Gauge invariance of the quantum Wilson loop

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We define the quantum operator for the Wilson loop in QCD and prove that it is gauge invariant. We also derive diagrammatically the Feynman rules in various gauges for the perturbative calculation of the Wilson loop. Our result solves the puzzle whether the Feynman rules in the axial gauge are consistent with those in other gauges, to all order of the coupling constant and for all contours in the space-time region. In particular, we show that one must include the contribution of ghost loops if the principal-value prescription is used for the axial-gauge propagator.

I. INTRODUCTION

The Feynman rules in the axial gauge[1] for non-Abelian gauge field theories have always been suspect. In this paper, we shall derive sets of Feynman rules including those in the axial gauge which give the same value for the quantum Wilson loop as that in the Feynman gauge.

First, we give a brief account on past difficulties. In the temporal axial gauge, the gluon propagator in the momentum space is singular at $k_0=0$. (There is a similar singularity in the case of the spatial axial gauge.) The path-integral formulation is unable to determine whether one should use the principal-value prescription. Indeed, the principal-value prescription is especially called to doubt by the work of Caracciolo, Curci, and Menotti,[2] who calculated the rectangular Wilson loop in the temporal axial gauge to the order of g^4. The sides of the rectangle are parallel to the x^0 and x^1 axes with the lengths of T and L, respectively. These authors stated that, in the asymptotic region of $T/L \gg 1$, the result is not consistent with that[3,4] obtained in the Feynman gauge or the Coulomb gauge, if the principal-value prescription is used. They also gave an axial-gauge propagator which yields the correct value for the Wilson loop (to the order g^4 and in the region of $T/L \gg 1$).

In this paper we shall establish the Feynman rules for the Wilson loop in any gauge, including the axial gauge. We shall prove the validity of these rules to arbitrary orders of the coupling constant, and for arbitrary loops in the space-time region. This will be demonstrated in two independent ways.

The first way is diagrammatic. The diagrammatic method will lead to the following theorem of equivalence of Feynman rules in non-Abelian gauge field theories:[5] let the gluon propagator be

$$\tilde{D}^{ab}_{\mu\nu} = -\frac{i\delta^{ab}}{k^2+i\epsilon'}(g_{\mu\nu}-a_\mu k_\nu - b_\nu k_\mu) , \tag{1.1}$$

with $a_\mu(k)$ and $b_\mu(k)$ related by

$$a_\mu(k) = -b_\mu(-k) , \tag{1.2}$$

and with ϵ' infinitesimal, let the ghost-ghost-gluon vertex be

$$G^\mu(k) = i\frac{[(a\cdot k)-1]k^\mu - k^2 a^\mu(k)}{k^2+i\epsilon'} , \tag{1.3}$$

where k^μ is the momentum of the outgoing ghost [the ghost propagator has been included in (1.3)], then the on-shell scattering amplitude and the Wilson loop calculated with the rest of the Feynman rules in a non-Abelian gauge field theory are independent of $a_\mu(k)$. In particular, all of these sets of Feynman rules are equivalent to the set of Feynman rules with $a_\mu(k)=0$, the latter being the Feynman rules in the Feynman gauge. In the above, we have made the implicit assumption that, in the gauge considered, the Feynman integrals are sufficiently convergent so that a change of loop-momentum variables are allowed. Otherwise, the rules must be amended by including the contribution of anomalous terms.[5] That this theorem holds for on-shell scattering amplitudes has already been proven elsewhere.[5,6] We shall prove below that this theorem holds for the Wilson loop as well.

The second way to study the Wilson loop is on the basis of canonical quantization and is in fact nonperturbative. We shall show that there exists a correspondence formula[7] which equates the Wilson loop in the temporal axial gauge to the Wilson loop in the effective theory with ghost fields. A perturbative expansion of this correspondence formula leads to the equivalence of the Feynman rules in the temporal axial gauge with those in the covariant gauge. This equivalence can be extended to other gauges such as the Coulomb gauge and the spatial axial gauge.

II. DIAGRAMMATIC PROOF

In a quantum non-Abelian gauge field theory, the Wilson loop is defined to be

$$W = \left\langle \mathrm{tr} PT \exp\left[-ig \oint_c A_\mu dx^\mu\right]\right\rangle . \tag{2.1}$$

In (2.1), $\langle\ \rangle$ denotes the vacuum expectation value, c is a closed contour in the four-dimensional configuration space, g is the coupling constant, and A_μ is the matrix of field operator in the Heisenberg representation:

$$A_\mu = A^a_\mu T^a , \tag{2.2}$$

where T^a is one of the infinitesimal group generators in the adjoint representation. The symbol T signifies that the operators A^a_μ in (2.1) are time ordered [see the discussion in the paragraph following Eq. (3.2)] while the symbol P signifies that the group matrices in (2.1) are path ordered. Thus, expanding the exponential in (2.1), we have

$$W=\mathrm{tr}I+\sum_{n=2}^{n=\infty}\frac{(-ig)^n}{n!}\mathrm{tr}P(T^{a_1}\cdots T^{a_n})\oint_c dx_1^{\mu_1}\cdots\oint_c dx_n^{\mu_n}G^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n)\;, \tag{2.3}$$

where

$$G^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n)=\langle TA^{a_1}_{\mu_1}(x_1)\cdots A^{a_n}_{\mu_n}(x_n)\rangle \tag{2.4}$$

is the n-point Green's function. The term corresponding to $n=1$ is omitted from (2.3) as $\mathrm{tr}T^a=0$. We shall prove that, in the perturbative calculation of the expression in (2.3), any $a_\mu(k)$ can be used in the gluon propagator (1.1) and the corresponding ghost (1.3) (absence of anomalous terms assumed).

The lowest-order nontrivial term in (2.3) is of the order of g^2. It comes from the lowest-order (g^0) term in the two-point Green's function in (2.3) and is equal to

$$-\frac{g^2}{2!}\mathrm{tr}P(T^aT^b)\oint_c dx_1^\mu\oint_c dx_2^\nu D^{ab}_{\mu\nu}(x_1-x_2)\;, \tag{2.5}$$

where, by (1.1),

$$D^{ab}_{\mu\nu}(x)=\delta_{ab}[-ig_{\mu\nu}\Delta_F(x)-\partial_\nu\alpha_\mu(x)+\partial_\mu\alpha_\nu(-x)]\;, \tag{2.6}$$

with

$$\alpha_\mu(x)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}a_\mu(k)\;, \tag{2.7}$$

and with $\Delta_F(x)$ the Fourier transform of $(k^2+i\epsilon')^{-1}$. It is easy to prove that the expression in (2.5) is independent of α. We have

$$\mathrm{tr}T^aT^b=\mathrm{tr}T^bT^a \tag{2.8}$$

[hence we may drop the symbol P in (2.5)]; thus a term such as $\partial_\nu\alpha_\mu(x)$ in $D^{ab}_{\mu\nu}(x)$ is integrated to zero over the closed contour c.

The next-order terms in (2.3) are of the order g^4, and come from the term of the order g^2 in the two-point Green's function, the term of the order of g in the three-point Green's function, and the term of the order of g^0 in the four-point Green's function. These terms are illustrated in Fig. 1. Let us first consider Fig. 1(e). In this figure, the four-point Green's function is approximated by

$$D^{ab}_{\mu\nu}(x_1-x_2)D^{cd}_{\rho\sigma}(x_3-x_4)\;, \tag{2.9}$$

and the others obtained from it by permuting the coordinates. Let us first prove that the α functions in $D^{cd}_{\rho\sigma}(x_3-x_4)$ are canceled by some corresponding terms in the three-point Green's function. We note that the group factor $\mathrm{tr}P(T^aT^bT^cT^d)$ associated with Fig. 1(e) depends on the ordering of x_1,x_2,x_3,x_4 on the closed contour c. In particular, the difference of the group factors corresponding to the two orderings illustrated in Figs. 2(a) and 2(b) is that of the three-point Green's function in Fig. 2(c). Let the coordinates x_1, x_2, and x_3 be ordered as in Fig. 2(a) and let us carry out the following integration over x_4:

$$\begin{aligned}\mathrm{tr}P(T^aT^bT^cT^d)\oint dx_4^\sigma\frac{\partial}{\partial x_4^\sigma}\alpha_\rho(x_3-x_4)\delta^{cd}\delta^{ab}=&\ \mathrm{tr}(T^aT^cT^dT^b)\int_{x_2}^{x_1}dx_4^\sigma\frac{\partial}{\partial x_4^\sigma}\alpha_\rho(x_3-x_4)\delta^{cd}\delta^{ab}\\&+\mathrm{tr}(T^aT^cT^bT^d)\int_{x_1}^{x_2}dx_4^\sigma\frac{\partial}{\partial x_4^\sigma}\alpha_\rho(x_3-x_4)\delta^{cd}\delta^{ab}\;.\end{aligned} \tag{2.10}$$

We get

$$[\alpha_\rho(x_3-x_1)-\alpha_\rho(x_3-x_2)]\times\text{group factor of Fig. 2(c)}\;. \tag{2.11}$$

Next we consider the diagram in Fig. 3 and concentrate on the term

$$\frac{\partial\alpha_\rho(x_3-y)}{\partial y^{\rho'}} \tag{2.12}$$

in the propagator $D_{\rho\rho'}$. In momentum space, $-i\partial/\partial y^{\rho'}$ is equal to $p_{\rho'}$, where p is the momentum for the gluon line joining x_3 with y in Fig. 3. Now we may easily check that

$$p^{\rho'}\Gamma_{\rho'\mu'\nu'}(p,q,k)=(q^2g_{\mu'\nu'}-q_{\mu'}q_{\nu'})-(k^2g_{\mu'\nu'}-k_{\mu'}k_{\nu'})\;, \tag{2.13}$$

where $\Gamma_{\rho'\mu'\nu'}(p,q,k)$ is the three-point interaction in Fig. 3. The term in the first parentheses in (2.13), multiplied by the propagator (in momentum space) for the line joining x_1 with y, becomes

$$-i(q^2g_{\mu'\nu'}-q_{\mu'}q_{\nu'})\frac{g_{\mu\mu'}-q_\mu a_{\mu'}-q_{\mu'}b_\mu}{q^2}=-ig_{\mu\nu'}-iG_{\nu'}(q)q_\mu\;, \tag{2.14}$$

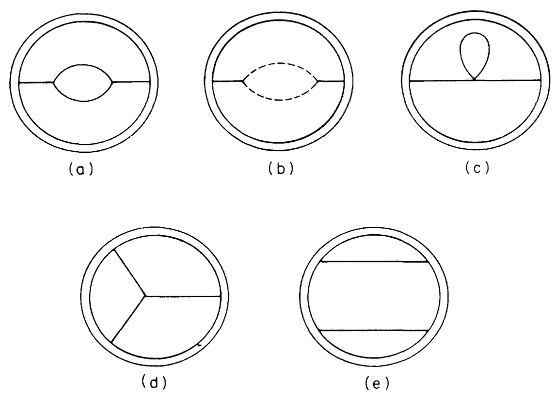

FIG. 1. g^4 terms in the Wilson loop. The double-line circle represents a contour c in space and time. A solid line represents a gluon and a dashed line represents a ghost.

where $G_{\nu'}(q)$ is the ghost-ghost-gluon vertex given by (1.3). In configuration space, the term $-ig_{\mu\nu'}$ in (2.14) corresponds to

$$-ig_{\mu\nu'}\delta^{(4)}(x_1-y)\ . \tag{2.15}$$

Therefore, this term gives, together with the factor $\alpha_\rho(x_3-y)$ in (2.12), an expression which cancels the first term in the square brackets of (2.11). Similarly, the term in the second parentheses in (2.13), multiplied by the propagator for the gluon line joining x_2 and y, produces an expression which cancels the second term in the square brackets of (2.11). Finally the ghost terms such as the second term on the right side of (2.14), as well as terms such as $\partial\alpha_{\rho'}(x_3-y)/\partial x_3^\rho$ in the propagator $D_{\rho\rho'}(x_3-y)$ produce expressions which cancel terms belonging to Fig. 1(a). Specifically, these latter terms are the ones generated by the $[\partial_\mu\alpha_\nu(x)-\partial_\nu\alpha_\mu(-x)]$ terms of the propagators in the virtual loop in Fig. 1(a) if we repeat the procedure above.

If we choose

$$a_\mu(k)=\frac{n_\mu k_0-\frac{1}{2}k_\mu}{k_0^2+\epsilon^2}\ , \tag{2.16}$$

where n_μ is the temporal vector (1,0,0,0), then

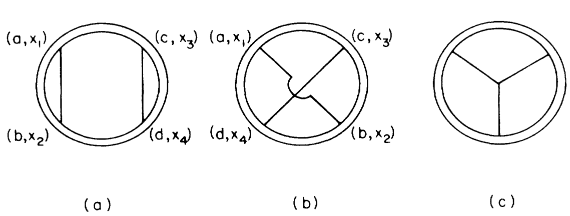

FIG. 2. The difference of the group factors corresponding to (a) and (b) is equal to that corresponding to (c).

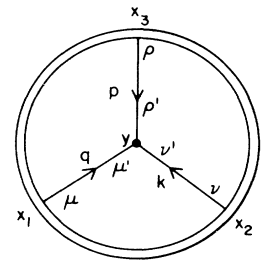

FIG. 3. The lowest-order diagram for the three-point function with x_1, x_2, and x_3 in counterclockwise order. The dot at y denotes a three-gluon vertex.

$$\tilde{D}^{ab}_{\mu\nu}(k)=\frac{-i\delta_{ab}}{k^2+i\epsilon'}\left[g_{\mu\nu}-\frac{(n_\mu k_\nu+n_\nu k_\mu)k_0-k_\mu k_\nu}{k_0^2+\epsilon^2}\right] \tag{2.17}$$

and

$$G_\mu(k)=-\frac{i}{k^2+i\epsilon'}\,\frac{k^2k_0n_\mu+\epsilon^2k_\mu}{k_0^2+\epsilon^2}\ . \tag{2.18}$$

We may regard the propagator (2.17) in the limit of $\epsilon\to 0$ as the gluon propagator in the temporal axial gauge. There are, however, two points to keep in mind.

(i) From (2.17) we have

$$\tilde{D}^{ab}_{00}(k)=-\frac{i\delta_{ab}\epsilon^2}{(k^2+i\epsilon')(k_0^2+\epsilon^2)}\ . \tag{2.19}$$

We emphasize that, while $\tilde{D}^{ab}_{00}(k)$ vanishes in the limit $\epsilon\to 0$ with k_0 fixed, it is of the order of unity in the limit $\epsilon\to 0$ with k_0/ϵ fixed. It has been verified that there exist diagrams with temporal gluons giving nonzero contribution in the neighborhood of $k_0=0$. Therefore, the propagator $\tilde{D}^{ab}_{0}(k)$ may not be set to zero at the beginning. Rather, we should take the limit $\epsilon\to 0$ at the end of each diagrammatic calculation.

(ii) From (2.18) we have in the limit $\epsilon\to 0$,

$$G_\mu(k)=-in_\mu\mathcal{P}\left[\frac{1}{k_0}\right]\ ,$$

where $\mathcal{P}$ signifies that the principal value is taken. Since

$$\int_{-\infty}^{\infty}dk_0\mathcal{P}\prod_{n=1}^{N}\frac{1}{k_0+a_n}=0\ ,$$

$N\geq 2$ and $a_n\neq a_m$ if $n\neq m$, it may appear that one need not, in the axial gauge, take into account ghost loops, as asserted by the path-integral approach. This conclusion is erroneous for the same reason just given, and we must include ghost vertices in a diagrammatic calculation, taking the limit $\epsilon\to 0$ only at the end.

The choice (2.17) for the axial-gauge propagation is not the only one possible. We may use, for example, the propagator

$$\tilde{D}^{ab}_{\mu\nu} = -\frac{i\delta_{ab}}{k^2+i\epsilon'}\left[g_{\mu\nu}-(\delta_{\mu 0}k_\nu+\delta_{\nu 0}k_\mu)\frac{1}{2}\left[\frac{1}{k_0+i\epsilon}+\frac{1}{k_0-i\epsilon}\right]+k_\mu k_\nu\frac{1}{2}\left[\frac{1}{(k_0+i\epsilon)^2}+\frac{1}{(k_0-i\epsilon)^2}\right]\right] . \tag{2.20}$$

In the limit $\epsilon\to 0$, this propagator is traditionally regarded as the temporal-gauge propagator in the principal-value prescription. The corresponding ghost vertex remains to be given by (2.18). This propagator together with the corresponding ghost again give the correct value for any Wilson loop, if the limit $\epsilon\to 0$ is taken at the end of a calculation, not at the beginning as was done in the literature. We have verified that, in this limit, both the temporal gluons and ghosts contribute. This explains why Caracciolo, Curci, and Menotti[2] did not obtain the correct answer for the g^4 term of the Wilson loop, using the conventional principal-value prescription.

It is also possible to choose the axial-gauge propagator to be

$$\tilde{D}^{ab}_{\mu\nu} = -\frac{i\delta_{ab}}{k^2+i\epsilon'}\left[g_{\mu\nu}-\frac{n_\mu k_\nu}{k_0+i\epsilon}-\frac{n_\nu k_\mu}{k_0-i\epsilon} + \frac{k_\mu k_\nu}{k_0{}^2+i\epsilon}\right] . \tag{2.21}$$

The ghost-ghost-gluon vertex corresponding to (2.21) is

$$\frac{-k^2 n^\mu - i\epsilon k_\mu}{k_0+i\epsilon}\frac{i}{k^2+i\epsilon'} . \tag{2.22}$$

For such a vertex, the integrand for a ghost loop has singularity at $k_0=-i\epsilon$ only. Therefore, we may choose to close the contour integration in the upper-half k_0 plane, obtaining zero. In other words, there is, in this prescription, no contribution from ghosts. However, the contribution from temporal gluons remains to be nonzero.

III. THE CORRESPONDENCE FORMULA

Finally, we shall derive the correspondence formula which equates the Wilson loop in the temporal axial gauge to that in the Feynman gauge. To do this, we first discuss the meaning of the Wilson loop as a quantum operator. It is well known that if

$$A'_\mu = uA_\mu u^{-1} + (ig)^{-1}u\partial_\mu u^{-1} , \tag{3.1}$$

where u is a matrix in the group space, then

$$P\exp\left[-ig\int_{x_1}^{x_2} A'_\mu(x)dx^\mu\right] = u(x_2)P\exp\left[-ig\int_{x_1}^{x_2}A_\mu(x)dx^\mu\right]u^{-1}(x_1) . \tag{3.2}$$

For a closed loop, $x_1=x_2$. Therefore, in the classical theory, one may prove from (3.2) that the Wilson loop is unchanged under a gauge transformation, as the factors $u(x_2)$ and $u^{-1}(x_1)$ cancel each other when the trace is taken. In quantum theory, one must further take note of the fact that the matrix elements of u may be quantum operators, and one is not allowed to move $u(x_2)$ from the leftest position to the rightest position as one takes the trace. It turns out that this minor complication can be handled by ordering the quantum operators in the Wilson loop according to time, as is given by Eq. (2.1). In this way the matrix elements of $u(x_2)$ and $u^{-1}(x_1)$ are adjacent if $t_2=t_1$, and the Wilson loop is unchanged under $A_\mu \to A'_\mu$.

As we have mentioned, $A_\mu(x)$ in the Wilson loop is in the Heisenberg representation. It remains to specify the Hamiltonian used in this representation. Let us consider the Wilson loop with the Hamiltonian being the one corresponding to the effective Lagrangian with ghost fields:[7]

$$L_{\text{eff}} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \tfrac{1}{2}\partial_\mu A^a_\nu \partial^\nu A^{\mu,a} - i(\partial^\mu\eta)^a(D_\mu\xi)^a , \tag{3.3}$$

where η and ξ are Hermitian ghost fields. This is because there exists, in this theory, a conserved [Becchi-Rouet-Stora (BRS)] charge which generates the gauge transformation (3.8) below. The symmetry of the Wilson loop under this gauge transformation will enable us to derive the correspondence formula (3.17) which equates this Wilson loop with that in the original Yang-Mills gauge theory. For clarity of presentation, it will prove convenient to make a unitary transformation represented by

$$U=\exp\left[-i\int A^a_0(\mathbf{x})\nabla\cdot\mathbf{A}^a(\mathbf{x})d^3x\right] \tag{3.4}$$

for the operators in the quantum theory of (3.3). We shall indicate the transformed operator by an overbar, e.g.,

$$\bar{H}_{\text{eff}} = UH_{\text{eff}}U^{-1} ,$$

while H_{eff} is the Hamiltonian corresponding to (3.3). Specifically, we have

$$\bar{H}_{\text{eff}} \equiv \bar{H}_F + \int d^3x[\, i\pi^a_\eta\pi^a_\xi + gf^{abc}A^b_0\xi^c\pi^a_\xi - i(\nabla\eta)^a\cdot(\mathbf{D}\xi)^a] , \tag{3.5}$$

where

$$\bar{H}_F = \int d^3x[\,\tfrac{1}{2}\boldsymbol{\pi}^a\cdot\boldsymbol{\pi}^a + \tfrac{1}{2}\mathbf{B}^a\cdot\mathbf{B}^a + A^a_0(\mathbf{D}\cdot\boldsymbol{\pi})^a - \tfrac{1}{2}\pi^a_0\pi^a_0 - \pi^a_0\nabla\cdot\mathbf{A}^a] . \tag{3.6}$$

In the integrand of (3.6), a term which is equal to a spatial divergence has been dropped, as it does not contribute to $\bar{H}_F$. The BRS charge is

$$\bar{Q} = \int d^3x[-(\mathbf{D}\xi)^a\cdot\boldsymbol{\pi}^a + i\pi^a_\eta\pi^a_0 + \tfrac{1}{2}gf^{abc}\pi^a_\xi\xi^b\xi^c] . \tag{3.7}$$

It is easy to verify from (3.7) that

$$[\bar{Q}, A^{\mu,a}] = -i(D^\mu \xi)^a , \tag{3.8}$$

$$[\bar{Q}, \xi^a]_+ = -\tfrac{1}{2} ig f^{abc} \xi^b \xi^c , \tag{3.9}$$

$$[\bar{Q}, \eta^a]_+ = \pi_0^a , \tag{3.10}$$

and

$$[\bar{Q}, \pi_\xi^a]_+ = -i(\mathbf{D}\cdot\boldsymbol{\pi})^a - ig f^{abc} \pi_\xi^b \xi^c . \tag{3.11}$$

Since $\bar{Q}$ commutes with $\bar{H}_{\rm eff}$, the fields in Eqs. (3.8)–(3.11) can be replaced by their Heisenberg representations. Therefore, (3.8) means that $\bar{Q}$ is the operator which generates the infinitesimal gauge transformation with the group parameters $\xi^a(x)$, where

$$\xi^a(x) = \exp(i\bar{H}_{\rm eff} x_0)\xi^a(\mathbf{x})\exp(-i\bar{H}_{\rm eff} x_0) .$$

It then follows from (3.8) and (3.2) that

$$[\bar{Q}, \bar{W}_{\rm eff}(x_2,x_1)] = gT[\bar{W}_{\rm eff}(x_2,x_1)\xi(x_1) - \xi(x_2)\bar{W}_{\rm eff}(x_2,x_1)] , \tag{3.12}$$

where

$$\xi \equiv \xi^a T^a ,$$

and where

$$\bar{W}_{\rm eff}(x_2,x_1) \equiv PT \exp\left[-ig \int_{x_1}^{x_2} A_\mu(x) dx^\mu\right] , \tag{3.13}$$

with

$$A_\mu(x) \equiv e^{i\bar{H}_{\rm eff} x_0} A_\mu(\mathbf{x}) e^{-i\bar{H}_{\rm eff} x_0} . \tag{3.14}$$

In particular, if $x_1 = x_2$, we have

$$[\bar{Q}, \bar{W}_{\rm eff}] = 0 , \tag{3.15}$$

where $\bar{W}_{\rm eff}$ is the Wilson loop operator in the theory of (3.3). More precisely

$$\bar{W}_{\rm eff} = {\rm tr}\,\bar{W}_{\rm eff}(x_2,x_1) , \tag{3.16}$$

where $x_1 = x_2$ and the contour of integration is a closed one.

We are now ready to prove the correspondence formula

$$\langle \bar{\psi}_1 | \bar{W}_{\rm eff} | \bar{\psi}_2 \rangle = \langle \psi_{w1} | W_w | \psi_{w2} \rangle_w , \tag{3.17}$$

where W_w is the Wilson loop operator in the temporal gauge in which $A_0 = 0$ and

$$\mathbf{A}(x) \equiv e^{iH_w x_0} \mathbf{A}(\mathbf{x}) e^{-iH_w x_0} ,$$

with

$$H_w = \tfrac{1}{2} \int d^3x\, (\boldsymbol{\pi}^a\cdot\boldsymbol{\pi}^a + \mathbf{B}^a\cdot\mathbf{B}^a) . \tag{3.18}$$

Equation (3.17) is valid for any states $|\bar{\psi}\rangle$ and $|\psi_w\rangle$ satisfying[7]

$$\bar{\psi}(A_\mu^a, \eta^a, \xi^a) = e^{\Omega} \psi_w(\mathbf{A}^a) , \tag{3.19}$$

where

$$\Omega = \Theta_1 + \Theta_2 , \tag{3.20}$$

$$\Theta_1 = -i \int d^3x\, \eta^a \frac{1}{\sqrt{-\nabla^2}} (\nabla\cdot\mathbf{D}\xi)^a , \tag{3.21}$$

and

$$\Theta_2 = \tfrac{1}{2} \int d^3x (\sqrt{-\nabla^2} A_0^a - i\nabla\cdot\mathbf{A}^a) \times \frac{1}{\sqrt{-\nabla^2}} (\sqrt{-\nabla^2} A_0^a - i\nabla\cdot\mathbf{A}^a) , \tag{3.22}$$

with

$$(\mathbf{D}\cdot\boldsymbol{\pi})\psi_w = 0 , \tag{3.23}$$

which is the Gauss law. It is straightforward to verify from (3.19)–(3.23) that[7]

$$\bar{Q}\bar{\psi} = 0 . \tag{3.24}$$

The proof of the correspondence formula (3.17) will involve the repeated use of (3.24) for both $|\bar{\psi}_2\rangle$ and $|\bar{\psi}_1\rangle$. The inner product on the left side of (3.17) involves functional integrations over all four polarizations of A_μ^a as well as the ghost fields, while the inner product on the right side of (3.17) involves functional integrations over $\mathbf{A}_T^a$ only.[7] More precisely, (3.17) can be written as

$$\int DA_\mu^a(\mathbf{x}) D\eta^a(\mathbf{x}) D\xi^a(\mathbf{x}) \bar{\psi}_1^*(A_\mu^a, \eta^a, \xi^a) \bar{W}_{\rm eff} \bar{\psi}_2(A_\mu^a, \eta^a, \xi^a) = \int D\,\mathbf{A}^a(\mathbf{x}) \delta(\nabla\cdot\mathbf{A}^a) \det\left[\frac{1}{\nabla^2}\nabla\cdot\mathbf{D}\right] \psi_{w1}^*(\mathbf{A}^a) W_w \psi_{w2}(\mathbf{A}^a) . \tag{3.25}$$

To prove (3.17), let us first prove that if the time components of x_1 and x_2 are equal, then

$$W(t) \equiv \langle \bar{\psi}_1 | {\rm tr}\,\bar{W}_{\rm eff}(x_1,x_2,>) e^{i\bar{H}_{\rm eff} t} e^{-iH_w t} W_w(x_2,x_1,<) | \bar{\psi}_2 \rangle \tag{3.26}$$

is independent of this time component denoted by t. In (3.26), the points on the contour of W_w ($\bar{W}_{\rm eff}$) are of time components smaller (larger) than t, as indicated by the sign $<$ ($>$) in (3.26). To prove this, we differentiate (3.26) with respect to time and get

$$i\frac{dW(t)}{dt} = \langle \bar{\psi}_1 | {\rm tr} P \bar{W}_{\rm eff}(x_1,x_2,>) e^{i\bar{H}_{\rm eff} t} [H_w - \bar{H}_{\rm eff} + gA_0(\mathbf{x}_1) - gA_0(\mathbf{x}_2)] e^{-iH_w t} W_w(x_2,x_1,<) | \bar{\psi}_2 \rangle . \tag{3.27}$$

The term in square brackets in (3.27) is equal to zero. To see this, we have from (3.8)–(3.11) that

$$\pi_0^a \pi_0^a = [\bar{Q}, \eta^a \pi_0^a]_+ , \tag{3.28}$$

$$\pi_0^a \nabla\cdot\mathbf{A}^a - i\eta^a(\nabla\cdot\mathbf{D}\xi)^a = [\bar{Q}, \eta^a \nabla\cdot\mathbf{A}^a]_+ , \tag{3.29}$$

and

$$A_0^a(\mathbf{D}\cdot\boldsymbol{\pi})^a + g f^{abc} A_0^b \xi^c \pi_\xi^a + i\pi_\eta^a \pi_\xi^a = [\bar{Q}, i\pi_\xi^a A_0^a]_+ . \tag{3.30}$$

From (3.28)–(3.30), (3.5), and (3.18), we get

$$H_w = \bar{H}_{\rm eff} = \int d^3x [\bar{Q}, \tfrac{1}{2}\eta^a \pi_0^a + \eta^a \nabla\cdot\mathbf{A}^a - i\pi_\xi^a A_0^a]_+ . \tag{3.31}$$

Because of (3.31), the term $H_w-\overline{H}_{\rm eff}$ in (3.27) becomes, after we make use of (3.12) and (3.24),

$$\int d^3x[g\xi(\mathbf{x}_1)-g\xi(\mathbf{x}_2),\tfrac{1}{2}\eta^a\pi_0^a+\eta^a\nabla\cdot\mathbf{A}^a-i\pi_\xi^a A_0^a]_+ = gA_0(\mathbf{x}_2)-gA_0(\mathbf{x}_1)\ . \quad (3.32)$$

In deriving (3.32), we have made use of the facts that $\overline{Q}$ commutes with H_w, (3.24) holds for ψ_w, and

$$[\overline{Q},W_w(x_2,x_1)]=gTW_w(x_2,x_1)[\xi(x_1)-\xi(x_2)]\ , \quad (3.33)$$

which is the counterpart of Eq. (3.12). From (3.27) and (3.32), we find that $W(t)$ is independent of t. Therefore,

$$W(T_f)=W(T_i)\ , \quad (3.34)$$

where T_f (T_i) is the largest (smallest) time on the contour c in the Wilson loop. Equation (3.34) is, explicitly,

$$\langle\overline{\psi}_1\,|\,\overline{W}_{\rm eff}e^{i\overline{H}_{\rm eff}T_i}e^{-iH_wT_i}\,|\,\overline{\psi}_2\rangle = \langle\overline{\psi}_1\,|\,e^{i\overline{H}_{\rm eff}T_f}e^{-iH_wT_f}W_w\,|\,\overline{\psi}_2\rangle\ . \quad (3.35)$$

As we have proven in Ref. 7, we may replace $\overline{H}_{\rm eff}$ in (3.35) by H_w. This is because $e^{-iH_wT_i}\,|\,\overline{\psi}_2\rangle$, $e^{-iH_wT_f}W_w\,|\,\overline{\psi}_2\rangle$, $|\,\overline{\psi}_1\rangle$, and $\overline{W}_{\rm eff}\,|\,\overline{\psi}_1\rangle$ satisfy (3.24), and a correspondence formula holds for the matrix elements of $e^{i\overline{H}_{\rm eff}t}$ between states satisfying (3.24) (Ref. 7). Equation (3.35) then becomes

$$\langle\overline{\psi}_1\,|\,\overline{W}_{\rm eff}\,|\,\overline{\psi}_2\rangle=\langle\overline{\psi}_1\,|\,W_w\,|\,\overline{\psi}_2\rangle\ . \quad (3.36)$$

To reduce (3.36) to (3.25), we need to carry out the functional integration over A_0 as well as the ghost fields in the right side of Eq. (3.36). These integrations cannot be done yet, as the operator W_w operates on the e^Ω factor in the initial wave function. To move this e^Ω factor to the left of W_w, let us define, similar to (3.26), an intermediate quantity,

$$\Lambda(t)\equiv\langle e^\Omega\psi_{w1}\,|\,{\rm tr}W_w(x_1,x_2,>)e^{iH_wt}e^\Omega \times e^{-iH_wt}W_w(x_2,x_1,<)\,|\,\psi_{w2}\rangle\ , \quad (3.37)$$

where t is the time component of x_1 and x_2. We shall show that $\Lambda(t)$ is independent of t. The derivative of $\Lambda(t)$ with respect to t is

$$-i\frac{d\Lambda(t)}{dt}=\langle e^\Omega\psi_{w1}\,|\,{\rm tr}W_w(x_1,x_2,>)e^{iH_wt}[H_w,e^\Omega] \times e^{-iH_wt}W_w(x_2,x_1,<)\,|\,\psi_{w2}\rangle\ . \quad (3.38)$$

The commutator on the right side of Eq. (3.38) can be decomposed as

$$[H_w,e^\Omega]=\tfrac{1}{2}[(\pi_i^a)^2,e^\Omega] =\tfrac{1}{2}\pi_i^a[\pi_i^a,\Omega]e^\Omega+\tfrac{1}{2}[\pi_i^a,\Omega]e^\Omega\pi_i^a\ . \quad (3.39)$$

We shall show that (3.39) vanishes upon being inserted to (3.38). Applying Eqs. (3.8)–(3.10) to (3.21), we get

$$\Theta_1=\int d^3x\left[\overline{Q},\eta^a\frac{1}{\sqrt{-\nabla^2}}\nabla\cdot\mathbf{A}^a\right]_+ -\int d^3x\ \pi_0^a\frac{1}{\sqrt{-\nabla^2}}\nabla\cdot\mathbf{A}^a\ . \quad (3.40)$$

Thus (3.20) can be written as

$$\Omega=\Omega_1+\Omega_2\ , \quad (3.41)$$

where

$$\Omega_1\equiv\int d^3x\left[\overline{Q},\eta^a\frac{1}{\sqrt{-\nabla^2}}\nabla\cdot\mathbf{A}^a\right]_+\ , \quad (3.42)$$

$$\Omega_2\equiv-\int d^3x\ \pi_0^a\frac{1}{\sqrt{-\nabla^2}}\nabla\cdot\mathbf{A}^a+\Theta_2\ , \quad (3.43)$$

and Θ_2 is expression (3.22).

It is straightforward to show that

$$[\pi_i^a,\Omega_2]e^\Omega F(\mathbf{A})=0\ , \quad (3.44)$$

where F is any function of $\mathbf{A}$. This can be done by computing $[\pi_i^a,\Omega_2]$ explicitly and observing the explicit dependence of π_0 in this commutator and that of A_0 in Ω. The factor $[\pi_i^a,\Omega]$ in both terms of the right side of Eq. (3.39) can therefore be replaced by $[\pi_i^a,\Omega_1]$ when (3.39) is substituted into (3.38). Therefore the right-hand side of (3.38) can be rewritten as

$$\tfrac{1}{2}\langle e^\Omega\psi_{w1}\,|\,{\rm tr}W_w(x_1,x_2,>)e^{iH_wt}(\pi_i^a[\pi_i^a,\Omega_1]e^\Omega+[\pi_i^a,\Omega_1]e^\Omega\pi_i^a)e^{-iH_wt}W_w(x_2,x_1,<)\,|\,\psi_{w2}\rangle\ . \quad (3.45)$$

Next we substitute the right side of (3.42) for Ω_1 in (3.45). We may then replace $\overline{Q}$ in the resulting expression by $gT(\xi(x_1)-\xi(x_2))$. This can be proved by writing out all the terms in $[\pi_i^a,\Omega_1]$, with Ω_1 replaced by the right side of (3.42), moving $\overline{Q}$ to the left (right) for the term in which $\eta^a(1/\sqrt{-\nabla^2})\nabla\cdot\mathbf{A}^a$ lies to the right (left) of $\overline{Q}$, and making use of

$$[\overline{Q},e^\Omega]=0,\quad \overline{Q}\,|\,\psi_w\rangle=0,\quad [\overline{Q},\pi_i^a]=-igf^{abc}\xi^b\pi_i^c\ .$$

Since $\xi(x_1)-\xi(x_2)$ anticommutes with $\eta^a(1/\sqrt{-\nabla^2})\nabla\cdot\mathbf{A}^a$, expression (3.45) vanishes. Thus $\Lambda(t)$ is a constant and we obtain the conclusion

$$\Lambda(T_f)=\Lambda(T_i)$$

or

$$\langle\overline{\psi}_1\,|\,W_we^{iH_wT_i}e^\Omega e^{-iH_wT_i}\,|\,\psi_{w2}\rangle=\langle\overline{\psi}_1\,|\,e^{iH_wT_f}e^\Omega e^{-iH_wT_f}W_w\,|\,\psi_{w2}\rangle\ . \quad (3.46)$$

Finally, we may move the e^{Ω} to the left of $e^{iH_w T_i}$ or $e^{iH_w T_f}$ in the expression above. This is because we may prove, just as in the above, that the commutator $[H_w, e^{\Omega}]$ vanishes when it is inserted between two states satisfying (3.24). Thus

$$\langle \bar{\psi}_1 \mid W_w \mid \bar{\psi}_2 \rangle = \langle \bar{\psi}_1 \mid e^{\Omega} W_w \mid \psi_{w2} \rangle \ . \tag{3.47}$$

By carrying out the functional integrations for the ghost fields and the A_0 field, we obtain

$$\int DA_\mu^a(\mathbf{x}) D\eta^a(\mathbf{x}) D\xi^a(\mathbf{x}) \bar{\psi}_1^*(A_\mu^a, \eta^a, \xi^a) \overline{W}_{\text{eff}} \bar{\psi}_2(A_\mu^a, \eta^a, \xi^a)$$
$$= \int D\,\mathbf{A}^a(\mathbf{x}) \exp\left[-\int d^3x\, (\nabla\cdot\mathbf{A}^a) \frac{1}{\sqrt{-\nabla^2}} (\nabla\cdot\mathbf{A}^a)\right] \det\left[\frac{1}{\nabla^2}\nabla\cdot\mathbf{D}\right] \psi_{w1}^*(\mathbf{A}^a) W_w \psi_{w2}(\mathbf{A}^a)\ , \tag{3.48}$$

which is equivalent to the correspondence formula (3.25) (see Ref. 7).

Equation (3.17) is an exact formula, and is valid between states $\mid \psi_w \rangle$ satisfying the Gauss law. If we choose $\mid \psi_{w1} \rangle$ and $\mid \psi_{w2} \rangle$ to be vacuum states, and if we make a perturbative expansion of (3.17) and use the adiabatic hypothesis, the left (right) side of (3.17) is the perturbation series for the Wilson loop in the Feynman (temporal axial) gauge. The propagator for the longitudinal gluon on the right side of (3.17) is

$$D_{LL}(x,x') = -i\delta^{(3)}(\mathbf{x}-\mathbf{x}')t_> \ , \tag{3.49}$$

where $t_>$ is the larger of x_0 and x_0'. Also, for the transverse gluon, we have

$$D_{ij}^T(x,x') = i\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\epsilon'} \left[\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right] . \tag{3.50}$$

In addition

$$D_{0\mu}(x,x') = D_{\mu 0}(x,x') = 0 \ . \tag{3.51}$$

This is the propagator given by Eq. (8) of Ref. 2 with $\alpha = -1$. The other choice of $\alpha = 1$ in Ref. 2, also giving the correct answer, can be obtained from our formalism by reversing the roles of ψ_{w1}^* and ψ_{w2}. Note that, in this prescription, we pay the price of translational invariance for the propagator but gain the advantage that $D_{0\mu}$ is strictly zero.

Finally, we mention that the correspondence formula (3.17) can be extended to the covariant α gauge, the spatial axial gauge, and the Coulomb gauge.

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