of integration which are required to sustain the gauge conditions and finally determine the correct fluxes.

In Sec. II we develop the energy-loss expression, accounting for the potential contributions from discontinuities in the derivatives of the metric. We then show how it is connected to the ADM energy expression. The ADM energy-loss expression with harmonic coordinates is used in Sec. III to calculate electromagnetic radiation flux from an electric charge dipole oscillator. In Sec. IV the more complicated problem of the mass quadrupole oscillator is addressed, also in harmonic coordinates. In both examples, the usual answer results. This might suggest that harmonic and admissible coordinates can generally be connected by transformations which merge with the identity transformation as $r \to \infty$. This is because the energy-momentum complex is invariant under such a transformation and the admissible coordinate condition assures that discontinuity terms will not contribute in the energy expression.

We end with a summary and concluding remarks in Sec. V.

II. EXPRESSIONS FOR ENERGY LOSS

A straightforward approach to the energy-momentum complex and associated conservation laws is given by Weinberg.[9] The metric tensor g_{ik} is decomposed into a

Minkowskian part η_{ik} and a remainder h_{ik} (latin indices range from 0 to 3, greek indices from 1 to 3),

$$g_{ik}=\eta_{ik}+h_{ik} \tag{2.1}$$

and is substituted into the Einstein field equations. The part of the Einstein tensor which is linear in h_{ik} is retained on the left-hand side and the nonlinear part is grouped into a "pseudotensor" t_{ik}. This is brought to the right-hand side where, in conjunction with the nongravitational energy-momentum tensor T_{ik}, it serves as part of the source of h_{ik} in a nonlinear manner:

$$R_{ik}^{(1)}-\tfrac{1}{2}\eta_{ik}R^{(1)}{}_j^j=8\pi G(T_{ik}+t_{ik}) \ . \tag{2.2}$$

The composite source is labeled τ_{ik} and its indices, as well as those of other nontensorial quantities, are raised (and lowered) with the Minkowski metric η^{ik} (η_{ik}). It is noted that since the left side of (2.2) has a vanishing ordinary divergence, the right-hand side does as well,

$$\tau^{ik}_{,k}=0 \ , \tag{2.3}$$

and hence there is a conservation law of the form

$$\frac{d}{dt}\int_V \tau^{0i}dV=-\oint_S \tau^{i\mu}dS_\mu \tag{2.4}$$

for a total energy-momentum four-"vector" P^i (which is ultimately a Lorentz four-vector):

$$P^i=\int_V \tau^{0i}dV \tag{2.5}$$

for all matter and fields including gravitation. From (2.4), $\tau^{i\mu}$ is interpreted as the corresponding energy-momentum flux density. Various other useful properties are cited by Weinberg as well to support the identification.

All of this appears to be reasonable as it stands but what has been overlooked in the derivation is an implicit assumption of overall continuity. Indeed, let us assume that over some closed surface D (within V), which has velocity $\mathbf{v}$, that τ^{ik} is discontinuous. It is simple to envisage such a situation. If there is matter with sharp boundaries within V, then T^{00} will be discontinuous. For other situations, it is still commonplace for the second derivatives of the metric to harbor discontinuities and since the gravitational part of τ^{ik} has second-derivative terms, it is only reasonable to allow for discontinuities in τ^{ik}.

The present authors[10] have developed the required machinery for this in theorems for time derivatives and an extended Gauss theorem relating to retarded integrals with discontinuities. Here we require only their simple forms over constant t slices:

$$\frac{d}{dt}\int_V f\,dV=\int_V\left[\frac{\partial f}{\partial t}\right]dV+\oint_D d\mathbf{S}\cdot\mathbf{v}f \ , \tag{2.6}$$

$$\int_V \nabla\cdot\mathbf{F}\,dV=\oint_S d\mathbf{S}\cdot\mathbf{F}+\oint_D d\mathbf{S}\cdot\mathbf{F} \ , \tag{2.7}$$

$$\frac{d}{dt}\oint_D d\mathbf{S}\cdot\mathbf{F}=\oint_D d\mathbf{S}\cdot\mathbf{F}_{,0}+\oint_D d\mathbf{S}\cdot\mathbf{v}(\nabla\cdot\mathbf{F}) \ . \tag{2.8}$$

In (2.6)–(2.8), $\oint_D$ denotes the discontinuity in the integral in the sense of inner minus outer over the surface D in which f and $\mathbf{F}$ are discontinuous.

Equations (2.3), (2.6), and (2.7) yield

$$\frac{d}{dt}\int_V \tau^{0i}dV=-\oint_S \tau^{i\mu}dS_\mu-\oint_D(\tau^{i\mu}-v^\mu\tau^{i0})dS_\mu \tag{2.9}$$

as a general restatement of (2.4). At this point, it is easy to see that there will be discontinuity terms for which their casual neglect as in (2.4) ultimately has no effect. For example, in the T^{ik} parts of τ^{ik} for nonrelativistic situations, the integrand of the D term vanishes because energy and linear momentum flux densities are, respectively, equal to velocity times energy and linear momentum densities. However, there is no apparent basis for ignoring the D term in general.

Let us now focus upon the $i=0$ energy expression and develop an expression for the time rate of change of energy in a volume which approaches infinity while we allow for some loss due to radiation which crosses the sphere S at infinity. It is simplest to work in harmonic coordinates expressed through the gauge conditions

$$\psi^{ik}_{,k}=0, \quad \psi_{ik}\equiv h_{ik}-\tfrac{1}{2}\eta_{ik}h_j{}^j \ . \tag{2.10}$$

The Einstein equations then simplify to inhomogeneous wave equations

$$\Box\psi^{ik}=16\pi G\tau^{ik}, \quad \Box\equiv\nabla^2-\frac{\partial^2}{\partial t^2} \ . \tag{2.11}$$

With Eqs. (2.6)–(2.11) we find after a moderately lengthy calculation, that the energy loss can be expressed as

$$\begin{aligned}\dot{E}&=\frac{d}{dt}\int_V \tau^{00}dV\\&=\frac{1}{16\pi G}\frac{\partial}{\partial t}\left[\oint_S dS_\mu(\psi^{0\mu}_{,0}+\psi^{00}_{,\mu})\right.\\&\qquad\left.+\oint_D dS_\mu(\psi^{0\mu}_{,0}+\psi^{00}_{,\mu})\right] \ .\end{aligned} \tag{2.12}$$

From this expression we are in a position to invoke a simple sufficient condition which casts the energy-loss expression into the form of a surface integral over the bounding infinite sphere. If the first partials $\psi^{0\mu}_{,0}$ and $\psi^{00}_{,\mu}$ are always continuous over D, then the $\oint_D$ term in (2.12) vanishes and we are left with a simple time derivative of an S integral. It is interesting to note that through the gauge conditions (2.10), this integral reduces to

$$E=\frac{1}{16\pi G}\oint_S(h^\alpha_{\alpha,\beta}-h^\alpha_{\beta,\alpha})dS_\beta \tag{2.13}$$

which is precisely the ADM energy expression. The ADM energy expression also arises more generally,[9] without the assumption of harmonic coordinates, but the implicit continuity conditions are more complicated.

The efficacy of choosing continuity conditions on first derivatives suggests another approach. We first note that the τ^{ik} complex can be expressed as an ordinary divergence of a third-rank quantity Q^{jik} which involves first partials of h^{ik}:

$$\tau^{ik}=\frac{1}{8\pi G}Q^{jik}_{,j}, \quad Q^{jik}=-Q^{ijk} \ . \tag{2.14}$$

Then, using (2.8), the D term of (2.9) which we now call A is

$$A=\frac{1}{8\pi G}\left[\frac{d}{dt}\oint_D dS_\mu Q^{0\mu 0}+\oint_D dS_\mu Q^{\alpha\mu 0}_{\ \ ,\alpha}\right]. \tag{2.15}$$

From the antisymmetry property in (2.14), it follows that $Q^{\alpha\mu 0}{}_{,\alpha}$ has a vanishing divergence, $Q^{\alpha\mu 0}_{\ \ ,\alpha\mu}=0$, and hence $Q^{\alpha\mu 0}{}_{,\alpha}$ can be expressed as a curl. Explicitly,

$$Q^{\alpha\mu 0}_{\ \ ,\alpha}=\tfrac{1}{2}(\nabla\times\mathbf{F})^\mu , \tag{2.16}$$

where

$$F^1=h^{20}_{,3}-h^{30}_{,2},\quad F^2=h^{30}_{,1}-h^{10}_{,3},$$
$$F^3=h^{10}_{,2}-h^{20}_{,1} \tag{2.17}$$

and hence the second integral in (2.15) vanishes by Stokes's theorem. Moreover, since Q^{jik} is constructed from first derivatives of the metric, it follows that in Cartesian coordinates which are also admissible in the sense of Lichnerowicz,[11] namely, that the metric and its first derivatives are continuous in these coordinates, the first term in A vanishes as well and hence A is zero. In this case, the traditional (2.4) describes the energy loss. Now we note that asymptotically, τ^{ik} is invariant to first order in ϵ under the transformation

$$x^i\to x'^i=x^i+\epsilon^i(x) , \tag{2.18}$$

where $\epsilon^i(x)\to 0$ as $r\to\infty$. This follows from Weinberg:[9] the change induced in Q^{jik} is

$$\Delta Q^{jik}=D^{ljik}_{\ \ \ \ ,l} , \tag{2.19}$$

where D^{ljik} is totally antisymmetric in its first three indices. Hence, under (2.18),

$$\tau^{ik}\to\tau'^{ik}=\tau^{ik}+\Delta\tau^{ik} , \tag{2.20}$$

where

$$8\pi G\Delta\tau^{ik}=\Delta Q^{jik}_{\ \ \ ,j}=D^{ljik}_{\ \ \ \ ,lj}=0 , \tag{2.21}$$

where the last equality follows from the antisymmetry in l and j of D^{ljik}.

Hence it follows that (2.4) can indeed by used to compute the energy loss in any system which can be transformed into admissible Cartesian coordinates by (2.18). At this point, we are in a position to see what has been implicitly assumed in various calculations in the past. It is often most convenient to use harmonic coordinates and this has been the standard practical method. When these are used with (2.4) to compute an energy flux, the correct answer will be obtained if these harmonic coordinates are transformable into admissible Cartesian coordinates by (2.18). This is a sufficient condition, and has been implicitly assumed to hold. The discontinuity term might vanish for some other reason but at any rate, the question of its presence of absence clearly must be addressed.

III. LINEAR CHARGE DIPOLE OSCILLATOR

We first consider a system with which we are completely familiar from Maxwell theory, an electric dipole oscillating with frequency ω along the z (polar) axis:

$$\mathbf{d}=\hat{\mathbf{k}}\alpha\cos\omega t . \tag{3.1}$$

Interestingly, we can compute the *electromagnetic* energy loss from this system using general relativity. Since the system radiates electromagnetic energy, it loses mass and hence this loss must show up in the change in the ADM expression. This is a very difficult method to apply to such a simple problem but it is very useful for the imparted insight and confidence in the methods described earlier. It is most convenient to use harmonic coordinates for the calculation and $\dot{E}$ in this system is then given by (2.12). Let us assume that they are also admissible or, at least more weakly, that on a time average, the rate of change of the D term in (2.12) vanishes. Then, if the resultant answer is correct, we have justified the assumption of the neglect of the D term.

On this basis, the Einstein equations are in the convenient form of (2.11) where the dominant parts of the source for ψ^{ik} within the expanding wave front are the appropriate components of the electromagnetic field energy-momentum tensor $\eta^{ia}\eta^{kb}T_{ab}$ formed from the oscillating electric and magnetic field vectors orthogonal to the flux:

$$T^{00}=\frac{1}{8\pi}(E^2+H^2),\quad T^{0\mu}=\frac{1}{4\pi}(\mathbf{E}\times\mathbf{H})^\mu , \tag{3.2}$$

where

$$\mathbf{H}=\frac{1}{R_0}\ddot{\mathbf{d}}\times\hat{\mathbf{n}},\quad \mathbf{E}=\mathbf{H}\times\hat{\mathbf{n}} \tag{3.3}$$

and $\hat{\mathbf{n}}$ is a unit radial vector. From Eqs. (3.1)–(3.3) and (2.11), the required components of ψ^{ik} satisfy

$$\begin{aligned}\Box\psi^{0k}&=\frac{4\alpha^2}{R^2}g^k(\theta,\phi)\cos^2\omega(t-R) ,\\ g^k&=\sin^2\theta(1,\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta) .\end{aligned} \tag{3.4}$$

The asymptotic solution of (3.4) can be expressed as

$$\psi^{0k}=-\frac{\alpha^2}{R}g^k\left[\left[\xi+\frac{\sin 2\omega\xi}{2\omega}\right]\ln\eta+B(\xi)+C(\eta)\right] , \tag{3.5}$$

$\xi\equiv t-R$, $\eta\equiv t+r$; B,C are functions of integration.

The functions of integration are determined by the demand that ψ^{0k} satisfy the gauge conditions (2.10) which are the basis for (2.11). This gives

$$B=-\left[\xi+\frac{\sin 2\omega\xi}{2\omega}\right](1+\ln 2R_0),\quad C=0 \tag{3.6}$$

in the neighborhood of the sphere $R=R_0\to\infty$. It is interesting to note that the logarithmic term which the gauge conditions require for the function of integration combines with the logarithmic term in the solution (3.5) for ψ^{0k} to avert a divergence.

If we assume that the D term can be neglected, then $\dot{E}$ should be given by the rate of change of the ADM mass, the first integral of (2.12). This assumption gives

$$\dot{E}=-\tfrac{2}{3}\ddot{\mathbf{d}}^2 \tag{3.7}$$

which is precisely the well-known dipole energy flux rate calculated by traditional methods of Maxwell theory. Thus, at least for this problem, the neglect of the D integral has been justified. Electromagnetic energy loss has been calculated using general relativity. It is to be noted that it is a linear problem in the sense that the lowest-order time-varying part of the metric determines the energy flux. The source of ψ^{0k} in the wave front is the "matter" part T^{0k} of the total complex τ^{0k}, constructed from the electromagnetic wave fields. The nonlinear t^{0k} part would contribute only higher-order corrections to the dominant dipole loss (3.7). This is to be compared with the gravitational-radiation energy loss described by a simple example in the next section.

IV. LINEAR MASS QUADRUPOLE OSCILLATOR

We now consider the closest physical mass analogue to the charge dipole: two masses in oscillation along the z axis.[12–14] As in the electromagnetic example, we wish to describe the gravitational radiation energy loss from this system by (2.12) rather than by (2.9). This is a much more involved calculation and it is instructive to trace it through.

The nature of the energy which is carried on the wave front is now gravitational rather than electromagnetic as in the charge dipole example. The fields which play a role analogous to that formerly played by **E** and **H** are the lowest-order time-dependent parts of the metric which have as source, the energy-momentum tensor T^{ik} of the mass quadrupole oscillator. In harmonic coordinates, this is described by (2.11) with τ^{ik} equal to T^{ik} since the pseudotensor vanishes to this order. The solution is well known:[12,14]

$$\psi^{33}=A\cos\omega(t-R),\quad \psi^{30}=n_3\psi^{33}, \qquad \psi^{00}=n_3{}^2\psi^{33}, \tag{4.1}$$

$$A\equiv\frac{2\omega^2GI}{R}. \tag{4.2}$$

It is important to note that it is not this ψ^{ik} which gives the energy loss via (2.12). Clearly, (4.1) in (2.12) would vanish on a time average. In traditional calculations, (4.1) is used to construct the pseudotensor and the energy loss is then calculated via (2.9). This is in fact the direct analogue of the Poynting vector calculation in electromagnetism. The pseudotensor has terms which are bilinear in the ψ^{ik} components and these have a nonvanishing time average. Clearly, to match this with (2.12), we require the *second-order* dynamic solution of ψ^{ik} and this is what distinguishes this problem as being nonlinear in contrast with the former problem of the charge dipole.

The second-order ψ^{ik} is found using (2.11) but now the source within the wave front is the t^{ik} formed from (4.1). (Note that the T^{ik} part of the τ^{ik} vanishes in this region in contrast with the former problem.) The form of t_{ik} in terms of ψ^{ik} is given in Eq. (4.3) of Ref. 15:

$$\begin{aligned}-8\pi Gt_{ik}={}&\tfrac{1}{2}\psi^{lm}(\psi_{ik,lm}+\psi_{lm,ik}-\psi_{li,mk}-\psi_{mk,li})+\tfrac{1}{4}(\psi_{lm,k}-\psi_{lk,m}+\psi_{mk,l})(\psi_i^{m,l}+\psi^{ml}{}_{,i}-\psi_i^{l,m})\\&-\tfrac{1}{8}\eta_{ik}(\psi_{lm,n}-\psi_{ln,m}+\psi_{mn,l})(\psi^{lm,n}-\psi^{ln,m}+\psi^{mn,l})\\&-\tfrac{1}{4}[\eta_{ik}(\psi^{lm}\psi_{,lm}-\psi\psi_{,l}{}^{,l}-\tfrac{3}{4}\psi_{,l}\psi^{,l}+2\psi^{lm}\psi_{lm,n}{}^{,n})+\psi\psi_{,ik}+\tfrac{1}{2}\psi_{,i}\psi_{,k}\\&\qquad+\psi\psi_{ik,l}{}^{,l}+\psi_{ik}\psi_{,l}{}^{,l}+2\psi_{ik,l}\psi^{,l}-\psi_i^l\psi_{,lk}-\psi_k^l\psi_{,li}-\psi_{li,k}\psi^{,l}-\psi_{lk,i}\psi^{,l}]+O(\psi^3),\\ \psi\equiv{}&\eta_{lm}\psi^{lm}.\end{aligned} \tag{4.3}$$

After a rather lengthy but straightforward calculation, Eqs. (4.3) with (4.1) yield

$$\frac{-8\pi G}{\omega^2A^2}t_{0\alpha}=(1-n_3{}^2)^2\frac{n_\alpha}{8}[3\cos^2\omega(t-R)-1], \qquad \frac{-8\pi Gt_{00}}{\omega^2A^2}=-\frac{(1-n_3{}^2)^2}{8}[3\cos^2\omega(t-R)-1] \tag{4.4}$$

($\alpha=1,2,3$). From this point, the calculation proceeds in a very similar manner to that of Sec. III. Equations (2.11) and (4.4) yield the analogue of (3.4):

$$\Box\psi^{0k}=\left[\frac{GI\omega^3}{R}\right]^2 j^k[3\cos^2\omega(t-R)-1], \qquad j^k=\sin^4\theta(1,\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta), \tag{4.5}$$

and the solution in the neighborhood of $R=R_0\to\infty$ which satisfies the gauge conditions (2.10) is

$$\psi^{0k}=-\frac{(GI\omega^3)^2}{4R}\left[\left[\xi+\frac{3}{2\omega}\sin2\omega\xi\right]\left[\ln\frac{\eta}{2R_0}-1\right]\right]. \tag{4.6}$$

This solution is the analogue of (3.5) and (3.6) but unlike the dipole solution, it is the iterated first nonlinear dynamic part of the metric.

Finally, if we neglect the D term, (2.12) with (4.6) yield the time-averaged energy-loss rate

$$\dot{E}=-\tfrac{1}{15}GI^2\omega^6 \tag{4.7}$$

which is the energy-loss rate found from (2.4), or even more simply, by the quadrupole formula.

V. SUMMARY AND CONCLUDING REMARKS

In this paper we have illustrated that the standard energy-loss picture balanced by a flux over the asymptotic sphere implicitly assumes the vanishing of certain discontinuity integrals. The flux in that case is seen to be the time rate of change of the ADM integral. Although the traditional approach is to treat the ADM integral as the invariant mass, this presupposes that one evaluate it beyond any wave front. However, if we allow the wave front to cross the ADM surface, then the ADM expression can be treated like a universal Poynting vector.

From Weinberg's equations, we showed that the τ^{ik}

complex is invariant under coordinate shifts (2.18) which merge asymptotically with the identity. This led to the conclusion that the asymptotic flux integral does measure the energy loss in any system which can be so related to admissible coordinates for it is in the latter that the first partials of the metric are continuous.

We then used the ADM integral to compute the electromagnetic radiation energy loss from an electric dipole oscillator. Since the radiative flux implies an energy loss and hence a diminution in mass, the flux in the ADM integral, evaluated with the metric within the radiative wave front, must measure this mass loss. Harmonic coordinates were used to find the metric and logarithmic terms which threatened to induce divergences were seen to be eliminated by corresponding functions of integration required to sustain the gauge conditions. The ADM mass loss was seen to be precisely the standard dipole formula (3.7) calculated very simply in Maxwell theory.

The corresponding gravitational-radiation energy flux was then computed with the ADM integral. This was seen to be a considerably more involved problem in that the source of the required order field in the ADM integral was the gravitational stress-energy pseudotensor in the wave front, which is quadratic in the lowest-order dynamic metric. This is in contrast with the electric dipole problem where the source is the much simpler energy-momentum tensor of the radiative electromagnetic field within the wave front. In this sense, the electromagnetic problem is seen to be linear in contrast with the gravitational problem which is nonlinear. Apart from this, the two calculations, including the logarithmic aspects, are totally analogous and in the gravitational problem, the usual quadrupole answer results.

It is tempting to conclude that these results vindicate the neglect in general of the surface discontinuity terms discussed earlier. This would be overly simplistic. In the electromagnetic problem, although the actual dipole source could have a discontinuous charge-electrovacuum boundary, the flux for such a system must be virtually identical to that generated by the same source with the charge at the interface smoothed by a continuous charge density over a thin layer. Hence there really could not have been any expectation of a difference in the flux from that of (3.7).

In the mass quadrupole problem, the corresponding smoothing of the mass distribution might not necessarily lead to the smoothing of the more complicated τ^{ik} source complex. The fact that the answer coincides with the usual one derived by other methods is an inadequate justification because those other methods themselves rely upon the very same implicit smoothness. While the likelihood is that these discontinuity terms ultimately do not contribute in the energy balance equations, a rigorous demonstration would be useful. This could be achieved, for example, if it could be demonstrated that harmonic coordinates can be found which are simultaneously admissible or linking a harmonic system to an admissible one by (2.18) which asymptotically merges with the identity.

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