Renormalizability of the time-dependent variational equations in quantum field theory

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The field-theoretic time-dependent variational approximation allows the study of dynamics by a formalism which reflects some of the nonlinearities of the full theory. We study renormalizability of the time-dependent variational equations which are obtained when a Gaussian trial wave functional is used. Renormalizability depends on the specified initial data. We develop reasonable criteria for limiting the initial data, and show that the time-dependent variational equations are made finite by the renormalization prescription used in the vacuum sector.

I. INTRODUCTION

Field-theoretic descriptions of natural processes often suffer from a serious shortcoming; the usual approach for solving dynamical equations is the perturbative expansion, in which nonlinear features of the quantum field theory cannot easily be seen.

The field-theoretic time-dependent variational approximation allows studying dynamics by a formalism which reflects some of the nonlinearities of the full theory. Furthermore, calculations are performed in the Schrödinger picture, where one has a clear description of the system's time evolution.

In this paper we shall study renormalizability of the time-dependent variational equations, which are obtained when a Gaussian trial wave functional is used. Although the formalism for this field-theoretic approximation was introduced by Jackiw and Kerman[1] some time ago, the issue of renormalizability was not addressed. Renormalization of a time-dependent system is intrinsically interesting, but our main motivation is to apply this approximation scheme to the dynamics of the inflation-driving scalar field[2] in the inflationary scenario.[3,4]

In Ref. 5 the range of validity of the time-dependent Gaussian variational approximation has been examined: The method is applied to various one-dimensional quantum-mechanical problems and the approximate results are compared with exact ones, obtained by numerical calculation. The approximation is found to be very accurate for quantum roll processes, in which a particle rolls down from the maximum of a potential.

This is similar to what happens in the new inflationary universe. In the original scenario,[4] it is assumed that the initial state of the inflation-driving scalar field was in thermal equilibrium and that the system was initially described by a density matrix. As the temperature decreases, the scalar field begins to roll down the hill of the potential diagram. At this stage the system is no longer in thermal equilibrium and can be described by a single wave functional.

Moreover, in the recently modified picture of the new inflationary universe, one expects the initial configuration of the inflation-driving scalar field to be a random, nonthermal configuration. The reason for this is that proper density fluctuations require the inflation-driving scalar field to interact extremely weakly with a coupling constant of order 10^{-12} or smaller.[6] Such a weakly interacting field cannot be in thermal equilibrium with other fields. It remains an open question whether inflation can actually arise from a random initial configuration.[7] For this problem, our variational approximation may be used for the entire time evolution of the scalar field.

We shall first describe the time-dependent variational approximation where the trial wave functional is a Gaussian. In the functional Schrödinger picture for field theory, an abstract quantum-mechanical state $|\psi(t)\rangle$ is replaced by a wave functional $\Psi(\phi,t)$, which is a functional of a c-number field $\phi(\mathbf{x})$ at a fixed time:

$$|\psi(t)\rangle \rightarrow \Psi(\phi,t) . \tag{1.1}$$

The action of the operator $\phi(\mathbf{x})$ on $|\psi(t)\rangle$ is realized by multiplying $\Psi(\phi,t)$ by $\phi(\mathbf{x})$:

$$\phi(\mathbf{x})\,|\psi(t)\rangle \rightarrow \phi(\mathbf{x})\Psi(\phi,t) . \tag{1.2}$$

The action of the canonical momentum $\pi(\mathbf{x})$ is realized by functional differentiation:

$$\pi(\mathbf{x})\,|\psi(t)\rangle \rightarrow -i\hbar\frac{\delta}{\delta\phi(\mathbf{x})}\Psi(\phi,t) . \tag{1.3}$$

The functional Schrödinger equation, which governs how a given initial state evolves with time, may be obtained by considering the effective action

$$\Gamma = \int dt\,\langle\psi(t)\,|\,i\hbar\partial_t - H\,|\,\psi(t)\rangle \tag{1.4}$$

and demanding that it be stationary against arbitrary variations of $|\psi(t)\rangle \equiv \Psi(\phi,t)$. This requirement produces

$$i\hbar\frac{\partial\Psi(\phi,t)}{\partial t} = H\Psi(\phi,t) = \int_x \left[-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right]\Psi(\phi,t) . \tag{1.5}$$

As an approximation we take a Gaussian trial wave functional in the evaluation of Γ:

$$\Psi(\phi,t)=\exp\left\{-\left[\int_{\mathbf{x},\mathbf{y}}[\phi(\mathbf{x})-\hat{\phi}(\mathbf{x},t)]\left[\frac{G^{-1}(\mathbf{x},\mathbf{y},t)}{4\hbar}-\frac{i}{\hbar}\Sigma(\mathbf{x},\mathbf{y},t)\right][\phi(\mathbf{y})-\hat{\phi}(\mathbf{y},t)]+\frac{i}{\hbar}\int_{\mathbf{x}}\hat{\pi}(\mathbf{x},t)[\phi(\mathbf{x})-\hat{\phi}(\mathbf{x},t)]\right]\right\}. \tag{1.6}$$

A normalization factor has been suppressed. The meaning of the various quantities in this wave functional is seen from the following:

$$\langle\phi(\mathbf{x})\rangle=\hat{\phi}(\mathbf{x},t)\ , \tag{1.7a}$$

$$\left\langle-i\hbar\frac{\delta}{\delta\phi(x)}\right\rangle=\hat{\pi}(\mathbf{x},t)\ , \tag{1.7b}$$

$$\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle=\hat{\phi}(\mathbf{x},t)\hat{\phi}(\mathbf{y},t)+\hbar G(\mathbf{x},\mathbf{y},t)\ , \tag{1.7c}$$

$$\left\langle i\hbar\frac{\partial}{\partial t}\right\rangle=\int_{\mathbf{x}}\hat{\pi}(\mathbf{x},t)\dot{\hat{\phi}}(\mathbf{x},t)+\hbar\int_{\mathbf{x},\mathbf{y}}\Sigma(\mathbf{x},\mathbf{y},t)\dot{G}(\mathbf{y},\mathbf{x},t)\ . \tag{1.7d}$$

[Here and in (1.4) expectation values are evaluated by functional integration.] Ψ is Gaussian centered at $\hat{\phi}$ with width given by G. The conjugate momentum of $\hat{\phi}$ is $\hat{\pi}$ and Σ plays a role of the conjugate momentum of G. The variational parameters are $\hat{\phi}$, $\hat{\pi}$, G, and Σ. The effective action in this approximation is

$$\begin{aligned}\Gamma=\int dt\Bigg[&\int_{x}[\hat{\pi}\dot{\hat{\phi}}-\tfrac{1}{2}\hat{\pi}^2-\tfrac{1}{2}(\nabla\hat{\phi})^2-V(\hat{\phi})]\\&+\hbar\left[\int_{\mathbf{x},\mathbf{y}}\Sigma\dot{G}-2\int_{\mathbf{x},\mathbf{y},\mathbf{z}}\Sigma G\Sigma-\int_{x}[\tfrac{1}{8}G^{-1}(\mathbf{x},\mathbf{x},t)-\tfrac{1}{2}\nabla_{\mathbf{x}}{}^2G(\mathbf{x},\mathbf{y},t)|_{\mathbf{x}=\mathbf{y}}+\tfrac{1}{2}V^{(2)}(\hat{\phi})G(\mathbf{x},\mathbf{x},t)]\right]\\&-\hbar^2\tfrac{1}{8}V^{(4)}(\hat{\phi})\int_{\mathbf{x}}G(\mathbf{x},\mathbf{x},t)G(\mathbf{x},\mathbf{x},t)\Bigg]\ ,\end{aligned} \tag{1.8}$$

where $V^{(n)}(\hat{\phi})\equiv[d^nV(\hat{\phi})/d\hat{\phi}^n]$. Notice that the first integral is the familiar classical action. The variational equations are then

$$\frac{\delta\Gamma}{\delta\hat{\phi}(\mathbf{x},t)}=0\rightarrow\dot{\hat{\pi}}(\mathbf{x},t)=\nabla^2\hat{\phi}(\mathbf{x},t)-V^{(1)}(\hat{\phi})-\frac{\hbar}{2}G(\mathbf{x},\mathbf{x},t)V^{(3)}(\hat{\phi})\ , \tag{1.9a}$$

$$\frac{\delta\Gamma}{\delta\hat{\pi}(\mathbf{x},t)}=0\rightarrow\hat{\pi}(\mathbf{x},t)=\dot{\hat{\phi}}(\mathbf{x},t)\ , \tag{1.9b}$$

$$\begin{aligned}\frac{\delta\Gamma}{\delta G(\mathbf{x},\mathbf{y},t)}=0\rightarrow&\dot{\Sigma}(\mathbf{x},\mathbf{y},t)+2\int_{\mathbf{z}}\Sigma(\mathbf{x},\mathbf{z},t)\Sigma(\mathbf{z},\mathbf{y},t)\\&=\tfrac{1}{8}G^{-2}(\mathbf{x},\mathbf{y},t)+[\tfrac{1}{2}\nabla_{\mathbf{x}}{}^2-\tfrac{1}{2}V^{(2)}(\hat{\phi})-\tfrac{1}{4}\hbar V^{(4)}(\hat{\phi})G(\mathbf{x},\mathbf{x},t)]\delta^3(\mathbf{x}-\mathbf{y})\ ,\end{aligned} \tag{1.9c}$$

$$\frac{\delta\Gamma}{\delta\Sigma(\mathbf{x},\mathbf{y},t)}=0\rightarrow\dot{G}(\mathbf{x},\mathbf{y},t)=2\left[\int_{\mathbf{z}}[G(\mathbf{x},\mathbf{z},t)\Sigma(\mathbf{z},\mathbf{y},t)+\Sigma(\mathbf{x},\mathbf{z},t)G(\mathbf{z},\mathbf{y},t)]\right]\ . \tag{1.9d}$$

Henceforth we set $\hbar$ to unity.

II. RENORMALIZATION OF EFFECTIVE POTENTIAL

Before studying renormalizability of the time-dependent system, we shall briefly discuss how the static effective potential in the Gaussian variational approximation is renormalized.[8] When the potential $V(\phi)$ in Eq. (1.5) is given by

$$V(\phi)=\tfrac{1}{2}\mu^2\phi^2+\frac{1}{4!}\lambda\phi^4\ , \tag{2.1}$$

the effective potential in the Gaussian approximation is

$$\begin{aligned}V_{\text{eff}}(\hat{\phi},G)=&\tfrac{1}{2}\mu^2\hat{\phi}^2+\frac{1}{4!}\lambda\hat{\phi}^4+\tfrac{1}{4}G^{-1}(\mathbf{x},\mathbf{x})\\&-\frac{\lambda}{8}G(\mathbf{x},\mathbf{x})G(\mathbf{x},\mathbf{x})\ ,\end{aligned} \tag{2.2}$$

we take $\hat{\phi}$ to be homogeneous ($\mathbf{x}$ independent). G is the translation-invariant solution of the gap equation:

$$\tfrac{1}{4}G^{-2}(\mathbf{x},\mathbf{x}')=[-\nabla^2+\mu^2+\tfrac{1}{2}\lambda\hat{\phi}^2+\tfrac{1}{2}\lambda G(\mathbf{x},\mathbf{x})]\delta^3(\mathbf{x}-\mathbf{x}')\ . \tag{2.3}$$

Infinities of this system are contained in $G(\mathbf{x},\mathbf{x})$ which is

$$\begin{aligned}G(\mathbf{x},\mathbf{x})&=\int_k G(\mathbf{k})\\&=\int_k\frac{1}{2[k^2+\mu^2+\frac{1}{2}\lambda\hat{\phi}^2+\frac{1}{2}\lambda G(\mathbf{x},\mathbf{x})]^{1/2}}\ ,\end{aligned} \tag{2.4}$$

where $\int_k\equiv\int(d^3k)/(2\pi)^3$ and we have defined the Fourier transform of a function $f(\mathbf{x})$ as $f(\mathbf{x})=\int_k e^{i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{k})$.

Since the approximation is very similar to the large-N approximation, where n-point functions are also expressed in terms of only one- and two-point functions, we shall use the three-dimensional analogue of the renormalization prescription used in Ref. 9, for the large-N effective potential. Renormalized quantities (subscript R) are defined by

$$\frac{\mu_R^2}{\lambda_R}=\frac{\mu^2}{\lambda}+\tfrac{1}{2}I_1,\quad I_1\equiv\int_k\frac{1}{2k},$$

$$\frac{1}{\lambda_R}=\frac{1}{\lambda}+\tfrac{1}{2}I_2(M), \tag{2.5}$$

$$I_2(M)\equiv\frac{1}{M^2}\int_k\left[\frac{1}{2k}-\frac{1}{2(k^2+M^2)^{1/2}}\right],$$

where M is an arbitrary mass, at which the renormalization is performed.

We introduce a mass m defined by

$$m^2\equiv\mu^2+\frac{\lambda}{2}\hat{\phi}^2+\frac{\lambda}{2}\int_k G(\mathbf{k}), \tag{2.6}$$

which is finite by virtue of Eq. (2.5):

$$\begin{aligned}m^2&=\mu_R^2+\frac{\lambda_R}{2}\hat{\phi}^2+\\&\quad+\frac{\lambda_R}{2}\int_k\left[\frac{1}{2(k^2+m^2)^{1/2}}-\frac{1}{2k}+\frac{m^2}{M^2}\left[\frac{1}{2k}-\frac{1}{2(k^2+M^2)^{1/2}}\right]\right]\\&=\mu_R^2+\frac{\lambda_R}{2}\hat{\phi}^2+\frac{\lambda_R}{32\pi^2}m^2\ln\frac{m^2}{M^2}.\end{aligned} \tag{2.7}$$

The usual vacuum sector renormalized mass m_R is given by m^2 at $\hat{\phi}=0$:

$$m_R^2\equiv\mu_R^2+\frac{\lambda_R}{32\pi^2}m_R^2\ln\frac{m_R^2}{M^2}, \tag{2.8}$$

which, when we choose the arbitrary mass to be $M=m_R$, becomes $m_R^2=\mu_R^2$. By expressing $G(\mathbf{x},\mathbf{x})$ and $G^{-1}(\mathbf{x},\mathbf{x})$ in terms of m^2

$$G(\mathbf{x},\mathbf{x})=-m^2I_2(m)+I_1, \tag{2.9a}$$

$$G^{-1}(\mathbf{x},\mathbf{x})=-\frac{1}{32\pi^2}m^4+2m^2I_1-m^4I_2(m), \tag{2.9b}$$

we obtain a finite expression of V_{eff}:[10]

$$V_{\text{eff}}=\frac{m^4-\mu_R^4}{2\lambda_R}+\frac{m^4}{64\pi^2}\left[\ln\frac{M^2}{m^2}-\frac{1}{2}\right]-\tfrac{1}{12}\lambda\hat{\phi}^4. \tag{2.10}$$

(We have adjusted a constant $-\mu^4/2\lambda$.) In the limit of infinite cutoff $\lambda\rightarrow 0_-$.

III. FREE THEORY SOLUTION AND INITIAL STATES

A. Free theory solution

As we shall see later, the structure of the free theory provides the key ingredients for the time-dependent renormalization. In the simpler case where $\hat{\phi}(\mathbf{x},t)=0$ which we henceforth consider, our variational equations are

$$\dot{G}(\mathbf{k},t)=4\Sigma(\mathbf{k},t)G(\mathbf{k},t), \tag{3.1a}$$

$$\dot{\Sigma}(\mathbf{k},t)=\frac{1}{8}G^{-2}(\mathbf{k},t)-2\Sigma^2(\mathbf{k},t)-\frac{1}{2}\left[k^2+\mu^2+\frac{\lambda}{2}\int_k G(\mathbf{k},t)\right]. \tag{3.1b}$$

For $\lambda=0, G_0(\mathbf{k},t)$ is oscillatory about its equilibrium point

$$\frac{1}{2\omega_k}=\frac{1}{2(k^2+\mu^2)^{1/2}}.$$

This can be seen easily if we set $G_0\equiv Q^2$, whereupon Eqs. (3.1) become[8]

$$\ddot{Q}=\frac{1}{4Q^3}-\omega_k^2Q. \tag{3.2}$$

This may be regarded as a mechanical equation of motion for a particle, with coordinate Q, moving in a potential

$$V(Q)=\frac{1}{8Q^2}+\tfrac{1}{2}\omega_k^2Q^2. \tag{3.3}$$

The minimum of the potential is at $Q=1/\sqrt{2\omega_k}$ and the particle oscillates around this point.

Next, we present the most general solution to the free equation. For a given initial condition $G(k,0)$ and $\dot{G}(k,0)\equiv 4\Sigma(k,0)G(k,0)$, the solution $G_0(k,t)$ is

$$G_0(k,t)=\frac{1}{2\omega_k}\{1+2n_k-[(1+2n_k)^2-1]^{1/2}\cos 2[\omega_k t-\delta_0(k)]\}, \tag{3.4a}$$

where the average energy of the kth mode E_k is

$$E_k=(n_k+\tfrac{1}{2})\omega_k=\frac{\dot{G}^2(k,0)}{8G(k,0)}+\tfrac{1}{8}G^{-1}(k,0)+\tfrac{1}{2}\omega_k^2G(k,0), \tag{3.4b}$$

and the phase is given by

$$\cot 2\delta_0(k)=\left[\frac{\omega_k}{G(k,0)\dot{G}(k,0)}\left[G^2(k,0)-\frac{\dot{G}^2(k,0)+1}{4\omega_k^2}\right]\right]. \tag{3.4c}$$

(Everything may be taken to depend on the magnitude of the vector $\mathbf{k}$; therefore, we replace the vector $\mathbf{k}$ by k.) The energy E_k has been expressed in terms of the average kth-mode particle number n_k, in the initial state. This is defined by

$$\langle a^{\dagger}(\mathbf{k})a(\mathbf{k}')\rangle = n_k(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') , \tag{3.5}$$

where the annihilation operator has the representation

$$a(\mathbf{k}) = \int \frac{d^3x}{(2\omega_k)^{1/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\frac{\delta}{\delta\phi(\mathbf{x})} + \omega_k\phi(\mathbf{x}) \right] . \tag{3.6}$$

B. Initial states

The Gaussian trial states whose covariance satisfies Eqs. (3.1) are given in Eq. (1.6) with $\hat{\phi}(\mathbf{x},t)=\hat{\pi}(\mathbf{x},t)=0$. To obtain a solution initial data must be specified, and we fix at $t=0$ both $G(\mathbf{x}-\mathbf{y})$ and $\Sigma(\mathbf{x}-\mathbf{y})$; i.e., we select an initial Gaussian state. However, for the equations to be renormalizable, G and Σ cannot be arbitrary. Here we develop reasonable criteria for limiting these quantities.

We are interested only in those states which belong to the Fock space built on the vacuum. In particular, we want our trial states at initial time $t=0$ to be in that Fock space. Such initial states

$$\Psi = \exp\left[-\int_{\mathbf{x},\mathbf{y}} \phi(\mathbf{x})[\tfrac{1}{4}G^{-1}(\mathbf{x},\mathbf{y}) - i\Sigma(\mathbf{x},\mathbf{y})]\phi(\mathbf{y}) \right] , \tag{3.7}$$

must have a nonvanishing overlap with the vacuum state Ψ_V, which in the variational approximation is given by

$$\Psi_V = \exp\left[-\int_{\mathbf{xy}} \phi(x)\tfrac{1}{4}G_V{}^{-1}(\mathbf{x}-\mathbf{y})\phi(\mathbf{y}) \right] , \tag{3.8a}$$

where

$$G_V(\mathbf{x}-\mathbf{y}) = \int_k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{2(k^2+m_R{}^2)^{1/2}} , \tag{3.8b}$$

and m_R is the renormalized mass in the vacuum sector. (If Ψ's overlap with Ψ_V vanishes, then its overlap with all multiparticle states in the Fock space build on Ψ_V also vanishes.) We are assuming throughout that the order parameter vanishes at equilibrium; hence, $\hat{\phi}$ has been set to zero.

The overlap between Ψ and Ψ_V is

$$\left| \int \mathcal{D}\phi\, \Psi_V \Psi \right|^2 \sim e^{-N} = \exp\left[-\tfrac{1}{2}L^3 \int_k \ln(1+n_k) \right] , \tag{3.9}$$

where L^3 is the infinite-volume factor and n_k introduced in (3.5) is

$$n_k = \frac{1}{8\omega_k G}[(1-2\omega_k G)^2 + \dot{G}^2] . \tag{3.10}$$

(Interactions produce mode-mode coupling, so that the mode number and mode energy are not time independent.)

By "nonvanishing overlap" we shall mean

$$\frac{N}{L^3} = \tfrac{1}{2}\int \ln(1+n_k) < \infty , \tag{3.11}$$

which we shall take to imply that n_k must satisfy

$$\int_k n_k < \infty . \tag{3.12}$$

In other words, the average particle number density must be finite. We shall satisfy (3.12) by requiring n_k to decrease faster than k^{-3} at large k and shall choose our initial states so that they satisfy

$$n_k \underset{k\to\infty}{\sim} \frac{1}{k^4} . \tag{3.13}$$

[For simplicity, we ignore possible fractional powers, logarithmic behavior, or the possibility that (3.12) is finite because of oscillatory behavior.] From (3.10) we see that (3.13) requires the following large-k behavior for $G(k,0)$ and $\dot{G}(k,0)$:

$$\lim_{k\to\infty} G(k,0) = \frac{1}{2k}\left[1 + O\left[\frac{1}{k^2} \right] \right] , \tag{3.14a}$$

$$\lim_{k\to\infty} \dot{G}(k,0) = O\left[\frac{1}{k^2} \right] . \tag{3.14b}$$

In general, the terms of $O(1/k^2)$ may have the following structure:

$$O\left[\frac{1}{k^2} \right] = \frac{1}{k^2}[A + B(k)] , \tag{3.14c}$$

where A is a constant and $B(k)$ is an oscillating function. Therefore, we shall parametrize the initial value of G by

$$G(k,0) = \frac{1}{2(k^2+\overline{m}^2)^{1/2}}[1+f(k)] , \tag{3.15a}$$

where

$$\lim_{k\to\infty} G(k,0) = \frac{1}{2k}\left[1 - \frac{\overline{m}^2}{2k^2} + \frac{g\cos\alpha(k)}{k^2} \right] \tag{3.15b}$$

and

$$\lim_{k\to\infty} \dot{G}(k,0) = \frac{A + B\cos\beta(k)}{k^2} , \tag{3.15c}$$

with nonoscillatory α and β and k-independent constants g, A, B, and $\overline{m}$, the last being a mass parameter that we shall specify shortly.

IV. RENORMALIZATION OF THE TIME-DEPENDENT VARIATIONAL EQUATION

In this section we shall study renormalizability of the time-dependent variational equation for the states $\Psi(\phi,t)$ presented in Sec. III. We shall consider the simple case where $\hat{\phi}(\mathbf{x},t)=0$. Generalization of our result to the case where $\hat{\phi}(\mathbf{x},t)=\hat{\phi}(t)\neq 0$ is straightforward, but will not be discussed here because renormalizability is unaffected by the value of $\hat{\phi}(t)$. We show that the time-dependent variational equations are made finite by the static vacuum renormalization described in Sec. II.

Our variational equation in terms of G alone is

$$\ddot{G}=\tfrac{1}{2}G^{-1}+\tfrac{1}{2}G^{-1}\dot{G}^{2}-2\left[k^{2}+\mu^{2}+\frac{\lambda}{2}\int_{k}G(k,t)\right]G\ . \tag{4.1}$$

The possible divergences of this equation are contained in the integral $\int_k G(k,t)$. As in the static case, let us define an effective mass term which is now time dependent:

$$\begin{aligned} m^{2}(t)&=\mu^{2}+\frac{\lambda}{2}\int_{k}G(k,t)\\ &=\mu^{2}+\frac{\lambda}{2}\int_{k}G_{V}(k)+\frac{\lambda}{2}\int[G(k,t)-G_{V}(k)]\\ &=m^{2}+\tilde{m}^{2}(t)\ . \end{aligned} \tag{4.2}$$

$G_V(k)$ is the value of the G in the vacuum state; therefore,

$$m^{2}\equiv\mu^{2}+\frac{\lambda}{2}\int_{k}G_{V}(k)=\mu^{2}+\frac{\lambda}{2}\int_{k}\frac{1}{2(k^{2}+m^{2})^{1/2}}\ , \tag{4.3}$$

and m is the effective static mass in the vacuum sector discussed in Sec. II, and

$$\tilde{m}^{2}(t)\equiv\frac{\lambda}{2}\int_{k}[G(k,t)-G_{V}(k)]\ . \tag{4.4}$$

By the static renormalization prescription given in Eq. (2.4) we then obtain

$$m^{2}(t)=m_{R}^{2}+\tilde{m}^{2}(t)\ , \tag{4.5}$$

where $m_R=m$ is the usual renormalized mass in the vacuum sector given in Eq. (2.8). Therefore,

$$G_{V}(k)=\frac{1}{2(k^{2}+m_{R}^{2})^{1/2}} \tag{4.6}$$

and

$$\tilde{m}^{2}(t)=\frac{\lambda_{R}}{2}\int_{k}[G(k,t)-G_{V}(k)]+\tilde{m}^{2}(t)I_{2}(M)\ . \tag{4.7}$$

Our goal is to show that Eq. (4.7) is finite for all t when we choose our initial conditions in the form given in Eqs. (3.15).

Let us first consider finiteness of $\tilde{m}^2(0)$. From Eq. (4.7) we have

$$\begin{aligned} \tilde{m}^{2}(0)&=\frac{\lambda_{R}}{2}\int_{k}\left[\frac{1}{2(k^{2}+\overline{m}^{2})^{1/2}}[1+f(k)]-\frac{1}{2(k^{2}+m_{R}^{2})^{1/2}}+\frac{\tilde{m}^{2}(0)}{M^{2}}\left[\frac{1}{2k}-\frac{1}{2(k^{2}+M^{2})^{1/2}}\right]\right]\\ &=\frac{\lambda_{R}}{2}\left[\int_{k}\left[-\frac{\overline{m}^{2}}{4k^{3}}+\frac{m_{R}^{2}}{4k^{3}}+\frac{\tilde{m}^{2}(0)}{4k^{3}}\right]+\int_{k}\frac{g\cos\alpha(k)}{2k^{3}}+\text{finite terms}\right]\ . \end{aligned} \tag{4.8}$$

The integral $\int_k [g\cos\alpha(k)]/2k^3$ has an ultraviolet structure which may lead to a logarithmic divergence if $\lim_{k\to\infty}\alpha(k)=\text{const}$. We shall choose $\alpha(k)$ such that

$$\lim_{k\to\infty}\alpha(k)\sim k^{n},\quad n>0\ , \tag{4.9}$$

and therefore the integral is finite.

Next we observe that $\tilde{m}^2$ is finite only if

$$\overline{m}^{2}=m^{2}(0)=m_{R}^{2}+\tilde{m}^{2}(0)\ . \tag{4.10}$$

Namely, the mass $\overline{m}$ in the initial G is self-consistently determined by Eqs. (4.5)–(4.7) at $t=0$. We notice that if $f(k)=0$ in the initial G, then $G(k,0)$ is just G_V; $f(k)$ determines the value $m^2(0)=\overline{m}^2$ and describes the excitations in the initial $G(k,0)$ relative to the G_V in the vacuum state.

Now we shall show that $\tilde{m}^2(t)$ is finite for all t. To do this let us first assume that $\tilde{m}^2(t)$, as given in Eq. (4.7), is finite and look for the solution of Eq. (4.1), which may be written as

$$\ddot{G}=\tfrac{1}{2}G^{-1}+\tfrac{1}{2}G^{-1}\dot{G}^{2}-2[k^{2}+m_{R}^{2}+\tilde{m}^{2}(t)]G\ . \tag{4.11}$$

First we observe that the $k\to\infty$ limit of this equation coincides to the $k\to\infty$ limit of the free equation when mass terms are ignored. Next we conjecture that when the next-to-leading terms, which depend on the time-dependent mass are included, the solution of Eq. (4.11) at large k is still given by the free solution

$$\begin{aligned} \lim_{k\to\infty}G(k,t)&=\frac{1}{2k}\left[1-\frac{m_{R}^{2}+\tilde{m}^{2}(t)}{2k^{2}}+\frac{g\cos\alpha(k)}{k^{2}}\cos 2kt+\frac{A+B\cos\beta(k)}{k^{2}}\sin 2kt+O\left[\frac{1}{k^{3}}\right]\right]\\ &=\frac{1}{2k}\left[1-\frac{m_{R}^{2}+\tilde{m}^{2}(t)}{2k^{2}}-2\sqrt{\tilde{n}_{k}}\cos[2kt-2\delta(k)]+O\left[\frac{1}{k^{3}}\right]\right]\ , \end{aligned} \tag{4.12}$$

where $\tilde{n}_k$ is the average number of particles with mass $m^2(0)$ in the kth modes of the initial state. That the conjecture is true is straightforwardly established by checking that the expression in Eq. (4.12) indeed satisfies Eq. (4.11) for large k up to terms of $O(1/k)$.

Now let us examine finiteness of $\tilde{m}^2$ at all t. From Eqs. (4.7) and (4.12) we have

$$\tilde{m}^{2}(t)=\frac{\lambda_R}{2}\left\{\int_k\left[\frac{1}{2k}\left[1-\frac{m_R{}^2+\tilde{m}^{2}(t)}{2k^2}+\frac{g\cos\alpha(k)}{k^2}\cos 2kt+\frac{A+B\cos\beta(k)}{k^2}\sin 2kt\right]\right.\right.$$
$$\left.\left.-\frac{1}{2k}\left[1-\frac{m_R{}^2}{2k^2}\right]+\frac{\tilde{m}^{2}(t)}{4k^3}\right]+\text{finite terms}\right\}$$
$$=\frac{\lambda_R}{2}\left[\int_k\left[\frac{g\cos\alpha(k)}{2k^3}\cos 2kt+\frac{A+B\cos\beta(k)}{2k^3}\sin 2kt\right]+\text{finite terms}\right]. \tag{4.13}$$

The integral involving $\cos 2kt$ has a structure of a possible divergence at $t=0$, but as discussed earlier such a possibility does not occur because of the oscillatory behavior of $\cos\alpha(k)$. The integral involving $\sin 2kt$ term is finite for all t. Therefore, $\tilde{m}^{2}(t)$ is finite.

We have thus shown that for the initial states, which belong to the Fock space built on the vacuum, the time-dependent variational equation is made finite by the static renormalization used in the vacuum sector. A detailed perturbative proof that $G(k,t)$ given in Eq. (4.12) is correct and that $\tilde{m}^{2}(t)$ is finite is given in the Appendix. (A brief discussion on the two space-time dimensional case is given in Ref. 11.)

Finally we would like to point out that one may alternatively renormalize the time-dependent variational equations at initial time $t=0$. Here, the initial renormalized mass has no communication with the static mass in the vacuum sector. At $t=0$, with $\hat{\phi}(t)=0$, we define our renormalized mass as

$$m^2(0)=\mu^2+\frac{\lambda}{2}\int_k G(k,0)\ . \tag{4.14}$$

Then the time-dependent mass has the expression

$$m^2(t)=m^2(0)+\frac{\lambda}{2}\int_k[G(k,t)-G(k,0)]$$
$$=m^2(0)+\hat{m}^2(t)\ . \tag{4.15}$$

With the same coupling-constant renormalization used before we have

$$\hat{m}^{2}(t)=\frac{\lambda_R}{2}\left[\int_k[G(k,t)-G(k,0)]-\hat{m}^{2}(t)I_2(M)\right], \tag{4.16}$$
$$\hat{m}^{2}(0)=0\ .$$

Since we already know the large-k behavior of our solution $G(k,t)$, it is straightforward to check finiteness of $\hat{m}^{2}(t)$. We find that $\hat{m}^{2}(t)$ is finite only if $\overline{m}^{2}$ of the initial G is given by $m^2(0)$. Therefore, when we use a mass renormalization prescription which is not related to the vacuum mass renormalization, the initial mass $\overline{m}$ is not self-consistently determined, rather it coincides with the collection of infinite quantities [Eq. (4.14)] defined to be finite (renormalized) in the sum. Furthermore, although the renormalization prescription described in Eq. (4.14) makes our equation finite, one cannot calculate physical quantities in terms of the renormalized mass in the vacuum sector.

In order that the energy density of the system be finite, stronger constraints than the ones considered here are needed; specifically in Eqs. (3.15) $f(k)\sim_{R\to\infty}O(1/k^3)$ or smaller and $\dot{G}(k,0)\sim_{k\to\infty}O(1/k^3)$ or smaller.

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APPENDIX

We shall present the perturbative calculation which shows that $m^2(t)$ given in Eq. (4.2) is finite and that $G(k,t)$ given in Eq. (4.12) is the large-k asymptotic solution of our variational equation. For our perturbative calculation we find that it is simpler to use a combined quantity of G and Σ:

$$\Omega(k,t)=\tfrac{1}{2}G^{-1}(k,t)-2i\Sigma(k,t)\ , \tag{A1}$$

which determines the covariance of the Gaussian wave functional.

In terms of Ω, our variational equations become a first-order equation in time:

$$i\frac{\partial\Omega}{\partial t}=\Omega^2-\left[k^2+\mu^2+\frac{\lambda}{4}\int_k\frac{1}{\mathrm{Re}\Omega}\right]. \tag{A2}$$

The time-dependent mass, in terms of m_R, λ_R, and Ω is

$$m^2(t)=m_R{}^2+\frac{\lambda_R}{2}\left[\int_k\left[\frac{1}{2\,\mathrm{Re}\Omega(k,t)}-G_V(k)\right]\right.$$
$$\left.+\tilde{m}^{2}(t)I_2(M)\right]$$
$$=m_R{}^2+\tilde{m}^{2}(t)\ . \tag{A3}$$

By expanding $m^2(t)$ and $\Omega(k,t)$ as

$$m^2(t)=\sum_{n=0}\lambda_R{}^2 m_n{}^2(t)=m_R{}^2+\sum_{n=1}\lambda_R{}^n\tilde{m}_n{}^2(t)\ , \tag{A4}$$

$$\Omega(k,t)=\sum_{n=0}\lambda_R{}^n\Omega_n(k,t)\ , \tag{A5}$$

we obtain

$$m_0{}^2(t)=m_R{}^2 , \tag{A6a}$$

$$\tilde{m}_1{}^2(t)=\frac{1}{2}\left[\int_k \frac{1}{2\,\mathrm{Re}\Omega_0}-G_V\right] , \tag{A6b}$$

$$\tilde{m}_2{}^2(t)=\frac{1}{2}\left[\int_k \frac{1}{2\,\mathrm{Re}\Omega_0}\left[-\frac{\mathrm{Re}\Omega_1}{\mathrm{Re}\Omega_0}\right]+\tilde{m}_1{}^2(t)I_2(M)\right] , \tag{A6c}$$

$$\tilde{m}_3{}^2(t)=\frac{1}{2}\left\{\int_k \frac{1}{2\,\mathrm{Re}\Omega}\left[-\frac{\mathrm{Re}\Omega_2}{\mathrm{Re}\Omega_0}+\left[\frac{\mathrm{Re}\Omega_1}{\mathrm{Re}\Omega_0}\right]^2\right]+\tilde{m}_2{}^2(t)I_2(M)\right\} , \tag{A6d}$$

$\ldots,$

$$\tilde{m}_n{}^2(t)=\frac{1}{2}\left\{\int_k \frac{1}{2\,\mathrm{Re}\Omega_0}\left[-\frac{\mathrm{Re}\Omega_{n-1}}{\mathrm{Re}\Omega_0}+\left[\text{terms involving products of two or more}\frac{\mathrm{Re}\Omega_i}{\mathrm{Re}\Omega_0},\ 1\le i\le n-2\right]\right]\right.$$
$$\left.+\tilde{m}_{n-1}{}^2(t)I_2(M)\right\} \tag{A6e}$$

and

$$i\frac{\partial\Omega_0}{\partial t}=\Omega_0{}^2-(k^2+m_R{}^2) , \tag{A7a}$$

$$i\frac{\partial\Omega_1}{\partial t}=2\Omega_0\Omega_1-\tilde{m}_1{}^2(t) , \tag{A7b}$$

$$i\frac{\partial\Omega_2}{\partial t}=2\Omega_0\Omega_2+\Omega_1{}^2-\tilde{m}_2{}^2(t) , \tag{A7c}$$

$$i\frac{\partial\Omega_3}{\partial\tau}=2\Omega_0\Omega_3+2\Omega_1\Omega_2-\tilde{m}_3{}^2(t) , \tag{A7d}$$

$\ldots,$

$$i\frac{\partial\Omega_n}{\partial t}=2\Omega_0\Omega_n+(\text{all possible }\Omega_i\Omega_j\text{ with }1\le i,j\le n-1\text{ and }i+j=n)-\tilde{m}_n{}^2(t) . \tag{A7e}$$

Our initial states are subject to the same constraints given in Eqs. (3.15). In terms of $\mathrm{Re}\Omega(k,0)$ and $\mathrm{Im}\Omega(k,0)$, Eqs. (3.15) are

$$\lim_{k\to\infty}\mathrm{Re}\Omega(k,0)=k\left[1+\frac{\tilde{m}^2}{2k^2}-\frac{g\cos\alpha(k)}{k^2}\right] , \tag{A8a}$$

$$\lim_{k\to\infty}\mathrm{Im}\Omega(k,0)=-\frac{1}{k}[A+B\cos\beta(k)] . \tag{A8b}$$

In perturbation expansion, we shall expand

$$\tilde{m}^2=m_R{}^2+\sum_{n=1}\lambda_R{}^n\tilde{m}_n{}^2 , \tag{A9a}$$

$$g\cos\alpha(k)=\left[\sum_{n=0}\lambda_R{}^n g_n\right]\cos\left[\sum_{n=0}\lambda_R{}^n\alpha_n(k)\right] , \tag{A9b}$$

$$A=\sum_{n=0}\lambda_R{}^n A_n , \tag{A9c}$$

$$B\cos\beta(k)=\left[\sum_{n=0}\lambda_R{}^n B_n\right]\cos\left[\sum_{n=0}\lambda_R{}^n\beta_n(k)\right] . \tag{A9d}$$

Then, we find that

$$g\cos\alpha(k)=g_0\cos\alpha_0(k)+\sum_{n=1}\lambda_R{}^n\bar{g}_n\cos[\alpha_0(k)+\delta_n] , \tag{A10a}$$

where $\bar{g}_n$ and δ_n are functions of g_m, $0\le m\le n$, and α_l, $1\le l\le n$:

$$\bar{g}_n=\bar{g}_n(g_m,\alpha_l),\quad 0\le m\le n ,$$
$$\delta_n=\delta_n(g_m,\alpha_l),\quad 1\le l\le n . \tag{A10b}$$

The g_n, being expansion coefficients of g, are k independent. Moreover, in order that $\bar{g}_n$ be k independent, we choose $\alpha_l, l\ge1$ to be k independent, and in turn assume that the phase shifts δ_n also are k independent. Therefore, in each order of λ_R, we specify two new constants g_m and α_l to produce $g\cos\alpha(k)$ in the full initial condition. A similar expansion is obtained for $B\cos\beta(k)$:

$$B\cos\beta(k)=B_0\cos\beta_0(k)+\sum_{n=1}\lambda_R{}^n\bar{B}_n\cos[\beta_0(k)+\rho_n] , \tag{A10c}$$

where $\bar{B}_n$ and ρ_n are constant and functions of B_m, $0\le m\le n$, and β_l, $1\le l\le n$. Therefore, we have

$$\lim_{k\to\infty}\mathrm{Re}\Omega_0(k,0)=k\left[1+\frac{m_R{}^2}{2k^2}-\frac{g_0\cos\alpha_0(k)}{k^2}\right] , \tag{A11a}$$

$$\lim_{k\to\infty} \mathrm{Im}\Omega_0(k,0)=-\frac{1}{k}[A_0+B_0\cos\beta_0(k)]\,, \tag{A11b}$$

$$\lim_{k\to\infty} \mathrm{Re}\Omega_n(k,0)=\frac{\overline{m}_n{}^2}{2k}-\frac{\overline{g}_n\cos[\alpha_0(k)+\delta_n]}{k},\quad n\geq 1\,, \tag{A11c}$$

$$\lim_{n\to\infty} \mathrm{Im}\Omega_n(k,0)=-\frac{A_n}{k}-\frac{\overline{B}_n\cos[\beta_0(k)+\rho_n]}{k},\quad n\geq 1\,. \tag{A11d}$$

First, let us discuss the lowest-order solution $\Omega_0(k,t)$. The most general solution of (A7a) is

$$\Omega_0(k,t)=-i\omega_k\cot\{\omega_k t-[\delta_0(k)+i\epsilon_0(k)]\}$$
$$=\omega_k\frac{\sinh 2\epsilon_0(k)-i\sin 2[\omega_k t-\delta_0(k)]}{\cosh 2\epsilon_0(k)-\cos 2[\omega_k t-\delta_0(k)]}\,, \tag{A12}$$

where $\omega_k=(k^2+m_R{}^2)^{1/2}$ and $\delta(k)+i\epsilon(k)$ are complex integration constants which are related to our initial conditions. By comparing with the free solution $G_0(k,t)$, obtained earlier in Sec. III, with $\mathrm{Re}\Omega_0(k,t)=\frac{1}{2}G_0{}^{-1}(k,t)$ we identify $\epsilon_0(k)$ and $\delta_0(k)$ by

$$\frac{\cosh 2\epsilon_0(k)}{\sinh 2\epsilon_0(k)}=1+2n_k \underset{k\to\infty}{\sim} 1+O\left[\frac{1}{k^4}\right]\,, \tag{A13a}$$

$$\frac{1}{\sinh 2\epsilon_0(k)}=[(1+2n_k)^2-1]^{1/2} \underset{k\to\infty}{\sim} O\left[\frac{1}{k^2}\right]\,, \tag{A13b}$$

and

$$\frac{\cos 2\delta_0(k)}{\sinh 2\epsilon_0(k)} \underset{k\to\infty}{\sim} \frac{g_0\cos\alpha_0(k)}{k^2}\,, \tag{A13c}$$

$$\frac{\sin 2\delta_0(k)}{\sinh 2\epsilon(k)} \underset{k\to\infty}{\sim} \frac{A_0+B_0\cos\beta_0(k)}{k^2}\,. \tag{A13d}$$

The general solution to Eqs. (A7b)–(A7e) is given by

$$\Omega_n(k,t)=\frac{1}{h(k,t)}\left[\int_0^t if_n(k,\tau)h(k,\tau)d\tau+\Omega_n(k,0)\right],\quad n\geq 1\,, \tag{A14a}$$

where

$$h(k,t)\equiv\exp\left[i\int_0^t 2\Omega_0(h,t')dt'\right]=\frac{\sin^2(\omega_k t-\delta_0-i\epsilon_0)}{\sin^2(\delta_0+i\epsilon_0)} \tag{A14b}$$

and

$$f_1(k,t)=\tilde{m}_1{}^2(t)\,,$$
$$\ldots\,, \tag{A14c}$$
$$f_n(k,t)=\tilde{m}_n^2(t)-(\text{all possible }\Omega_i\Omega_j\text{ with }1\leq i,j\leq n-1\text{ and }i+j=n)\text{ for }n\geq 2\,.$$

Now let us examine finiteness of $\tilde{m}_1{}^2(t)$:

$$\tilde{m}_1{}^2(t)=\frac{1}{2}\int_k\left[\frac{1}{2\,\mathrm{Re}\Omega_0}-\frac{1}{2(k^2+m_R{}^2)^{1/2}}\right]$$
$$=\frac{1}{2}\int_k\left[\frac{1}{2k}-\frac{m_R{}^2}{4k^3}+\frac{g_0\cos\alpha_0(k)}{2k^3}\cos 2kt+\frac{A_0+B_0\cos\beta_0(k)}{2k^3}\sin 2kt-\frac{1}{2k}+\frac{m_R{}^2}{4k^3}\right]+\text{finite terms}\,. \tag{A15}$$

Equation (A15) is precisely the same form as Eq. (4.13). Since $\alpha(k)=\alpha_0(k)+\sum_{n=1}\lambda_R^n\alpha_n$ where α_n is k independent, $\lim_{k\to\infty}\alpha_0(k)\neq\text{const}$ by Eq. (4.9) and the integral involving $\cos\alpha_0(k)$ is finite for all t including $t=0$. Therefore, $\tilde{m}_1{}^2(t)$ is finite.

Now let us consider the large-k behavior of $\Omega_1(k,t)$. From the solution given in Eqs. (A14) we obtain

$$\lim_{k\to\infty}\Omega_1(k,t)=\left\{\left[\frac{1}{2k}\tilde{m}_1{}^2(t)+\left[\mathrm{Re}\Omega_1(k,0)-\frac{1}{2k}\tilde{m}_1{}^2(0)\right]\cos 2kt+\mathrm{Im}\Omega_1(k,0)\sin 2kt\right]\right.$$
$$\left.+i[\mathrm{Im}\Omega_1(k,0)\cos 2kt-\mathrm{Re}\Omega_1(k,0)\sin 2kt]+O\left[\frac{1}{k^2}\right]\right\}\,. \tag{A16}$$

From our initial condition $\Omega_1(k,0)$ given in Eqs. (A11),

$$\lim_{k\to\infty}\Omega_1(k,t)=\left\{\left[\frac{1}{2k}\tilde{m}_1{}^2(t)+\left[\frac{-\frac{1}{2}\tilde{m}_1{}^2(0)+\frac{1}{2}\overline{m}_1{}^2-\overline{g}_1\cos[\alpha_0(k)+\delta_1]}{k}\right]\cos 2kt\right.\right.$$
$$\left.\left.-\frac{1}{k}\{A_1+B_1\cos[\beta_0(k)+\rho_1]\}\sin 2kt\right]+iO\left[\frac{1}{k}\right]\right\}\,. \tag{A17}$$

Next, using the above expression for $\Omega_1(k,t)$ we immediately see that $\tilde{m}_2{}^2(t)$ is finite when we take $\overline{m}_1(0)=\tilde{m}_1(0)$:

$$\tilde{m}_2{}^2(t)=\tfrac{1}{2}\int_k\left[-\frac{\mathrm{Re}\Omega_1}{2\,\mathrm{Re}\Omega_0{}^2}+\frac{\tilde{m}_1{}^2(t)}{M^2}\left[\frac{1}{2k}-\frac{1}{2(k^2+M^2)^{1/2}}\right]\right]. \tag{A18}$$

Finally, we are ready to generalize our result to an arbitrary order in λ_R. The fact that $\lim_{k\to\infty}\Omega_1(k,t)\sim 1/k$ tells us the following.

(i) In the limit $k\to\infty$ the equation for $\Omega_2(k,t)$ becomes

$$i\frac{\partial\Omega_2}{\partial t}(k,t)=2\Omega_0\Omega_2-\tilde{m}_2{}^2(t)\ , \tag{A19}$$

since in (A7c) $\Omega_1{}^2(k,t)\sim_{k\to\infty}(1/k^2)$, and is negligible compared to $\tilde{m}_2{}^2(t)$ term. The above equation is exactly of the same form as the equation for Ω_1, and, therefore, the large-k behavior of $\Omega_2(k,t)$ is given by Eq. (A17) with all subscripts 1 replaced by 2.

(ii) In the limit $k\to\infty$, the expression for $\tilde{m}_3{}^2(t)$ given in Eq. (A6d) becomes exactly the same as that of $\tilde{m}_2{}^2(t)$ again with all subscripts 1 replaced by 2, since the term

$$\frac{\mathrm{Re}\Omega_1{}^2}{\mathrm{Re}\Omega_0^3}\underset{k\to\infty}{\sim}\frac{1}{k^5}$$

is negligible compared to

$$\frac{\mathrm{Re}\Omega_3}{\mathrm{Re}\Omega_0{}^2}\underset{k\to\infty}{\sim}\frac{1}{k^3}\ ,$$

and we have

$$\tilde{m}_3{}^2(t)\underset{k\to\infty}{\sim}\left[\int_k\frac{-\mathrm{Re}\Omega_2}{2\,\mathrm{Re}\Omega_0{}^2}+\tilde{m}_2{}^2(t)I_2(M)\right]; \tag{A20}$$

therefore $\tilde{m}_3{}^2(t)$ is finite when $\overline{m}_2{}^2=\tilde{m}_2{}^2(0)$.

The above two-step procedure can be continued indefinitely and to an arbitrary order in λ_R and we have, in the large-k limit,

$$i\frac{\partial\Omega_n(k,t)}{\partial t}=2\Omega_0\Omega_n-\tilde{m}_n{}^2(t) \tag{A21a}$$

and

$$\tilde{m}_{n+1}{}^2(t)=\frac{1}{2}\left[\int_k-\frac{\mathrm{Re}\Omega_n}{2\,\mathrm{Re}\Omega_0{}^2}+\tilde{m}_n{}^2(t)I_2(M)\right], \tag{A21b}$$

where

$$\Omega_n(k,t)=\left\{\left[\frac{1}{2k}\tilde{m}_n{}^2(t)+\left[\frac{-\frac{1}{2}\tilde{m}_n{}^2(0)+\frac{1}{2}\overline{m}_n{}^2-\overline{g}_n\cos[\alpha_0(k)+\delta_n]}{k}\right]\cos 2kt\right.\right.$$
$$\left.\left.-\frac{1}{k}\{A_n+\overline{B}_n\cos[\beta_n(k)+\rho_n]\}\sin 2kt\right]+iO\left[\frac{1}{k}\right]\right\}. \tag{A21c}$$

Therefore, $\tilde{m}_{n+1}{}^2(t)$ is finite when we take $\overline{m}_n{}^2=\tilde{m}_n{}^2(0)$.

Furthermore, the large-k behavior of the full solution $G(k,t)$ obtained from our perturbative calculation is, from Eqs. (A12) and (A21c),

$$\lim_{k\to\infty}G(k,t)=\frac{1}{2\,\mathrm{Re}\Omega_0(k,t)}-\sum_{n=1}\lambda_R{}^n\frac{\Omega_n(k,t)}{2\,\mathrm{Re}\Omega_0{}^2(k,t)}$$
$$=\frac{1}{2k}-\frac{1}{4k^3}\left[m_R{}^2+\sum_{n=1}\lambda_R^n\tilde{m}_n{}^2(t)\right]+\frac{1}{2k^3}\left[g_0\cos\alpha_0(k)+\sum_{n=1}\lambda_R^n\overline{g}_n\cos[\alpha_0(k)+\delta_n]\right]\cos 2kt$$
$$+\frac{1}{2k^3}\left[\sum_{n=0}\lambda_R^n A_n+B_0\cos\beta_0(k)+\sum_{n=1}\lambda_R^n\overline{B}_n\cos[\beta_n(k)+\rho_n]\right]\sin 2kt$$
$$=\frac{1}{2k}-\frac{m^2(t)}{4k^3}+\frac{1}{2k^3}g\,\cos\alpha(k)\cos 2kt+\frac{1}{2k^3}[A+B\,\cos\beta(k)]\sin 2kt\ . \tag{A22}$$

The above expression is precisely the one given in Eq. (4.12).

[1]R. Jackiw and A. Kerman, Phys. Lett. **71A**, 158 (1979).

[2]G. F. Mazenko, Phys. Rev. Lett. **54**, 2163 (1985); G. Semenoff and N. Weiss, Phys. Rev. D **31**, 689 (1985); A. Albrecht and R. Brandenberger, *ibid.* **31**, 1225 (1985); M. Evans and J. G. McCarthy, *ibid.* **31**, 1799 (1985); A. H. Guth and S.-Y. Pi, *ibid.* **32**, 1899 (1981).

[3]A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[4]A. D. Linde, Phys. Lett. **114B**, 431 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[5]F. Cooper, S.-Y. Pi, and P. Stancioff, Phys. Rev. D **34**, 3831 (1986).

[6]S. W. Hawking, Phys. Lett. **115B**, 295 (1982); A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); A. Starobinsky, Phys. Lett. **117B**, 175 (1982); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[7]A purely classical analysis has been carried out by A. Albrecht, R. Brandenberger, and R. Matzner, Phys. Rev. D **35**, 429 (1987).

[8]A. Kerman and D. Vautherin (private communication).

[9]S. Coleman, R. Jackiw, and D. Politzer, Phys. Rev. D **10**, 2491 (1974); W. Bardeen and M. Moshe, *ibid.* **28**, 1372 (1983).

[10]More on the structure of the renormalized ϕ^4 theory is found in Ref. 9 and in L. Abbott, J. Kang, and H. Schnitzer, Phys. Rev. D **13**, 2212 (1976).

[11]In two space-time dimensions, our criteria for choosing initial states becomes $\lim_{k\to\infty} n_k \sim 1/k^2$. Then the initial states have $G(k,0)$ and $\dot G(k,0)$ of the form

$$G(k,0)=[1/2(k^2+\overline{m}^{\,2})^{1/2}][1+f(k)]$$

with

$$\lim_{k\to\infty} f(k)=\frac{1}{k}[c+g\cos\alpha(k)]$$

and

$$\lim_{k\to\infty} \dot G(k,0)=\frac{1}{k}[A+B\cos\beta(k)]\ ,$$

where c, g, A, and B are k independent. In two dimensions the ultraviolet divergences in the vacuum sector require only mass renormalization, $\mu^2+(\lambda/2)\int_k G_V(k)=m_R{}^2$. Using the large-$k$ behavior of $G(k,t)$ given in Eq. (4.12) one can easily check that the above mass renormalization renders the time-dependent mass $m^2(t)$ finite at all t. Here renormalizability does not require the initial mass $\overline{m}$ to be m (0), but $\overline{m}=m(0)$ seems to be a logical choice and in that case $f(k)$ determines the excitation of $G(k,0)$ relative to $G_V(k)$.

[12]F. Cooper and E. Mottola have also studied the subject discussed in our paper [preceding paper, Phys. Rev. D **36**, 3114 (1987)].