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Renormalization of the Yang-Mills theories in the light-cone gauge

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The structure of the renormalization of the Yang-Mills theories in the light-cone gauge is investigated. It is shown that, despite the appearance of an infinite number of nonlocal divergent terms, the theory can be made finite to any order in the loop expansion by introducing a finite number of renormalization constants. Those constants can be interpreted as coefficients of a canonical transformation of fields and coupling constants in such a way that gauge invariance and unitarity of the renormalized theory are manifestly satisfied. In particular it is shown that the nonlocal structures are completely decoupled from the physical quantities.

I. INTRODUCTION

The light-cone gauge has been frequently used in perturbative calculations involving Yang-Mills theories. Its main virtue is the possibility of a partonic interpretation: after eliminating the redundant degrees of freedom, the number of the independent fields equals the number of "quanta" present in the theory. This circumstance makes this gauge choice particularly suited to study the properties of supersymmetric models[1] as well as in connection with noncovariant formulations of string theories.[2]

The resulting treatment is, however, quite singular; in particular, it cannot be considered as the limiting case of an axial-gauge formulation of the theory when the gauge vector n_μ becomes lightlike, owing to the presence of severe singularities in the Green's functions when $n^2 \to 0$.[3]

As a matter of fact, it was commonly believed that, within the light-cone gauge choice, as well as for the axial-gauge choice,[4] only physical transverse quanta were needed to describe the dynamics of the massless vector fields, at the price of losing manifest Lorentz covariance, due to the presence of the gauge vector n_μ, and of getting spurious singularities in the vector propagator (and in the whole set of the Green's functions), which need to be properly and carefully handled. The quantization procedure set up along the same lines of the axial case, as traditionally done for instance within the "null plane" formalism,[5] unfortunately leads to an inconsistent formulation of the theory, at least as far as the perturbative framework is concerned.[6] The point is that a resolution of the constraint equations in terms of the physical degrees of freedom, as it could be implemented by inverting the differential operator $n^\mu \partial_\mu$, in analogy with the axial case, *forces* the use of the principal-value prescription for the spurious singularities. This in turn results in a breakdown of the power-counting criterion[6] for the light-cone Feynman integrals, preventing the usual renormalization procedure for the Green's functions. Another drawback of the principal-value prescription, strictly related to the previous one, is the loss of simple rules, such as the familiar Wick rotation to get Euclidean Feynman integrals. As a consequence, this quantization scheme has to be abandoned.

More recently a different quantization based on the usual equal-time operator algebra has been proposed.[7] The main consequence of this approach has been the appearance of a new prescription, we shall call the Leibbrandt-Mandelstam (LM) prescription for handling the spurious singularity in the vector propagator, in a way which allows a Wick rotation without extra contributions and thereby a power-counting criterion for the convergence of the Feynman integrals. Such a prescription was previously proposed in Refs. 8 and 9 with precisely this motivation. For instance, if the β function in the supersymmetric (SUSY) $n=4$ model is computed from the self-energy at one loop using this prescription, it is found to vanish in agreement with the result obtained in other gauges.[6]

Within this new quantization scheme the Gauss law does not hold as an operator constraint equation in the whole Hilbert space, where a ghostlike degree of freedom propagating along the generating line of the light-cone is present. Nevertheless it was shown[7] that it is still possible to select in the Hilbert space, a physical subspace with positive-semidefinite metric where the Gauss law, the Lorentz covariance, and the unitarity of the formal S matrix are recovered, at least in the framework of perturbation theory. The essential point is that the ghostlike quanta are necessarily present to generate the nice LM prescription for the spurious singularities, but, as required from gauge invariance, they decouple from physical quantities.

We notice, however, that the above statements, although quite natural and satisfactory, are only of a formal character, due to the occurrence of the well-known ultraviolet divergences. It is the aim of this paper to investigate in detail the general structure of those ultraviolet divergences, and to set up a consistent renormalization procedure for the Green's functions, in such a way that the above-mentioned physical requirements are still manifestly satisfied.

It was soon realized that the accomplishment of that program is not a trivial application of general known results, owing to some unusual features specific of the light-cone gauge, as already emerging in the one-loop calculations.[9,10] The difficulties are twofold. The first one is connected with the presence in the LM prescription of a "dual" null gauge vector n_μ^* besides the original vector n_μ which defines the gauge choice. The occurrence of two gauge vectors allows the dangerous possible appearance of a very wide class of Lorentz-noncovariant divergent structures compatible with dimensional analysis and symmetry properties. The second, and more delicate problem to deal with, is the presence of singularities with a nonlocal character in the proper vertices: namely, of poles at $\omega=2$ (2ω being the dimensions of space-time) with residues which are not polynomials in the external momenta.[6,9] We stress that this pathology is peculiar to the light-cone gauge choice; indeed one can prove that other algebraic gauges do not share it. Nonlocal divergences require nonlocal counterterms; plenty of them can be envisaged starting from the basic nonlocal quantity $(n^\mu\partial_\mu)^{-1}n^\nu A_\nu$, which is homogeneous in n_μ and *dimensionless*, in a way compatible with the Lee-Ward identities; therefore, a rationale must be found to put the otherwise *a priori* arbitrary proliferation of those peculiar divergent structures under control. As a matter of fact, we shall succeed in determining the precise form of those counterterms, starting from basic symmetry principles and fully exploiting the characteristic features of this gauge. It will be found that, although their number is infinite, as they are present in one-particle-irreducible (1PI) vertices with any number of legs, their general form is quite specific and depends on few renormalization constants. We shall also show that those nonlocal counterterms, although necessary in order to make the 1PI vertices finite, always decouple from the physical sector.

After having reviewed the canonical quantization and introduced the basic notations in Sec. II, in Sec. III we review and comment on previous results showing how renormalization can be accomplished at the one-loop level. This is quite useful and actually necessary as a first step towards the inductive proof of renormalization at any order in the loop expansion. The latter is given in Secs. IV and V for the pure Yang-Mills theory and extended in Sec. VI to the case in which massive Dirac fermions are also present.

We stress that, after having understood the peculiarities of the light-cone gauge in a well-established theory (at least in the perturbative sense), we can now apply this gauge choice to more complicated situations, being able to disentangle true dynamical difficulties from pure gauge artifacts.

II. QUANTIZATION OF THE YANG-MILLS THEORIES IN THE LIGHT-CONE GAUGE

In this section we summarize the main results we obtained in Ref. 7. In doing so we also establish our notations and conventions. Starting from the Lagrangian density

$$\mathcal{L}=-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a}-\Lambda^a n^\mu A^a_\mu+\bar\psi(i\partial\!\!\!/-m)\psi+g\bar\psi A\!\!\!/^a\tau^a\psi\ , \tag{2.1}$$

where

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\ , \tag{2.2}$$

τ^a is a basis in the fundamental representation of the Lie algebra of the internal-symmetry group, Λ^a are Lagrange multipliers, and n_μ is the lightlike vector $(n_0,\mathbf{n})$, we derive the equations of motion and constraints:

$$D^{ab}_\nu F^{b,\mu\nu}+g\bar\psi\tau^a\gamma^\mu\psi=n^\mu\Lambda^a\ , \tag{2.3}$$

$$n^\mu A^a_\mu=0\ , \tag{2.4}$$

$$(i\mathcal{D}\!\!\!/-m)\psi=0\ , \tag{2.5}$$

where

$$D^{ab}_\mu=\partial_\mu\delta^{ab}-gf^{abc}A^c_\mu \tag{2.6}$$

and

$$\mathcal{D}_\mu=\partial_\mu\mathbb{1}-ig\tau^a A^a_\mu\ . \tag{2.7}$$

Applying the covariant derivative D^{ca}_μ to Eq. (2.3) and taking into account that the fermionic current is covariantly conserved, we get the basic equation

$$n_\mu\partial^\mu\Lambda^a=0 \tag{2.8}$$

for the Lagrange multipliers.

Equation (2.8) can *a priori* give rise to two different approaches. The first one is to consider a null-plane formalism[4,5] in which Eq. (2.8) is treated as a constraint; with suitable boundary conditions the differential operator $n\cdot\partial\equiv\partial_-$ can be inverted, thereby recovering the Gauss law, in a strong sense, in the Dirac terminology. Following this way, the spurious singularity in the vector propagator turns out to be defined as the Cauchy principal value. It leads, however, to trouble in the perturbative expansion for the Green's functions, as the superficial degree of convergence of an arbitrary Feynman diagram is no longer controlled by any power-counting criterion. Moreover, a computation of the first coefficient in the perturbative expansion for the β function of the renormalization group disagrees with the one obtained, e.g., in a covariant gauge.[6]

The second possibility is to look at Eq. (2.8) from the usual point of view of the time-evolution framework. Within this context the above equation is nothing but an equation of motion, so that the Gauss law has to be thought to hold in a weaker sense, as we shall better specify in the following.

This fact amounts to keeping extra degrees of freedom, namely, in dealing with a Hilbert space with indefinite metric, where ghostlike states are present. Nevertheless, we have shown in Ref. 7 that it is possible to select a subspace $\mathcal{H}_P$ of the physical states in which Λ^a vanish; i.e., the Gauss law is satisfied. This subspace has a positive-semidefinite metric and is stable under the Poincaré group generators, whose algebra closes in $\mathcal{H}_P$; furthermore, the restriction of the formal perturbative S matrix on $\mathcal{H}_P$ is a unitary operator.

Following this last quantization scheme a new prescription emerges for the spurious singularity of the vector propagator,[7-9] viz.,

$$\frac{1}{[n\cdot p]}\equiv\frac{n^*\cdot p}{(n\cdot p)(n^*\cdot p)+i\epsilon}=\frac{1}{n\cdot p+i\epsilon\,\mathrm{sgn}(n^*\cdot p)}\ , \tag{2.9}$$

where n^*_μ is the null vector $(n_0,-\mathbf{n})$. The LM prescription of Eq. (2.9) allows the usual Wick rotation and a power-counting criterion for the Feynman integrals. Equation (2.9) implies the presence of a ghost propagating along the generating line of the light cone, as already noticed.[7]

In conclusion, if one wants to perform perturbative calculations in the light-cone gauge, the LM prescription must be chosen, which, however, makes explicit use of the "dual" vector n^*_μ. It is just owing to the presence of the ghostlike degree of freedom and of the dual gauge vector, endowed in the LM prescription, that it becomes nontrivial to control the theory when loop corrections are taken into account. We shall see, however, in the remainder of this paper that all new objects, typical of the light-cone quantum theory, work very well in producing coherent and satisfactory results to any order in perturbation theory.

III. THE ONE-LOOP RENORMALIZATION

In this section we recall some known results concerning one-loop calculations with the aim of pointing out the most remarkable features of this gauge choice. In so doing we shall discuss the main difficulties of the theory which prevent a straightforward application of the standard renormalization procedure to this case.

The one-loop calculation of the divergent part of the self-energy tensor has been given in Ref. 9 and reads

$$\Pi_{\mu\nu}(p)=\frac{ig^2c_A}{16\pi^2(2-\omega)}\left[\tfrac{11}{3}(g_{\mu\nu}p^2-p_\mu p_\nu)-2\frac{p^2}{n\cdot n^*}(\hat n_\mu\hat n^*_\nu+\hat n^*_\mu\hat n_\nu)\right]\ , \tag{3.1}$$

$$\hat n_\mu=n_\mu-p_\mu n\cdot p/p^2,\quad \hat n^*_\mu=n^*_\mu-n_\mu n^*\cdot p/n\cdot p\ .$$

We notice that the presence of the vector n^*_μ allows several non-Lorentz-covariant structures; in addition a nonpolynomial coefficient appears which requires nonlocal subtractions in the Lagrangian. It is worth emphasizing that this term is unavoidable in order to satisfy the transversality condition

$$p_\mu\Pi^{\mu\nu}(p)=0\ , \tag{3.2}$$

which is one of the Ward identities. We also stress that out of the possible four independent transverse tensor structures allowed by Eq. (3.2), only two appear in Eq. (3.1).

The divergent part of the three-vector-irreducible proper vertex has been computed in Ref. 11. As it has been explicitly verified there, the terms which are present correctly map onto the corresponding ones of the self-energy as required by the Lee-Ward identity

$$p^\mu\Gamma^{abc}_{\mu\nu\rho}(p,q,r)=g\mu^{2-\omega}f^{abc}[\Pi_{\nu\rho}(r)-\Pi_{\nu\rho}(q)]\ . \tag{3.3}$$

Once again nonpolynomial coefficients arise in the following four structures:

$${}^{(1)}S^{abc}_{\mu\nu\rho}(p,q,r)=f^{abc}\left[g_{\nu\rho}n_\mu\left[\frac{n\cdot q\,n^*\cdot q-n\cdot r\,n^*\cdot r}{n\cdot n^*\,n\cdot p}\right]\right]_{(s)}\ , \tag{3.4a}$$

$${}^{(2)}S^{abc}_{\mu\nu\rho}(p,q,r)=f^{abc}\left[p_\mu n_\nu n_\rho\left[\frac{n^*\cdot q}{n\cdot q}-\frac{n^*\cdot r}{n\cdot r}\right]\frac{1}{n\cdot n^*}\right]_{(s)}\ , \tag{3.4b}$$

$${}^{(3)}S^{abc}_{\mu\nu\rho}(p,q,r)=f^{abc}\left[\frac{(q-r)_\mu n_\nu n_\rho}{n\cdot n^*}\left[\frac{n^*\cdot q}{n\cdot q}+\frac{n^*\cdot r}{n\cdot r}\right]\right]_{(s)}\ , \tag{3.4c}$$

$${}^{(4)}S^{abc}_{\mu\nu\rho}(p,q,r)=f^{abc}n_\mu n_\nu n_\rho\left[\frac{q^2n^*\cdot r-r^2n^*\cdot q}{n\cdot n^*\,n\cdot q\,n\cdot r}\right]_{(s)}\ , \tag{3.4d}$$

where the subscript (s) means cyclic symmetrization.

The trouble is that, just owing to the presence of nonpolynomial coefficients, the Lee-Ward identity (3.3) might be eluded by an infinite number of possible kernels, as for instance the tensor

$$K^{abc}_{\mu\nu\rho}(p,q,r)=f^{abc}\left[\left[g_{\nu\rho}-\frac{n_\nu q_\rho}{n\cdot q}\right]\left[r_\mu-\frac{n_\mu}{n\cdot p}p\cdot r\right]-\left[g_{\mu\rho}-\frac{n_\mu p_\rho}{n\cdot p}\right]\left[r_\nu-\frac{n_\nu}{n\cdot q}q\cdot r\right]\right]_{(s)}\ , \tag{3.5}$$

which satisfies the homogeneous equation

$$p^\mu K^{abc}_{\mu\nu\rho}(p,q,r)=0\ . \tag{3.6}$$

This critical feature of the light-cone gauge has been discovered in Ref. 12 in the fermion-fermion-vector vertex, where it really occurs at the one-loop level, and further discussed in Ref. 13; in this paper the existence of a fine-tuning mechanism to treat it has been explicitly recognized at the one-loop level.

One should also notice that, because kernels are possible, the equality of the coefficients of the standard covariant parts of the three-vector vertex and of the self-energy (implying $Z_1 = Z_3$) at one loop, should be regarded, at this level, as rather accidental.

In an analogous way, N-point 1PI vertices could exhibit divergences with nonpolynomial behavior. In particular, the complete divergent part of the 1PI four-vector vertex at one loop has never been explicitly computed to our knowledge. The renormalization procedure we shall develop in the following sections will give, as a byproduct, a precise prediction concerning this divergent term as reported in Appendix A. It might be interesting to check this prediction by means of a direct calculation. In summary, if one wants to make all the 1PI vertices finite, one has to perform nonlocal subtractions in a consistent way: namely, one has to prove that the theory can be renormalized by a suitable finite number of independent counterterms. The same troublesome features also occur in the Becchi-Rouet-Stora (BRS) approach; a program along this line has been recently attempted by some authors.[14]

As far as the one-loop Green's functions are concerned, we have shown that renormalization is possible by means of local counterterms;[15] the extension to the many-loop case was, however, lacking.

In the remainder of this paper we will be able to prove that, at any order in the loop expansion, the 1P1 vertices can be made finite by introducing nonlocal counterterms of a "vanishing" type and of a very special nature. The renormalized Lagrangian density can be obtained by performing a canonical transformation on the fields in analogy with the renormalization of the Yang-Mills theory in the planar gauge.[16]

As a byproduct, one can show that the Green's functions (and *a fortiori* the formal S matrix) can be renormalized only with local counterterms to any loop order.

IV. THE STRUCTURE OF THE DIVERGENCES IN THE LIGHT-CONE INTEGRALS

In this section we will treat for simplicity the pure Yang-Mills theory, deferring the general case to Sec. VI. Our aim is to prove the following basic property: (i) it is possible to make finite all the Green's functions by introducing only a finite number of local counterterms; (ii) the 1PI vertices can be made finite by subtracting a special kind of local and nonlocal counterterms which will be completely determined.

Let us introduce some useful definitions. We shall call the *effective part* of a 1PI graph G the corresponding dimensionally regularized analytic expression in which possible terms containing at least one gauge vector n_μ with external Lorentz index μ are disregarded. This definition is convenient, as only effective parts of a 1PI graph can contribute as subdivergences in a Green's function.

The second definition we need is an extension of the concept of the degree of divergence of a diagram. We recall that the prescription (2.9) allows the usual Wick rotation without extra contributions and thereby, eventually, a power-counting criterion for the convergence of the integrals is indeed respected. Actually two different kinds of power counting are needed to deal with light-cone integrals, one related to the total high-momentum behavior of the integrands not distinguishing among the different components of the loop momenta, and another concerning the high-momentum behavior with respect to the transverse components $k_\perp^{(j)} = (\hat{k}^{(j)}, n^* \cdot k^{(j)})$, $k^{(j)}$ being any one of the loop momenta and $\hat{k}^{(j)}$ being its components orthogonal to both n_μ and n_μ^*. This fact obviously stems from the noncovariant nature of the propagator. For example, the propagator contributes with power $\alpha = -2$ to the total high-momentum behavior of the integrand of a diagram, while the covariant and the spurious parts of the propagator contribute with powers, respectively, given by $\beta_\perp = -2$ and $\beta_\perp = -1$ to the transverse high-momentum behavior of the graph.[17]

The overall (or superficial) degrees of divergence $\deg(G)$ and $\deg_\perp(G)$ are the indices obtained by counting the total and transverse powers of loop momenta in a graph G. It is evident that the total index coincides with the usual index of the corresponding covariant version of the graph.

A sufficient condition for the convergence of a light-cone Feynman integral will be that *both* the above-mentioned overall indices be negative for the whole diagram as well as for any possible subdiagram.[18] The validity of this statement is evident if one realizes that (i) the two indices are what is needed to control the convergence of the integrals in the lack of covariance and (ii) the LM prescription (2.9) for the spurious denominators in the integrand is a Feynman-type prescription: namely, it can be simply Wick-rotated to give absolutely convergent integrals in Euclidean space. Thereby, with those ingredients, one can mimic step by step the reasoning used in the covariant case.

We also employ the usual concept of subtracted graph and Bogoliubov R and $\bar{R}$ operations acting on the dimensionally regularized integral corresponding to a graph G, as introduced for instance in Ref. 19. We can now state and prove the following lemma.

Lemma 1. The pole part (in the dimensional regularization) of a subtracted graph $\bar{R}(G)$ is a polynomial in the transverse components $p_{\perp,i}$ of the external momenta.

This simply follows by observing that the operation $\partial_\perp$ of derivation with respect to transverse external momenta entails a lowering of *both* degrees of divergence and, thereby, eventually leads to convergent integrals. As a consequence, the reasoning of Ref. 20 can be repeated to get the desired result.

As a *corollary* one can show, using dimensional analysis and homogeneity with respect to the vectors n_μ^* and n_μ, that the divergent parts of a graph with more than four external legs necessarily are of vanishing type: namely, they possess at least a factor n_{μ_i}, μ_i being an external Lorentz index.

We are thus led to investigate the graphs with two, three, and four external legs, according to the mentioned criteria.

First of all any subtracted two-point effective graph with n loops can only exhibit the following divergent tensorial structures:

$$\begin{aligned}
{}^{(n)}S^{(1)}_{\mu\nu}(p)&=\gamma_1(\epsilon)g_{\mu\nu}p^2\ ,\\
{}^{(n)}S^{(2)}_{\mu\nu}(p)&=\gamma_2(\epsilon)p_\mu p_\nu\ ,\\
{}^{(n)}S^{(3)}_{\mu\nu}(p)&=\gamma_3(\epsilon)g_{\mu\nu}\frac{n\cdot pn^*\cdot p}{n\cdot n^*}\ ,\\
{}^{(n)}S^{(4)}_{\mu\nu}(p)&=\gamma_4(\epsilon)n^*_\mu p_\nu n\cdot p/n\cdot n^*\ ,\\
{}^{(n)}S^{(5)}_{\mu\nu}(p)&=\gamma_5(\epsilon)n^*_\nu p_\mu n\cdot p/n\cdot n^*\ ,\\
{}^{(n)}S^{(6)}_{\mu\nu}(p)&=\gamma_6(\epsilon)n^*_\mu n^*_\nu\left[\frac{n\cdot p}{n\cdot n^*}\right]^2\ ,
\end{aligned}\tag{4.1}$$

where $\gamma_i(\epsilon)$ are divergent coefficients when $\epsilon\to 0$. All of them are local and none of them can be multiplied by functions of the dimensionless quantity $Q(p)=p^2/n\cdot pn^*\cdot p$ [of course, under multiplication by $Q(p)$, $S^{(3)}\to S^{(1)}$], as the already established polynomial behavior in the transverse momentum $p_\perp$ would be contradicted.

The singular part of any effective subtracted three-point proper graph with n loops can exhibit the following independent structures obeying the mentioned constraints:

$$\begin{aligned}
{}^{(n)}S^{abc}_{1\mu\nu\rho}(p,q,r)&=\gamma_7(\epsilon)f^{abc}g_{\nu\rho}(q-r)_\mu\ ,\\
{}^{(n)}S^{abc}_{2\mu\nu\rho}(p,q,r)&=\gamma_8(\epsilon)f^{abc}g_{\nu\rho}\frac{n\cdot(q-r)}{n\cdot n^*}n^*_\mu\ ,
\end{aligned}\tag{4.2}$$

and cyclic permutations thereof.

It is clear that factors involving functions of the dimensionless quantities $Q_{ijkl}=n\cdot p_i n^*\cdot p_j/n\cdot n^* p_k\cdot p_l$ are forbidden; however, functions of dimensionless quantities such as $Q_{ij}=n\cdot p_i/n\cdot p_j$ are still possible. An example of those dangerous coefficients is given by

$$S^{abc}_{\mu\nu\rho}(p,q,r)=\gamma_9(\epsilon)f^{abc}g_{\mu\nu}r_\rho\frac{n\cdot(p-q)}{n\cdot r}\ .\tag{4.3}$$

Such a nonpolynomial effective term, which would not even contribute to the on-shell formal S matrix [as $e^{(\alpha)}(r)\cdot r=0$], and analogous ones, which can be set up out of Q_{ij}-type variables, would jeopardize, if present, the renormalizability of the theory in this gauge. We shall shortly give an argument which will forbid the appearance of those unpleasant dimensionless ratios of external longitudinal momenta.

Finally, the singular effective part of the subtracted four-point graph with n loops can only exhibit the usual covariant structure

$${}^{(n)}S^{abcd}_{\mu\nu\rho\sigma}(p,q,r,t)=\gamma_{10}(\epsilon)f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})\ ,\tag{4.4}$$

always apart from cyclic permutations and from dimensionless ratios of Q_{ij} type which will be eventually excluded. Then, if we are able to rule out a possible dependence of the singular part of the subtracted effective 1PI graphs on those dimensionless ratios of external longitudinal momenta, we have proven the first part of the basic property we have announced.

To reach this goal we have first to prove a lemma which helps in making more transparent the relation between the light-cone gauge and other algebraic gauge choices. It is well known that the light-cone gauge cannot be obtained as the limit of an axial or planar gauge when $n^2\to 0$, as this limit is a highly singular one.[3] However, a kind of continuity still exists with respect to the component n^0 of the gauge vector n^μ. To understand this point, let us consider once more the spurious singularity of the vector propagator in the light-cone gauge with the LM prescription (2.9):

$$\frac{1}{[n\cdot k]}=\frac{n_0,k_0+\mathbf{n}\cdot\mathbf{k}}{(n_0k_0)^2-(\mathbf{n}\cdot\mathbf{k})^2+i\epsilon}\ ;\tag{4.5}$$

of course in the light-cone case $|n_0|=|\mathbf{n}|$; however Eq. (4.5) defines a distribution allowing Wick rotation without extra contributions *for any value* of the parameters n_0 and $\mathbf{n}$, provided that $\mathbf{n}\neq 0$. In other words, the prescription (4.5) avoids the pinches of the integration contours between Feynman and spurious singularities, so that the limit $n_0\to 0$, $|\mathbf{n}|\neq 0$, in the sense of distributions is a smooth one. This continuity property is even more apparent if we consider the structure of the Euclidean space version of a dimensionally regularized, absolutely convergent, subtracted integral corresponding to the effective part of a 1PI proper graph with n loops and no subdivergences, as it can be easily realized.

Now it is known that Feynman integrals with spacelike spurious denominators can be subtracted with polynomial counterterms. This theorem can be inductively proven, using for instance the argument given in Ref. 3. The proof is based on the possibility of writing the spurious term $k_\mu/n\cdot k$ as $(\partial/\partial n^\mu)\ln|k\cdot n|$ in any propagator appearing in a loop integral and of applying the Weinberg theorem,[21] after the Wick rotation has been performed. We note that, if $n_0=0$, we are guaranteed that the Wick rotation does not entail any problem. We also stress that, in the light-cone case, the proof, as expected, cannot go through; in fact if we use the logarithmic representation for the spurious singularity, we obtain the principal-value prescription which interferes with the Wick rotation, as already noticed in Ref. 9. If we instead correctly start from the LM prescription (2.9), a logarithmic representation is no longer possible. In conclusion we can enunciate the following lemma.

Lemma 2. Nonpolynomial singular coefficients in a subtracted effective 1PI graph must either vanish or become local in the limit $n_0\to 0$, $|\mathbf{n}|\neq 0$ (spacelike limit).[22]

As previously promised, this result excludes the appearance of dimensionless ratios of longitudinal external momenta of Q_{ij} type in a subtracted effective part of a 1PI graph, so that the first part of our basic property is established and the possibility of making finite all the Green's functions of the theory by performing only *local* subtractions is proven.

So far we have considered only effective parts of 1PI proper graphs; however, the previously established results can be used to control the divergences of the whole 1PI graphs, including their nonpolynomial evanescent (i.e., not contributing to the Green's functions) singular parts. Let us first observe that the complete one-loop

1PI graphs do satisfy continuity in the limit $n_0 \to 0$, $|\mathbf{n}| \neq 0$. As a consequence, the divergent parts are polynomial in the transverse external momenta and contain spurious poles, but only in special combinations which disappear in the spacelike limit, as it can be explicitly verified in the examples reported in the previous sections (note that different cancellation mechanisms work to implement this remarkable property; see in particular the calculations of Ref. 11). We shall call this very special type of singular coefficients in the 1PI proper graphs "quasipolynomial."

It is now not difficult to foresee that an inductive proof can be developed for 1PI complete proper graphs to get the following result: 1PI proper graphs can be subtracted order by order by means of quasipolynomial counterterms. There is nothing strange in these kinds of counterterms which are typical of the light-cone gauge: they satisfy, by construction, the properties of polynomial behavior with respect to transverse momenta and continuity with respect to the spacelike limit and this is all we need in reaching our basic results. As we shall see in the following sections, the number of those special counterterms is very limited and this fact will result in the possibility of finding important relations among renormalization constants.

V. RENORMALIZATION AND COUNTERTERMS

We shall again be concerned in this section with the pure Yang-Mills case, the generalization including fermions being deferred to the next section. Starting from the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \Lambda^a n^\mu A^a_\mu + A^a_\mu J^{\mu,a} + K^a\Lambda^a \ , \quad (5.1)$$

J^a_μ and K^a being external sources, we can define, as usual, the generating functional for the Green's functions

$$W[J^{\mu,a},K^a] = \mathcal{N}^{-1}\int \mathcal{D}[A^a_\mu,\Lambda^a]\exp\left[i\int \mathcal{L}\,d^4x\right] \ . \quad (5.2)$$

Then, introducing the generating functional for the connected Green's functions Z, and the "classical fields" a^a_μ and λ^a, we can construct the generating functional of the 1PI vertices $\Gamma(a^a_\mu,\lambda^a)$ and define

$$\tilde{\Gamma}(a^a_\mu,\lambda^a) = \Gamma(a^a_\mu,\lambda^a) + \int d^4x\, n^\mu a^a_\mu(x)\lambda^a(x) \ . \quad (5.3)$$

It is easy to show that $\tilde{\Gamma}$ obeys the Lee-Ward identity

$$D^{ab}_\mu \delta\tilde{\Gamma}/\delta a^b_\mu(x) = 0 \ . \quad (5.4)$$

Actually, the generating functional $\tilde{\Gamma}$ will be thought of as dimensionally regularized. Later on we shall see that the same form of the functional equation holds if the regularized functional $\tilde{\Gamma}$ is replaced by the renormalized one.

Equation (5.4), upon differentiation with respect to the classical fields, generates the whole set of Ward identities. As it stands, it simply means that $\tilde{\Gamma}$, besides satisfying the usual constraints coming from dimensional analysis and symmetry properties, must also be invariant under a gauge transformation. We notice that the decoupling of the Faddeev-Popov ghosts in the light-cone gauge leads to the simple functional Eq. (5.4) and thereby it is possible to understand renormalization without resorting to the general BRS analysis.[23,24] Nevertheless all our results could be rephrased in this language.

We first remark that the loop integrals do develop singularities with a nonpolynomial character in the 1PI vertices, which however, must be compatible with the lemmas of the preceding section. We are thus led to investigate possible nonlocal gauge-covariant structures which can emerge as divergent parts of $\tilde{\Gamma}$; in so doing the usual inductive procedure with respect to the number of loops always has to be kept in mind. One possibility is given by the solution Ω^b of the equation

$$n^\mu D^{ab}_\mu \Omega^b = \frac{n^\mu n^{*\nu}}{n\cdot n^*} f^a_{\mu\nu} \ , \quad (5.5)$$

where $f^a_{\mu\nu}$ are the field strengths in terms of the classical potentials a_μ (we recall that the operator nD^{ab} can be inverted in a perturbative sense). It is also immediate to verify that Ω^b is compatible with the lemma 2, as it vanishes in the limit $n_0 \to 0$.

Actually, a careful analysis shows that Ω^b is the only nonlocal possibility. As a matter of fact, covariants containing $(n\cdot D)^{-1}$ in the combination

$$\Omega_\nu = \frac{n^\mu (n^\rho D_\rho)^{-1}}{n\cdot n^*} f_{\mu\nu} \ , \quad (5.6)$$

or in more complicated expressions, are not allowed because they conflict with the result stated in lemma 2. To this regard we realize that the structure

$$\Omega^a = n^{*\mu}\Omega^a_\mu \quad (5.7)$$

is a crucial and unique one.

Dimensional analysis together with homogeneity with respect to n_μ and n^*_μ leads us to the following expression giving rise to the nonlocal divergent density of $\tilde{\Gamma}$:

$$\Delta_\Omega = n^\mu D^\nu f_{\mu\nu}\Omega \ . \quad (5.8)$$

Obviously Δ_Ω leads to an infinite set of 1PI vertices with any number of legs of vanishing type, according to the result of Sec. IV.

As far as local divergent terms are concerned, we have three possibilities:

$$\Delta^{(1)} = f^a_{\mu\nu} f^{\mu\nu,a} \ , \quad (5.9a)$$

$$\Delta^{(2)} = n^\rho f^a_{\rho\mu} n^{*\nu} f_\nu{}^{\mu,a}/n\cdot n^* \ , \quad (5.9b)$$

$$\Delta^{(3)} = (n^\mu n^{*\nu} f^a_{\mu\nu})^2/(n\cdot n^*)^2 \ . \quad (5.9c)$$

The density $\Delta^{(1)}$ is the standard invariant expression, whereas $\Delta^{(2)}$ and $\Delta^{(3)}$ are to be excluded, as they are incompatible with the Lorentz invariance of the (formal) S matrix.[16,25] Following the usual inductive procedure with respect to the number of loops, the most general structure of the divergent part of $\tilde{\Gamma}$ can be obtained starting from a linear combination of the invariants (5.8) and (5.9a). As a consequence, there is no freedom for constructing kernels of the Lee-Ward identities of the type of Eq. (3.5) and thereby no independent renormalization constant for the three-vector vertex: namely, we get $Z_1 = Z_3$. It should be stressed once more that, within the light-cone gauge choice, it is not possible by merely using gauge invariance of the quantum theory as expressed by the Lee-Ward identities to show the gauge invariance of the renormalized S matrix as well as the relation $Z_1 = Z_3$; to do this properly it is necessary to resort to the basic property we have proven in Sec. IV, which guarantees the uniqueness of the counterterm (5.8) as the *only* possible nonpolynomial one.

In conclusion, the general form of the density of the divergent part of $\tilde{\Gamma}$ reads

$$\Delta = a_3 f^a_{\mu\nu} f^{\mu\nu,a} + \tilde{a}_3 n^\mu \Omega^a D^{ab}_\nu f_\mu{}^{\nu,b} . \tag{5.10}$$

It is then remarkable that the renormalized Lagrangian, although nonlocal, can be generated by performing the canonical transformation

$$A^{a(0)}_\mu = Z_3{}^{1/2}[A^a_\mu - (1-\tilde{Z}_3{}^{-1}) n_\mu \Omega^a] , \tag{5.11a}$$

$$g_0 = Z_3{}^{-1/2} g , \tag{5.11b}$$

$$\Lambda^{a(0)} = Z_3{}^{-1/2} \Lambda^a , \tag{5.11c}$$

Z_3 and $\tilde{Z}_3$ being two renormalization constants related to physical and unphysical components of the bare potentials, respectively. Equation (5.10) is the light-cone counterpart of the canonical transformation introduced in Ref. 16 for the spacelike planar gauge.

It is easy to verify that the renormalized Lagrangian can be written in terms of the "bare fields" as

$$\mathcal{L}_R = -\tfrac{1}{4} F^{(0)}_{\mu\nu} \cdot F^{(0)\mu\nu} - \Lambda^{(0)} \cdot n^\mu A^{(0)}_\mu , \tag{5.12}$$

according to the general formulation of the gauge-invariant renormalization given in Ref. 26. However, at variance with the cited planar gauge case, the regularized and renormalized Lagrangians are invariant with respect to the same representation (apart from a scale factor for the coupling constant) of the gauge group, as it happens in the axial case, where the wave-function renormalization is instead purely multiplicative and local.[27] As a matter of fact, it is easy to check that the infinitesimal gauge transformation of the bare potentials

$$\delta A^{a(0)}_\mu = \partial_\mu \omega^a_0 + g_0 f^{abc} A^{b(0)}_\mu \omega^c_0 , \tag{5.13}$$

corresponds, through Eqs. (5.11a) and (5.11b), to the infinitesimal gauge transformation for the renormalized potentials

$$\delta A^a_\mu = \partial_\mu \omega^a + g f^{abc} A^b_\mu \omega^c . \tag{5.14}$$

We can say that the light-cone shares renormalization properties with both axial and planar gauges. In particular, the renormalized Lee-Ward identities take the same form of Eq. (5.4) where regularized quantities are replaced by renormalized ones. We remark that the structure of Eq. (5.11) manifestly guarantees that the Lagrange multiplier $\Lambda = Z_3{}^{1/2} \Lambda^{(0)}$ still obeys the free equation (2.8) thereby leading to the covariance and unitarity of the formal renormalized S matrix.[7] Indeed, since the second term on the right-hand side (RHS) of Eq. (5.10) is proportional to the classical equations of motion, it is known[26] that it never contributes to the "on-shell" quantities to any order in perturbation theory. Therefore, the on-shell counterterms are simply given by the standard local and Lorentz-invariant part in the RHS of Eq. (5.10). This last fact has been explicitly verified to one-loop order on Ref. 15 and, as we have seen, is true to any order in the loop expansion; it is owing to this result that we call the component along the dual gauge vector n^*_μ of the bare potential in Eq.(5.11a) unphysical. In particular, the renormalized Lagrangian (5.12) is Gaussian with respect to the unphysical potential $n^* \cdot A^{(0)}$, so that we can tolerate the presence in $n^* \cdot A$ of a non-Hermitian part without having troublesome physical consequences. On the other hand, the decoupling property of the nonlocal singular counterterms from the S matrix was already established when we showed that the Green's functions do not require nonlocal subtractions in the Lagrangian.

Another consequence of the decoupling property of the nonlocal sector from physical quantities is that, thanks to Eq. (5.11b) the β function can be read directly from the physical components of the full propagator. It is worthwhile to notice that, although the whole set of Green's functions with physical components can be obtained, as it is well known,[28] from a "two-component" formalism with related Feynman rules, the renormalization structure cannot be obtained so transparently as in the complete "four-component" formalism, owing to the lack of Lee-Ward-type identities for two-component Green's functions and 1PI vertices. From the renormalized Lagrangian (5.12) it is straightforward to check the one-loop renormalization of the self-energy and of the three-vector vertex (including the nonpolynomial term of vanishing type) and to give a prediction for the divergent part of the 1PI vertices with any number N of external legs. In particular, the case $N = 4$ is exhibited in Appendix A.

As a final remark we note that one might conceive an approach to renormalization based only on Green's functions. The obvious advantage would be the possibility of dealing with only local structures as counterterms; however, the price to be paid is twofold: on the one hand Lee-Ward identities for 1PI vertices are no longer directly useful anymore and, on the other hand, Ward identities for the Green's functions always involve the presence of external legs corresponding to the Lagrange multipler. This last feature makes it rather involved to extract the necessary relation among the coefficients of the local counterterms.

VI. RENORMALIZATION IN THE PRESENCE OF DIRAC FERMIONS

The introduction of Dirac fermions does not entail any essential new difficulty in renormalizing the theory. Nevertheless it deserves a separate treatment as a fine-tuning mechanism emerges as pointed out in Ref. 13 in the one-loop case. In the theory with fermions Eq. (5.4) becomes

$$D_\mu^{ab}\frac{\delta\tilde{\Gamma}}{\delta a_\mu^b}+g\mu^{2-\omega}\bar{\phi}\tau^a\frac{\delta\tilde{\Gamma}}{\delta\bar{\phi}}-g\mu^{2-\omega}\frac{\delta\tilde{\Gamma}}{\delta\phi}\tau^a\phi=0\ , \quad (6.1)$$

vhere ϕ and $\bar{\phi}$ are the classical Fermi fields and τ are the SU(N) Lie-algebra matrices in the fundamental representation. Again Eq. (6.1) means that $\tilde{\Gamma}$ is invariant under a gauge transformation of all the classical fields. As a consequence, the new admissible divergent structures, which obey all the previously mentioned conditions, are

$$\Delta_4=\bar{\phi}\mathcal{D}\phi,\quad \Delta_5=\bar{\phi}\frac{\not{n}n^*\cdot D}{n\cdot n^*}\phi\ ,$$
$$\Delta_6=\bar{\phi}\frac{\not{n}^*n\cdot\mathcal{D}}{n\cdot n^*}\phi,\quad \Delta_7=\bar{\phi}\frac{\not{n}\not{n}^*\mathcal{D}}{n\cdot n^*}\phi\ , \quad (6.2)$$
$$\Delta_8=m\bar{\phi}\phi,\quad \Delta_9=m\bar{\phi}\frac{\not{n}\not{n}^*}{n\cdot n^*}\phi\ ,$$

which are the local ones, and

$$\hat{\Delta}_\Omega=\bar{\phi}\not{n}\tau^a\phi\Omega^a\ , \quad (6.3)$$

as the only admissible nonlocal one. In analogy with what happened in a pure Yang-Mills case, the requirement of Lorentz invariance for the S-matrix elements imposes further constraints on the coefficients of the terms (6.2) and (6.3). In particular, the Lorentz covariance of the on-shell fermion self-energy imposes the combination $\Delta_5-\Delta_6$, while analogous considerations for the fermion-fermion-vector vertex rule out Δ_7 and Δ_9 and establish a relation between Δ_Ω and $\hat{\Delta}_\Omega$. As a matter of fact, it is easy to realize that, by iterating the inductive procedure with respect to the number of loops, a cancellation mechanism takes place between Lorentz-noncovariant divergent parts of the on-shell amputated Green's functions (see Appendix B). As a consequence the form of the density of the divergent parts of $\tilde{\Gamma}$ in the presence of Dirac fermions is given by

$$\hat{\Delta}=a_3f^a_{\mu\nu}f^{a,\mu\nu}+a_2\bar{\phi}(i\mathcal{D}-m+\delta m)\phi$$
$$+i\tilde{a}_2\bar{\phi}(n\cdot\mathcal{D}\not{n}^*-n^*\cdot\mathcal{D}\not{n})\phi/n\cdot n^*$$
$$+\tilde{a}_3\Omega^a n^\mu(D^\nu f^a_{\mu\nu}-g\mu^{2-\omega}\bar{\phi}\tau^a\gamma_\mu\phi)\ . \quad (6.4)$$

We notice that, at variance with the pure Yang-Mills case, where it has been shown that no kernel of the Lee-Ward identities is allowed, when fermions are present, kernels with a divergent coefficient are indeed switched on (see Sec. III for the one-loop case), as one can read from Eq. (6.4). For instance, for the fermion-fermion-vector vertex we have

$$K_\mu=\gamma^{(\epsilon)}\frac{\not{n}}{n\cdot n^*}\left[n^*_\mu-n_\mu\frac{n^*\cdot k}{n\cdot k}\right]\ , \quad (6.5)$$

where $\gamma(\epsilon)$ is a divergent coefficient when $\epsilon\to0$. The coefficients of those kernels are willingly controlled by the on-shell Lorentz covariance as the relevant term in Eq. (6.4) vanishes on the classical equations of motion.

It is remarkable that the renormalized Lagrangian $\mathcal{L}_R$ can be obtained by generalizing the canonical transformation of Eqs. (5.11) as

$$A_\mu^{(0)}=Z_3^{1/2}[A_\mu-(1-\tilde{Z}_3^{-1})n_\mu\Omega]\ ,$$
$$\psi^{(0)}=(Z_2\tilde{Z}_2)^{1/2}\left[\mathbb{1}-(1-\tilde{Z}_2^{-1})\frac{\not{n}^*\not{n}}{2n\cdot n^*}\right]\psi\ , \quad (6.6)$$
$$g_0=Z_3^{-1/2}g,\quad m_0=m-\delta m,\quad \Lambda^{(0)}=Z_3^{-1/2}\Lambda\ ,$$

$\mathcal{L}_R$ possessing the following expression in terms of the bare fields:

$$\mathcal{L}_R=-\tfrac{1}{4}F^{(0)}_{\mu\nu}F^{(0),\mu\nu}-\Lambda^{(0)}n^\mu A^{(0)}_\mu+\bar{\psi}^{(0)}(i\mathcal{D}^{(0)}-m_0)\psi^{(0)}\ . \quad (6.7)$$

Once again, from Eq. (6.4), the finiteness of the one-loop 1PI vertices (including nonpolynomial contributions) with Z_3, Z_2, $\tilde{Z}_3$, and $\tilde{Z}_2$ as given in Ref. 13, can be easily checked in the cases with few external legs.

Let us conclude this section with a remark. As is well known, even in the presence of fermions, it is possible to resort to a two-component formulation, by decomposing the Dirac spinor in two two-component spinors ψ_+ and ψ_-. It turns out that one can perform the functional integration with respect to one of the two-component spinors, say ψ_-, giving rise to a determinant which is purely kinematical (as $A_-=0$). In this way the Green's functions with ψ_--type external legs can be expressed in terms of the remaining "independent" ones. Now it is worth considering that, at variance with the pure Yang-Mills case, also the "dependent" Green's functions are obviously physically relevant, so that the on-shell renormalization for the spinor wave function is achieved by a genuine matrix transformation, involving both Z_2 and $\tilde{Z}_2$.

VII. CONCLUSION

The basic result of this paper is a constructive proof of the renormalizability of the Yang-Mills theory quantized in the light-cone gauge to any order in the loop expansion. We have indeed shown that, in spite of some unusual (and potentially pathological) features such as the presence of singularities with a nonpolynomial character with respect to the external momenta in the 1PI vertices, this theory can be made finite by means of the introduction of a small number of renormalization constants in a way that preserves the unitarity of the (formal) S matrix in the physical Hilbert space. As a matter of fact, the usual procedure based on the Lee-Ward identities can be repeated in this case even in the presence of some singular features we have stressed; in addition because this gauge is "physical," there is no need of Faddeev-Papov ghosts and of BRS fields. As a byproduct we have shown that the nonlocal terms in the effective action are of a vanishing type: namely, they do not contribute to the S matrix. As a consequence, only the local parts of the counterterms are relevant in making finite all the physical quantities, as expected on the basis of gauge invariance.

In Sec. IV we have characterized the possible singular structures that can (and do) appear in the course of any diagrammatic calculation. In Secs. V and VI we have learned how to handle them in a consistent way. In particular, we notice that one should not be afraid of the appearance of nonlocal terms in the counter-Lagrangian as they are just needed to cancel (in a natural way) analogous nonlocal terms that the theory itself produces in loop calculations. All of them are completely under control. We conclude by saying that the light-cone gauge can now be safely used with the given prescriptions in any kind of perturbative calculation, taking full advantage of its peculiar features.

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APPENDIX A

In this appendix we report the contribution to the four-vector 1PI vertex generated by the structure (5.8). A straightforward calculation gives

$$\begin{aligned}\Gamma^{abcd}_{\mu\nu\rho\sigma}(k,p,q,r)=-g^2 f^{abe}f^{cde}(n\cdot n^*)^{-1}\Bigg\{&\hat{n}^*_{\{\mu}n_{\rho\}}g_{\nu\sigma}+\hat{n}^*_{\{\nu}n_{\sigma\}}g_{\mu\rho}-\hat{n}^*_{\{\nu}n_{\rho\}}g_{\mu\sigma}-\hat{n}^*_{\{\mu}n_{\sigma\}}g_{\nu\rho}\\
&+n_\mu n_\nu g_{\rho\sigma}\frac{n\cdot(r-q)}{n\cdot(r+q)}\left[\frac{n^*\cdot k}{n\cdot k}-\frac{n^*\cdot p}{n\cdot p}\right]\\
&+n_\rho n_\sigma g_{\mu\nu}\frac{n\cdot(p-k)}{n\cdot(p+k)}\left[\frac{n^*\cdot q}{n\cdot q}-\frac{n^*\cdot r}{n\cdot r}\right]\\
&+2(n_\mu k_\nu-n_\nu p_\mu)\frac{n_\rho n_\sigma}{n\cdot(q+r)}\left[\frac{n^*\cdot r}{n\cdot r}-\frac{n^*\cdot q}{n\cdot q}\right]\\
&+2(n_\rho q_\sigma-n_\sigma r_\rho)\frac{n_\mu n_\nu}{n\cdot(k+p)}\left[\frac{n^*\cdot p}{n\cdot p}-\frac{n^*\cdot k}{n\cdot k}\right]\\
&+n_\mu n_\nu n_\rho n_\sigma\Bigg[\left[\frac{p^2}{n\cdot p}-\frac{k^2}{n\cdot k}\right]\frac{1}{n\cdot(q+r)}\left[\frac{n^*\cdot r}{n\cdot r}-\frac{n^*\cdot q}{n\cdot q}\right]\\
&\qquad+\left[\frac{r^2}{n\cdot r}-\frac{q^2}{n\cdot q}\right]\frac{1}{n\cdot(k+p)}\left[\frac{n^*\cdot p}{n\cdot p}-\frac{n^*\cdot k}{n\cdot k}\right]\Bigg]\Bigg\}\\
&+(b,\nu,p\to c,\rho,q)+(b,\nu,p\to d,\sigma,r)\ ,\end{aligned}\tag{A1}$$

where $\hat{n}^*_\mu$ has been defined in Eq. (3.1) and the symbol $\{\mu_1\mu_2\}$ means symmetrization with respect to those indices.

We notice that it is of vanishing type and that it becomes local in the limit $n_0\to 0$, as it should. It is also easy to check that it exactly matches, under the Lee-Ward identity,

$$\begin{aligned}ik^\mu\Gamma^{abcd}_{\mu\nu\rho\sigma}(k,p,q,r)=&-gf^{ab}\Gamma^{cde}_{\nu\rho\sigma}(-q-r,q,r)\\
&-gf^{ace}\Gamma^{dbe}_{\rho\sigma\nu}(-r-p,r,p)\\
&-gf^{ade}\Gamma^{bce}_{\sigma\nu\rho}(-p-q,p,q)\ ,\end{aligned}\tag{A2}$$

the corresponding structures of the three-vector 1PI vertex, explicitly exhibited in Ref. 11. It would be nice to check the structure (A1) by means of a direct calculation of the divergent part of the four-vector 1PI vertex at one loop.

APPENDIX B

In this appendix we give an example of the constraints (fine-tuning) forced upon the possible divergent structures appearing in $\tilde{\Gamma}$ by the request of Lorentz covariance. We consider the divergent part of the fermion-fermion-vector amputated subtracted Green's function in the nth order in the loop expansion (see Fig. 1). The effective divergent noncovariant part of the vector self-energy is

$$^{(n)}\Gamma^{\rm div}_{\mu\nu}(k)\,|_{\rm eff}=-\tilde{a}_3\frac{n^*_\mu k_\nu}{n\cdot n^*}n\cdot k\ ,\tag{B1}$$

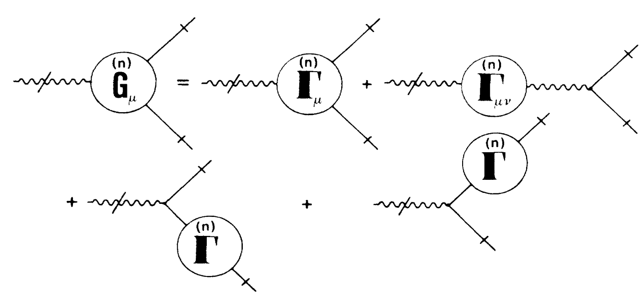

FIG. 1. Equation relating the fermion-fermion-vector amputated Green's functions to the corresponding 1PI vertices.

$\tilde{\alpha}_3$ being the divergent quantity multiplying the structure $\Omega n^\mu D^\nu f_{\mu\nu}$ [see Eq. (6.3)]. As far as the fermion-fermion proper vertex is concerned, we get

$$^{(n)}\Gamma^{\text{div}}(p)\,|_{\text{eff}}=\tilde{\alpha}_2\frac{\not{n}n^*\cdot p-\not{n}^*n\cdot p}{n\cdot n^*} \tag{B2}$$

[see again Eq. (6.3)]. Finally for the fermion-fermion-vector proper vertex we have

$$^{(n)}\Gamma^{\text{div}}_\mu(p)\,|_{\text{eff}}=(\tilde{\alpha}_2+\tilde{\alpha}_3)\frac{n^*_\mu\not{n}}{n\cdot n^*}\ , \tag{B3}$$

where we have denoted by $\tilde{\alpha}_3$ the (divergent) coefficient in front of the structure $g\Omega\bar{\phi}\tau A\phi$ [compare with Eq. (6.3)].

In the limit $k_\mu\rightarrow 0$ setting the two fermionic lines on shell, we obtain

$$^{(n)}G^{\text{div}}_\mu(p)\,|_{\text{noncov}}=(-\tilde{\alpha}_2-\tilde{\alpha}_3+\tilde{\alpha}_2+\tilde{\alpha}_2)\times\frac{n^*_\mu n\cdot p}{n\cdot n^*}\frac{\bar{w}(p)w(p)}{m}\ , \tag{B4}$$

which is requested to vanish by Lorentz covariance. Hence

$$\tilde{\alpha}_3=\tilde{\alpha}_3 \tag{B5}$$

as promised.

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$$D_{\mu\nu}(k)=\frac{1}{k^2+i\epsilon}\left[-g_{\mu\nu}+\frac{n_\mu k_\nu+n_\nu k_\mu}{[n\cdot k]}\right].$$

It is easy to recognize that under the rescaling $k_\mu\rightarrow\lambda k_\mu$, it is a homogeneous function of degree $\alpha=-2$. On the contrary, under the rescaling $k_\perp\rightarrow\lambda k_\perp$, for large λ, the covariant part still exhibits a degree $\beta_\perp=-2$, whereas the noncovariant part has the "transverse components" with degree $\beta_\perp=-1$. Therefore, if, for instance, one considers a self-energy graph, viz.,

$$\Pi^{ad}_{\mu\sigma}(p)=\int d^{2\omega}_k k V^{abc}_{\mu\nu\rho}(p,-k,-p+k)\times V^{bdc}_{\lambda\sigma\tau}(k,-p,p-k)D_{\nu\lambda}(k)D_{\rho\tau}(p-k)$$

taking into account that the vertex $V^{abc}_{\mu\nu\rho}$ has total degree $+1$, the integrand can be decomposed into terms possessing different high-momentum behaviors.

[18]Let us consider, for instance, the integral

$$I(q_1,q_2,q_3)=\int\frac{d^{2\omega}k}{(k+q_3)^2}\times\frac{1}{[n\cdot k][n\cdot(k+q_1)][n\cdot(k+q_2)]}\ ,$$

2ω being the space-time dimensions. Now $\deg(I)=2\omega-5$ and, hence, is negative when $\omega\rightarrow 2$, whereas $\deg_\perp(I)=2\omega-3$; as a consequence I exhibits a divergent part when $\omega\rightarrow 2$ given by

$$I^{\text{div}}=\frac{2i\pi^2}{2-\omega}\frac{1}{n\cdot n^*}\frac{1}{n(q_2-q_1)}\left[\frac{n^*\cdot q_2}{n\cdot q_2}-\frac{n^*\cdot q_1}{n\cdot q_1}\right].$$

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