PHYSICAL REVIEW D VOLUME 36, NUMBER 10 15 NOVEMBER 1987

Initial-value problems in quantum field theory in the large-N approximation

Fred Cooper and Emil Mottola
Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545
(Received 6 April 1987)

We derive the time-evolution equations appropriate to initial-value problems in $\lambda(\phi_a\phi_a)^2$ field theory at large N. The Heisenberg equations of motion for this theory are compared to the Schrödinger equation for a wave functional constrained to be in a Gaussian state [the time-dependent Hartree-Fock (TDHF) ansatz]. The TDHF ansatz corresponds to a special choice of initial conditions for the general large-N Heisenberg equations of motion. The renormalization of the theory is discussed in both approaches and a simple method is given to arrive at finite differential equations suitable for numerical integration forward in time. A necessary and sufficient general condition for these equations to be finite is that the initial state contain a finite average number of particles and/or correlated particle pairs per unit volume.

I. INTRODUCTION

In this paper we study initial-value problems in quantum field theory. Initial-value problems in classical physics are very familiar. In nonrelativistic quantum mechanics, the Schrödinger equation may be solved in terms of the initial data on the wave function. When one studies relativistic quantum field theory, however, one generally does not find a discussion of initial-value problems, and a quite different approach to relativistic quantum field theory is found in the standard texts.

There are two basic reasons for this. First of all, in quantum field theory one encounters ultraviolet divergences in perturbation theory for the first time. The technical problem of the isolation and removal of these divergences (i.e., renormalization) was not well understood historically until relativistically covariant methods of calculation were developed by Feynman and Schwinger. Conceptually, this is still the best way to handle the problem of divergences. The noncovariant Schrödinger Hamiltonian method, though more familiar and better suited to the initial-value formulation, is much more cumbersome for that purpose.

The second reason for the relative unimportance of initial-value problems in quantum field theory is that the great majority of problems to which the theory has been applied do not require knowledge of the detailed time evolution of the wave function. Instead, one is typically interested only in scattering matrix elements for the calculation of cross sections, decay rates, etc. A precise initial-value formulation is mostly superfluous for such applications.

In the last several years, there has been an increase of effort on various problems where the time evolution of initial data is not only relevant, but actually forms the heart of the matter. Our particular interest has been in the early Universe and inflationary cosmology,[1-3] but many problems of current interest in nuclear many-body theory,[4] condensed-matter physics, and quantum chemistry[5] fall into this class as well. The common feature seems to be the applicability of semiclassical or mean-field methods to quantum problems with infinite (or very many) degrees of freedom, where the time evolution is important. Because of the broad nature of such problems, a wide variety of techniques have been developed with little communication between the various subfields or understanding of the alternative approaches.

Accordingly, a major purpose of this paper is to attempt to bridge the gap between those who prefer the more intuitive Schrödinger wave-functional approach and those who prefer more elegant covariant treatments, but are less likely to have considered the initial-value formulation of field theory in any detail. A large fraction of the paper is devoted to a review of different formulations and the relationship between them. Our new contribution is (we believe) a simple and elegant treatment of the renormalization problem, entirely within the context of the (noncovariant) initial-value formulation of quantum field theory.

This present work is an outgrowth of a previous investigation of one of us[6] (F.C.), which reconsidered the time-dependent variational principle of Dirac,[7] later studied also by Jackiw and Kerman.[8] We considered in Ref. 6 some simple quantum-mechanical models of tunneling and quantum "rolling" and then extended the formalism to include field theory. The equations obtained in that paper, using Gaussian trial wave functionals are related to those recently posited by Mazenko[9] to study the inflationary epoch of the early Universe. Mazenko's method involved a large-N approximation to the correlation functions of field theory.

Although Mazenko derived the evolution equations at large N in the Heisenberg picture, he did not present a general strategy for solving the initial-value problem for arbitrary initial data. He also did not discuss the problem of mass and coupling-constant renormalization. Instead, he realized that if the initial data were a thermal distribution, normal ordering with respect to the vacuum would suffice to make the theory finite. The main ingredient missing from these earlier approaches was a clear understanding of the renormalization problem for arbitrary initial data, which we treat carefully in this pa-

per.

Because of this background and the diverse applications possible, we have decided to concentrate on a specific field theory to clarify the various approaches. We consider an interacting $\lambda(\phi_\alpha\phi_\alpha)^2/(8N)$ field theory in flat spacetime and we also assume spatial homogeneity.

The label α runs from 1 to N. This allow us to discuss the large-N approximation and its relationship to the time-dependent Hartree-Fock (TDHF) approximation to the wave functional. We show in Sec. II that the latter Gaussian ansatz is actually just a special case of the more general large-N Heisenberg equations of motion. In Sec. III we give a general specification of the initial-value problem for the noninteracting theory ($\lambda=0$) in the Heisenberg picture and derive an intuitive way of parametrizing the initial data by an equivalent specification of the particle number density and density of correlated pairs present in the initial state of the system. We then discuss the same problem in the Schrödinger picture and obtain the restriction on the initial data when the initial wave functional is Gaussian. When the initial wave functional is Gaussian, specifying the real and imaginary parts of the width of the Gaussian wave functional at time zero corresponds to a specific choice of particle and correlated-pair densities which are not independent. Sections IV and V are devoted to the renormalization question. There we show that the natural choice of the renormalized-mass parameter is the effective mass at $t=0$, which we take to be finite and well defined by definition. The relationship to covariant approaches to mass renormalization is also discussed. We conclude with a discussion of extensions and applications of our methods to a variety of physical problems.

II. INITIAL-VALUE PROBLEMS IN THE LARGE-N APPROXIMATION

In this section we would like first to rederive in the Heisenberg picture the large-N approximation to the time evolution of the two-point functions following the work of Mazenko.[9] In the large-N limit, the theory, which consists of an infinite number of coupled Green's-function equations, reduces to the solution of coupled equations for the one- and two-point functions.[10–14] Since the two-point function obeys a second-order equation in t and t', there are three independent initial data which one needs to specify at $t=0$. If we confine ourselves to equal-time Green's functions, one choice of variables (chosen by Mazenko) are the $\phi\phi$, $\phi\pi$, and $\pi\pi$ Wightman functions. For simplicity, let us first consider the case $\langle i \mid \phi_\alpha(x,t) \mid i \rangle = 0$ (we will lift this restriction later). Here $\mid i \rangle$ is an arbitrary initial state in the Heisenberg picture and is time independent. In what follows we suppress the i in the expectation value.

For the case $\langle \phi_\alpha \rangle = 0$ the equal-time Green's functions are proportional to $\delta_{\alpha\beta}$ and so we define three Green's functions G, D, and K as

$$\langle \phi_\alpha(x,t)\phi_\beta(y,t) \rangle \equiv G(x,y,t)\delta_{\alpha\beta} , \tag{2.1}$$

$$\langle \phi_\alpha(x,t)\pi_\beta(y,t) \rangle \equiv D(x,y,t)\delta_{\alpha\beta} , \tag{2.2}$$

$$\langle \pi_\alpha(x,t)\pi_\beta(y,t) \rangle \equiv K(x,y,t)\delta_{\alpha\beta} . \tag{2.3}$$

Using the Hamiltonian

$$H = \int d^3x \left[\tfrac{1}{2}\pi_\alpha\pi_\alpha + \tfrac{1}{2}(\nabla\phi_\alpha)\cdot(\nabla\phi_\alpha) + \tfrac{1}{2}\mu^2\phi^2 + \lambda(\phi_\alpha\phi_\alpha)^2/(8N) \right] \tag{2.4}$$

and the commutation relations

$$[\phi_\alpha(x,t),\phi_\beta(y,t)] = i\delta^3(x-y)\delta_{\alpha\beta} , \tag{2.5}$$

we have the operator equations of motion

$$\dot{\phi}_\alpha = \pi_\alpha , \tag{2.6}$$

$$\dot{\pi}_\alpha = \nabla^2\phi_\alpha - \mu^2\phi_\alpha - \lambda\phi_\alpha\phi_\lambda\phi_\lambda/(2N) . \tag{2.7}$$

The exact equations of motion we obtain for G, D, and K are then (assuming translational invariance of the initial state)

$$\dot{G}(x-y,t) = 2D(x-y,t) - i\delta(x-y) , \tag{2.8}$$

$$\dot{D}(x-y,t)\delta_{\alpha\beta} = \delta_{\alpha\beta}K(x-y,t) + \langle \phi_\alpha(x,t)[\nabla^2\phi_\beta(y,t) - \mu^2\phi_\beta(y) - \lambda\phi_\beta(y)\phi_\lambda(y)\phi_\lambda(y)/(2N)] \rangle , \tag{2.9}$$

$$\dot{K}(x-y,t)\delta_{\alpha\beta} = \langle [\nabla^2\phi_\alpha(y,t) - \mu^2\phi_\alpha(y) - \lambda\phi_\alpha(y)\phi_\lambda(y)\phi_\lambda(y)/(2N)]\pi_\beta \rangle + \langle \pi_\alpha[\nabla^2\phi_\beta(y,t) - \mu^2\phi_\beta(y) - \lambda\phi_\beta(y)\phi_\lambda(y)\phi_\lambda(y)/(2N)] \rangle . \tag{2.10}$$

At large N when $\langle \phi_\alpha \rangle = 0$ we have the factorization

$$\langle \phi_\alpha(x,t)\phi_\beta(y,t)\phi_\lambda(y,t)\phi_\lambda(y,t) \rangle = G_{\alpha\beta}(x-y,t)G_{\lambda\lambda}(0,t) + 2G_{\alpha\lambda}(x-y,t)G_{\beta\lambda}(0,t) = \delta_{\alpha\beta}(N+2)G(x-y,t)G(0,t) , \tag{2.11}$$

$$\langle \phi_\alpha(x,t)\phi_\lambda(x,t)\phi_\lambda(x,t)\pi_\beta(y,t) \rangle = \delta_{\alpha\beta}(N+2)G(0,t)D(x-y,t) . \tag{2.12}$$

The terms discarded are of order $1/N$ or smaller as $N\to\infty$. If we have spatial translational invariance we can introduce the Fourier transforms for the various functions via

$$f(k,t) = \int d^3x\, e^{-ik(x-x')} f(x-x',t) .$$

We obtain

$$\dot{G}(k,t) = 2D(k,t) - i , \tag{2.13}$$

$$\dot{D}(k,t)=K(k,t)-\Gamma G(k,t) , \tag{2.14}$$

$$\dot{K}(k,t)=-2\Gamma D(k,t)+i\Gamma , \tag{2.15}$$

where Γ is given by

$$\Gamma=k^2+m^2(t)=k^2+\mu^2+\tfrac{1}{2}\lambda\int[dk]G(k,t)=\omega_k{}^2(t)$$
$$=\omega_k{}^2+\tfrac{1}{2}\lambda\int[dk]G(k,t) \tag{2.16}$$

and $[dk]=d^3\mathbf{k}/(2\pi)^3$.

We emphasize that, in general, at large N there are three independent functions G, D, and K which require three independent pieces of initial data at $t=0$. As we shall show below, at $t=0$ the quantities G, D, and K can be related to the number density $n(k)$ and the real and imaginary part of the correlated-pair density $F(k)$ as

$$G(k,0)=[1+2n(k)+2\,\mathrm{Re}F(k)]/[2\omega_k(0)] ,$$
$$D(k,0)=i/2+2\,\mathrm{Im}F(k) , \tag{2.17}$$
$$K(k,0)=\tfrac{1}{2}\omega_k(0)[1+2n(k)+2\,\mathrm{Re}F(k)] .$$

If at $t=0$ we assume that one has a thermal distribution, then

$$n(k)=(e^{\beta\omega(k)}-1)^{-1},\quad F(k)=0 . \tag{2.18}$$

This was the case discussed by Mazenko[9] in a curved background geometry. In flat space, when $F(k)=0$, everything becomes time independent (i.e., a thermal distribution stays thermal). For an initial thermal distribution, one can prove that renormalization proceeds exactly as for the vacuum state using, for example, path-integral methods recently reviewed and elaborated upon in detail by Semenoff and Weiss.[15] In this paper we would like to consider arbitrary initial $n(k)$ and $F(k)$ which are consistent with the physical requirements that the initial state have a finite number of particles and correlated pairs per unit volume.

We now would like to study the same theory in the Schrödinger picture. To do this we extend our previous analysis of the time-dependent Hartree-Fock approximation in the Schrödinger picture to the case of ϕ^4 field theory with O(N) symmetry. In the Schrödinger picture the abstract quantum-mechanical state $|\psi(t)\rangle$ is described by a wave functional over the field configuration $\phi(x)$ at a given time t (the ϕ representation):

$$|\psi(t)\rangle\Longrightarrow\Psi(\phi,t)=\langle\Phi|\psi\rangle . \tag{2.19}$$

This is the analog of

$$\Psi(x,t)=\langle x|\Psi\rangle ,$$

for the Schrödinger wave function of a one-dimensional quantum-mechanical system.

The action of the operator $\Phi_\alpha(x)$ on $|\psi(t)\rangle$ is realized by multiplying $\Psi(\phi,t)$ by $\phi_\alpha(x)$ ($\alpha=1,2,\ldots,N$):

$$\Phi_\alpha(x)|\psi(t)\rangle\Longrightarrow\phi_\alpha(x)\Psi(\phi,t) \tag{2.20}$$

and the action of the canonical momentum $\pi_\alpha(x)$ is realized by functional differentiation:

$$\pi_\alpha(x)|\psi(t)\rangle\Longrightarrow-i\frac{\delta}{\delta\phi_\alpha(x)}\Psi(\phi,t) . \tag{2.21}$$

The inner product is defined by functional integration:

$$1=\int d\phi\,|\Phi\rangle\langle\Phi| ,$$
$$\langle\psi_1(t)|\psi_2(t)\rangle=\int d\phi\,\Psi_1^*(\phi,t)\Psi_2(\phi,t) . \tag{2.22}$$

The expectations of functions of operators are determined by

$$\langle\psi(t)|A(\phi,\pi)|\psi(t)\rangle$$
$$=\int d\phi\,\Psi^*(\phi,t)A(\phi,-i\delta/\delta\phi)\Psi(\phi,t) . \tag{2.23}$$

In order to make variational approximations to the functional Schrödinger equation, one needs a variational principle from which the Schrödinger equation can be derived. This principle is due to Dirac.[7] The exact variational principle starts with the action

$$S=\int_{t_1}^{t_2}L(\Psi,\Psi^*)dt , \tag{2.24}$$

where $L(\Psi,\Psi^*)$ is given by

$$L(\Psi,\Psi^*)=\langle\psi|i\partial/\partial t-H|\psi\rangle/\langle\psi|\psi\rangle . \tag{2.25}$$

The variational principle is that $\delta S=0$, which leads to the exact functional Schrödinger equation for this system in the ϕ representation:

$$i\partial\Psi(\phi,t)/\partial t=H\Psi(\phi,t)$$
$$=\int d^3x\{-\tfrac{1}{2}\delta^2/[\delta\phi_\alpha(x)]^2+\tfrac{1}{2}(\nabla\phi_\alpha\nabla\phi_\alpha)$$
$$+V(\phi)\}\Psi(\phi,t) . \tag{2.26}$$

In order to obtain tractable approximation schemes in quantum field theory in the Schrödinger picture one restricts the variation of Ψ to a subspace of the Hilbert space. The easiest way of doing this is by parametrizing a trial wave function, which we denote by Ψ_V. The TDHF approximation is a specific choice of a trial variational wave functional Ψ_V which is a Gaussian with variational parameters G and Σ related to the real and imaginary parts of the width of the Gaussian, and $\bar{\phi}$ and $\bar{\pi}$ which are the expectation values of the operators ϕ and π in the trial wave functional. Of course, in the variational method, one is not restricted to such a simple ansatz and by increasing the number of variational parameters cleverly one expects to get a better approximation to the field theory. Specifically, in TDHF one assumes, for O(N) ϕ^4 field theory,

$$\Psi(\phi,t)_V=\mathcal{N}\exp\Big[-\Big\{\int_{x,y}[\phi_\alpha(x)-\bar{\phi}_\alpha(x,t)][G^{-1}{}_{\alpha\beta}(x,y,t)/4-i\Sigma_{\alpha\beta}(x,y,t)][\phi_\beta(y)-\bar{\phi}_\beta(y,t)]$$
$$+i\int_x\bar{\pi}_\alpha(x,t)[\phi_\alpha(x)-\bar{\phi}_\alpha(x,t)]\Big\}\Big] , \tag{2.27}$$

where $\mathcal{N}$ is a normalization factor. The four variational functions G, Σ, $\bar{\phi}$, and $\bar{\pi}$ have the following significance:

$$\langle \phi_\alpha(x) \rangle_V = \bar{\phi}_\alpha(x,t) , \tag{2.28}$$

$$\langle -i\delta/\delta\phi_\alpha(x) \rangle_V = \bar{\pi}_\alpha(x,t) , \tag{2.29}$$

$$\langle \phi_\alpha(x)\phi_\beta(y) \rangle_V = \bar{\phi}_\alpha(x,t)\bar{\phi}_\beta(y,t) + G_{\alpha\beta}(x,y,t) , \tag{2.30}$$

$$\langle i\partial/\partial t \rangle_V = \int_x \bar{\pi}_\alpha(x,t)\partial\bar{\phi}_\alpha(x,t)/\partial t - \int_{x,y} \dot{\Sigma}_{\alpha\beta}(x,y,t)G_{\beta\alpha}(x,y,t) , \tag{2.31}$$

where the expectation values are determined as in (2.23) using the trial variational wave functional.

Thus, Ψ_V is a Gaussian centered at $\bar{\phi}$ with the real part of the width given by G. The conjugate momentum of $\bar{\phi}$ is $\bar{\pi}$, and Σ plays the role of the conjugate momentum of G.

Defining the effective action in the trial state as

$$\Gamma_{\text{eff}}(\bar{\phi},\bar{\pi},G,\Sigma) = \int dt_V \langle \psi(t) \mid (i\partial/\partial t - H) \mid \psi(t) \rangle_V , \tag{2.32}$$

we obtain

$$\Gamma_{\text{eff}} = \int dt \left[\int_x \bar{\pi}_\alpha(x,t)\partial\bar{\phi}_\alpha(x,t)/\partial t + \int_{x,y} \Sigma_{\alpha\beta}(x,y,t)\partial G_{\beta\alpha}(x,y,t)/\partial t - H_{\text{eff}} \right] , \tag{2.33}$$

where

$$H_{\text{eff}} = \int_x [\tfrac{1}{2}\bar{\pi}_\alpha\bar{\pi}_\alpha + \tfrac{1}{2}\nabla\bar{\phi}_\alpha\nabla\bar{\phi}_\alpha + \mathrm{Tr}\mathbf{G}^{-1}/8 + 2\,\mathrm{Tr}\boldsymbol{\Sigma}\mathbf{G}\boldsymbol{\Sigma} - \tfrac{1}{2}\nabla_x{}^2 G_{\alpha\alpha}(x,y,t) \mid_{x=y} + \langle V \rangle_V] .$$

The trace is over coordinate space as well as internal space and we have used boldface to indicate the matrix structure. We see that when we approximate the true wave function by the variational one, the approximate action (which we call the effective action because it is related to the large-N effective action of Cornwall, Jackiw, and Tomboulis[10]) has the form of a classical action for the variational fields $\bar{\phi}$ and G. When V is at most quartic we can use the method of the generating functional[6] to show that

$$\langle V \rangle = V(\bar{\phi}) + \tfrac{1}{2}G_{\alpha\beta}V_{\alpha\beta} + \tfrac{1}{24}V_{\alpha\beta\gamma\delta}(G_{\alpha\beta}G_{\gamma\delta} + G_{\alpha\gamma}G_{\beta\delta} + G_{\alpha\delta}G_{\beta\gamma}) , \tag{2.34}$$

where

$$V_{\alpha\beta} = \delta^2 V/\delta\phi_\alpha\delta\phi_\beta \mid_{\phi=\bar{\phi}} , \quad \text{etc.}$$

If we choose

$$V[\phi_\alpha] = \tfrac{1}{2}\mu^2\phi_\alpha\phi^\alpha + \lambda\phi_\alpha\phi^\alpha\phi_\beta\phi^\beta/(8N) , \tag{2.35}$$

we obtain

$$\langle V(\phi) \rangle = V(\bar{\phi}) + \tfrac{1}{2}\mu^2\mathrm{Tr}\mathbf{G} + \lambda G_{\alpha\beta}\bar{\phi}_\alpha\bar{\phi}_\beta/(2N) + \lambda(\mathrm{Tr}\mathbf{G})\bar{\phi}^2/(4N) + \lambda[(\mathrm{Tr}\mathbf{G})^2 + 2G_{\alpha\beta}G_{\alpha\beta}]/(8N) . \tag{2.36}$$

To obtain the large-N limit of this equation we write

$$G_{\alpha\beta} = \delta_{\alpha\beta}G + \Delta G_{\alpha\beta} \tag{2.37}$$

and rescale the field ϕ

$$\bar{\phi}_\alpha = N^{1/2}\tilde{\phi}_\alpha . \tag{2.38}$$

One can show by a detailed study of the large-N approximation that the correction term

$$\Delta G_{\alpha\beta} \propto \bar{\phi}_\alpha\bar{\phi}_\beta/(\mid\bar{\phi}_\alpha\mid \mid\bar{\phi}_\beta\mid)$$

and that this term does not contribute to the leading large-N behavior. After this rescaling and breakup of $G_{\alpha\beta}$ we plug into (2.36) and keep the leading behavior of $\langle V \rangle$ at large N and obtain

$$\langle V \rangle \Longrightarrow NV(\bar{\phi}) = N(\mu^2\tilde{\phi}^2/2 + \lambda\tilde{\phi}^4/8 + \mu^2G/2 + \lambda G^2/8 + \lambda G\tilde{\phi}^2/4) , \tag{2.39}$$

where $\tilde{\phi}^2 = \phi_\alpha\phi^\alpha$, etc.

If instead we just set $N=1$ in (2.34) we recover[1] the TDHF approximation for single-component field theory:

$$\langle V \rangle \|_{N=1} = \mu^2\phi^2/2 + \lambda\phi^4/8 + \mu^2G/2 + 3\lambda G^2/8 + 3\lambda G\phi^2/4 . \tag{2.40}$$

When $\bar{\phi}=0$, the $N=1$ TDHF approximation and the general large-N expansion give identical equations for G with a rescaling $\lambda/(8N) \rightarrow \lambda/24$. Since, at large N, the Hartree-Fock approximation becomes exact, we expect Eq. (2.39) to be an excellent approximation to large-N dynamics. We will therefore use the result of (2.39) with the rescaled fields $\tilde{\phi}$ as the $\langle V \rangle$ in this paper. Since N is in front of the entire action after rescaling we will suppress the N (and also the tilde in what follows) and write

$$\langle V \rangle = \mu^2\phi^2/2 + \lambda\phi^4/8 + \mu^2G/2 + \lambda G^2/8 + \lambda G\phi^2/4 . \tag{2.41}$$

The effective potential $V_{\text{eff}}[\phi,G]$ is defined as H_{eff} for constant ϕ and is

$$V_{\text{eff}}[\phi,G] = \langle V \rangle + \tfrac{1}{8}\mathrm{Tr}G^{-1} - \tfrac{1}{2}\nabla_x{}^2 G(x,y,t) \mid_{x=y} . \tag{2.42}$$

We will discuss the renormalization of the effective potential in the Appendix.

Having evaluated $\langle V \rangle$ we can now apply the variational principle $\delta\Gamma=0$. This variation leads to

$$\dot{\pi}(x,t) = \nabla^2\phi(x,t) - \partial\langle V \rangle/\partial\phi , \tag{2.43}$$

$$\dot{\phi}(x,t) = \pi(x,t) , \tag{2.44}$$

$$\dot{G}(x,y,t) = 2\left[\int_z [G(x,z,t)\Sigma(z,y,t) + \Sigma(x,z,t)G(z,y,t)] \right] , \tag{2.45}$$

$$\dot{\Sigma}(x,y,t) = -2\int_z \Sigma(x,z,t)\Sigma(z,y,t) + \tfrac{1}{8}G^{-2}(x,y,t) + (\tfrac{1}{2}\nabla_x{}^2 - \partial\langle V \rangle/\partial G)\delta^3(x-y) , \tag{2.46}$$

where $\langle V \rangle$ is given by (2.41).

Introducing the Fourier transforms at fixed time as before, we obtain (for $\bar{\phi}=0$)

$$\dot{G}(k,t)=4\Sigma(k,t)G(k,t) , \tag{2.47}$$

$$\dot{\Sigma}(k,t)=\tfrac{1}{8}[G(k,t)]^{-2}-2\Sigma^2(k,t)-\Gamma/2 , \tag{2.48}$$

or, in second-order form,

$$2\ddot{G}(k,t)G(k,t)-\dot{G}^2(k,t)+4\Gamma G^2(k,t)-1=0 , \tag{2.49}$$

where Γ is given by (2.16) above.

To compare with the general results of the large-N expansion we need to connect Σ and G to the variables defined previously. To do this we need to evaluate D and K in terms of G and Σ using the wave function (2.27). We find

$$\langle \phi_\alpha \phi_\beta \rangle_{\rm FT}=\delta_{\alpha\beta}G(k,t) , \tag{2.50}$$

$$\langle \phi_\alpha \pi_\beta \rangle_{\rm FT}=\tfrac{1}{2}i\delta_{\alpha\beta}+2G(k,t)\Sigma(k,t)\delta_{\alpha\beta} =D(k,t)\delta_{\alpha\beta} , \tag{2.51}$$

$$\langle \pi_\alpha \phi_\beta \rangle_{\rm FT}=-\tfrac{1}{2}i\delta_{\alpha\beta}+2G(k,t)\Sigma(k,t)\delta_{\alpha\beta} , \tag{2.52}$$

$$\langle \pi_\alpha \pi_\beta \rangle_{\rm FT}=\delta_{\alpha\beta}[G^{-1}(k,t)/4+4\Sigma(kt)G(kt)\Sigma(kt)] =K(k,t)\delta_{\alpha\beta} , \tag{2.53}$$

where the subscript FT stands for Fourier transformed. Using (2.50)–(2.53) we find that (2.13)–(2.15) are automatically satisfied as long as G and Σ obey (2.47) and (2.48). Thus, for Gaussian initial data, only *two* pieces of initial data, $G(0)$ and $\dot{G}(0)$, are needed instead of the more general case of arbitrary initial data where *three* pieces of initial data are needed [that is, in general, $\ddot{G}(0)$ is an independent piece of data]. We will show later that, in general, one can specify the initial-number density and real and imaginary parts of the density of pairs at $t=0$ as independent variables if we use the Heisenberg-equation approach. In the TDHF there is a constraint between these quantities. When one has initial data which comes from a Gaussian wave functional, the (in general) three independent functions necessary to specify the time development of the large-N approximation satisfy one relationship.

We have thus shown that, in the context of Gaussian initial data, the large-N approximation to the TDHF variational approximation is identical to the exact large-N approximation to the two-point function although it is not as general as the full large-N approximation. This result was found for static fields by Cornwall, Jackiw, and Tomboulis.[10] This results from the fact that at large N an initial Gaussian wave functional remains Gaussian as time evolves.

We would also like to study initial conditions where $\bar{\phi}$ is nonzero. For the case of spatially homogeneous configurations, $\bar{\phi}$ is a function only of time: $\bar{\phi}(t)=\langle \phi(x,t) \rangle$. At large N the equations we obtain using (2.41) and (2.43) are

$$\ddot{\bar{\phi}}(t)+\left[\mu^2+\tfrac{1}{2}\lambda\phi^2+\tfrac{1}{2}\lambda\int [dk]G(k,t)\right]\phi=0 , \tag{2.54}$$

$$2\ddot{G}(k,t)G(k,t)-\dot{G}^2(k,t)+4\Gamma G^2(k,t)-1=0 , \tag{2.55}$$

where the only difference in (2.55) is that now

$$\Gamma=k^2+\mu^2+\tfrac{1}{2}\lambda\phi(t)^2+\tfrac{1}{2}\lambda\int [dk]G(k,t) , \tag{2.56}$$

where $[dk]=d^3k/(2\pi)^3$.

III. INITIAL-VALUE PROBLEM WHEN $\lambda=0$

To understand the divergences in Γ, it is useful to solve for $G(k,t)$ in the case $\lambda=0$. Because of our interest in the new inflationary universe we will allow μ^2 to be positive or negative. First let us study the free field case in the Heisenberg picture which is more familiar to field theorists. We will use a parametrization which will generalize simply to the interacting case.

By translational invariance,

$$\langle i \mid \phi_\alpha(x,t)\phi_\beta(x',t') \mid i \rangle \equiv W_0(x-x',t,t')\delta_{\alpha\beta} . \tag{3.1}$$

In general (even in the interacting case) we have the following Fourier decomposition of ϕ at time t:

$$\phi_\alpha(x,t)=\int [dk][a_{k\alpha}f_k(t)e^{i(kx)} +a^\dagger_{k\alpha}f^*_k(t)e^{-i(kx)}] , \tag{3.2}$$

where the operators a and $a^\dagger$ obey the commutation relations

$$[a_{k\alpha},a^\dagger_{k'\beta}]=\delta^3(k-k')\delta_{\alpha\beta} \tag{3.3}$$

and the functions of $f_k(t)$ obey

$$f_k\dot{f}^*_k-f^*_k\dot{f}_k=i . \tag{3.4}$$

In free field theory, because of the free field equation, we may choose

$$f_k=e^{-i\omega_k t}/(2\omega_k)^{1/2} . \tag{3.5}$$

Spatial homogeneity then requires

$$\begin{aligned} &\langle i \mid a^\dagger_{k\alpha}a_{k'\beta} \mid i \rangle = n(k)(2\pi)^3\delta^3(k-k')\delta_{\alpha\beta} , \\ &\langle i \mid a_{k\alpha}a^\dagger_{k'\beta} \mid i \rangle = [n(k)+1](2\pi)^3\delta^3(k-k')\delta_{\alpha\beta} , \\ &\langle i \mid a_{k\alpha}a_{k'\beta} \mid i \rangle = F(k)(2\pi)^3\delta^3(k+k')\delta_{\alpha\beta} , \\ &\langle i \mid a^\dagger_{k\alpha}a^\dagger_{k'\beta} \mid i \rangle = F^*(k)(2\pi)^3\delta^3(k+k')\delta_{\alpha\beta} , \end{aligned} \tag{3.6}$$

where $n(k)$ is the phase-space particle number density in the initial state $\mid i \rangle$ and $F(k)$ is the pair density. After a little algebra we obtain, for the free-field-theory two-time Green's function,

$$W(k,t,t')=(2\omega_k)^{-1}\{[1+2n(k)]\cos\omega_k(t-t')-\tfrac{1}{2}i\sin\omega_k(t-t') +2\,\mathrm{Re}F(k)\cos\omega_k(t+t')+2\,\mathrm{Im}F(k)\sin\omega_k(t+t')\} , \tag{3.7}$$

so that the free equal-time two-point function $G(k,t)$ can be written as

$$G(k,t)=(2\omega_k)^{-1}\{[1+2n(k)]+2\,\mathrm{Re}F(k)\cos 2\omega_k t +2\,\mathrm{Im}F(k)\sin 2\omega_k t\}\ . \tag{3.8}$$

The initial data therefore has the following significance:

$$\begin{aligned} &G(k,0)=(2\omega_k)^{-1}[1+2n(k)+2\,\mathrm{Re}F(k)]\ ,\\ &\dot{G}(k,0)=2\,\mathrm{Im}F(k)\ ,\\ &\ddot{G}(k,0)=-4\omega_k\,\mathrm{Re}F(k)\ . \end{aligned} \tag{3.9}$$

When the initial wave functional is Gaussian, G and $\dot{G}$ are related to the real and imaginary parts of the width of the Gaussian so that when we specify the shape of the initial wave functional we automatically determine the number and correlated pair densities at $t=0$. From physical considerations we require that the average number of particles and pairs per unit volume is finite, which places integrability conditions on n and F:

$$\int n(k)[dk]<\infty,\quad \int F(k)[dk]<\infty\ . \tag{3.10}$$

This clearly places integrability conditions on the real and imaginary parts of the width of an initial Gaussian wave functional.

Now let us turn to the same problem in the Schrödinger picture when our initial data is a Gaussian wave functional that is not the vacuum wave functional.

In free field theory the conserved Hamiltonian density in momentum space is

$$H(k)=\dot{G}^2/(8G)+G^{-1}/8+(k^2+\mu^2)G/2\ . \tag{3.11}$$

Since energy is conserved, $\dot{G}=0$ determines the turning points of the G motion which oscillates between two values of G.

In free field theory the *vacuum* Green's function is time independent and is found by setting $\dot{G}=\ddot{G}=0$ in (2.49). We have

$$G_0(k)=1/(2\sqrt{\Gamma})=(2\omega_k)^{-1}=\tfrac{1}{2}(k^2+\mu^2)^{-1/2}\ . \tag{3.12}$$

More generally, at $t=0$,

$$G(k,t=0)=f_1(k)=(2\omega_k)^{-1}[1+2n(k)+2\,\mathrm{Re}F(k)]\ , \tag{3.13}$$

where $n(k)$ is the particle number density and $F(k)$ is the density of pairs.

In general, the solution for G can be parametrized as in (3.8):

$$G(k,t)=(2\omega_k)^{-1}\{[1+2n(k)]+2\,\mathrm{Re}F(k)\cos 2\omega_k t +2\,\mathrm{Im}F(k)\sin 2\omega_k t\}\ . \tag{3.14}$$

If $\mu^2<0$, then for $k^2<-\mu^2$ the cosine and sine functions become hyperbolic functions and ω_k is replaced by $|\omega_k|$.

Because of (2.49), when the initial data is a Gaussian, recall that the three initial data must satisfy one relationship. If we parametrize

$$G(k,0)=(1+\Delta)/(2\omega_k),\quad \Delta/2=n+\mathrm{Re}F\ ,$$

then

$$\begin{aligned} &n(k)=(\Delta^2+4\,\mathrm{Im}F^2)/[4(1+\Delta)]\ ,\\ &\mathrm{Re}F=(2\Delta+\Delta^2-4\,\mathrm{Im}F^2)/[4(1+\Delta)]\ . \end{aligned} \tag{3.15}$$

We see that if Δ is the leading correction to the vacuum contribution to the ultraviolet behavior of $G(k,t)$, then, if Δ is integrable, $n(k)$ which goes like Δ^2 is automatically integrable. So in the TDHF approximation one needs for integrability only to specify that the pair density is integrable. The more general parametrization of the Heisenberg picture allows one to study arbitrary initial data including initially thermal or near-equilibrium configurations at $t=0$. These more general initial states do *not* correspond to Gaussian wave functionals.

IV. RENORMALIZATION IN THE HEISENBERG PICTURE

The integral in the definition of Γ [Eq. (2.16)] has both quadratic and logarithmic divergences which are related to mass and coupling-constant renormalization. When λ is different from zero, then Eqs. (2.47)–(2.49) are meaningless because of these divergences. For the static problem (i.e., the ground state), the problem of renormalization at large N or in mean-field theory has been solved in several different contexts before.[5–8] In the Appendix, for completion, we show how to renormalize the static effective potential in the TDHF approximation.

In the time-dependent problem there are some new physical conditions that have to be considered. Firstly, one would like to parametrize the initial data at $t=0$ so that it is independent of the renormalization procedure. This can be done if we choose as our renormalized mass parameter the effective mass $m^2(0)$ defined by (2.16). Secondly, from physical requirements discussed earlier we will find that in order to make these equations finite, the initial data itself will have to be restricted to those initial data for which the average number of particles and correlated particle pairs unit volume is finite. In the TDHF large-N approximation, the equations we need to render finite are Eqs. (2.52)–(2.54).

Before analyzing these equations, which look quite noncovariant, let us first discuss renormalization in the Heisenberg picture at large N. Our approach to renormalization is to use a generalized WKB method, taking advantage of the fact that, in the large-N approximation, the equation for G is similar to the well studied time-independent Schrödinger equation of quantum mechanics. We will show that the ultraviolet properties of the Fourier transform of the Wightman function are determined by the WKB approximation to the Wightman function and, thus, an all order in λ renormalization can be explicitly done. Physically, the motivation for this approach is that the ultraviolet divergences are quite independent of the time variation and initial data, since they are present even in the vacuum state. Therefore, the first few orders of the WKB expansion contain all the divergences and lead to a natural technique for their removal. This WKB approach is related to other adiabatic methods found in the literature.[16]

At large N the covariant field equation is

$$[\Box+m^2(t)]\phi(x,t)=0 , \tag{4.1}$$

where $m^2(t)$ is given by $\mu^2+\frac{1}{2}\lambda\int[dk]G(k,t)$. Using the general Fourier decomposition (3.2) we obtain

$$[\partial_0{}^2+k^2+m^2(t)]f_k(t)=0 . \tag{4.2}$$

Since this equation resembles the time-independent Schrödinger equation [here time takes the place of space and the "potential" is $\omega^2(t)=k^2+m^2(t)$], we make the WKB ansatz for the wave function $f(t)$:

$$f_k(t)=(1/\sqrt{2\Omega})\exp[-iy(t)] , \tag{4.3}$$

where

$$y(t)=\int_0^t\Omega(x)dx \tag{4.4}$$

and we suppress the k label on Ω and ω.

We then obtain the *exact* equation for Ω:

$$\ddot{\Omega}/(2\Omega)-\tfrac{3}{4}(\dot{\Omega}/\Omega)^2+\Omega^2=\omega^2(t) . \tag{4.5}$$

Using this parametrization of f_k we obtain, for $W(k,t,t')$,

$$W(k,t,t')=\tfrac{1}{2}\Omega(t)^{-1/2}\Omega(t')^{-1/2}([1+2n(k)]\cos\{[y(t)-y(t')]\}-\tfrac{1}{2}i\sin\{[y(t)-y(t')]\}$$
$$+2\,\mathrm{Re}F(k)\cos\{[y(t)+y(t')]\}+2\,\mathrm{Im}F(k)\sin\{[y(t)+y(t')]\}) . \tag{4.6}$$

At equal times we obtain $W(k,t=t')=G(k,t)$,

$$G(k,t)=[2\Omega(t)]^{-1}\{[1+2n(k)]+2\,\mathrm{Re}F(k)\cos[2y(t)]$$
$$+2\,\mathrm{Im}F(k)\sin[2y(t)]\} . \tag{4.7}$$

This form is quite nice because it has the same structure as free field theory except the angle y replaces ωt, and the constant frequency ω_k found in free field theory in the denominator is replaced here by the time-dependent frequency $\Omega(t)$ which obeys a second-order differential equation.

It is convenient to choose, as initial data for Ω,

$$\Omega(0)=\omega(0),\quad \dot{\Omega}(0)=0 . \tag{4.8}$$

This selects the effective mass of the theory at $t=0$ as the natural renormalized mass parameter and chooses Ω at $t=0$ to be its WKB value. Since we have a general parametrization of the initial data in terms of $n(k)$, $F(k)$, and $\omega(0)$, there is no loss of generality in this choice of initial data for Ω or in the choice (4.3) for f.

The initial data are now identical to free field theory with ω_k replaced by $\omega(0)$, which we take to be a given fixed value. That is, we have, in the interacting case,

$$G(k,0)=[2\omega(0)]^{-1}[1+2n(k)+2\,\mathrm{Re}F(k)] , \tag{4.9a}$$

$$\dot{G}(k,0)=2\,\mathrm{Im}F(k) , \tag{4.9b}$$

$$\ddot{G}(k,0)=-4\omega(0)\mathrm{Re}F(k) . \tag{4.9c}$$

We can specify the initial data in two ways. We may give the value of G and its two derivatives at $t=0$ which for Gaussian initial data is related to specifying the real and imaginary parts of the width of the Gaussian. Or we may specify the number density and correlated pair density functions pertaining to the initial state as well as the effective mass of the state. The natural mass that enters in the initial-data problem is $m^2(0)$ and not $m_R{}^2$, the pole in the covariant vacuum two-point function. These parameters are finitely related, as we will see later. However it would be extremely complicated to discuss the initial-value problem in terms of $m_R{}^2$. Another benefit of using $m^2(0)$ is that all dependence of the initial data on λ is hidden in the implicit dependence of $m^2(0)$ on λ and $m_R{}^2$.

All the divergences in this approach come from the expression for $\omega(t)$:

$$\omega^2(t)=k^2+m^2(t)$$
$$=k^2+\mu^2+\tfrac{1}{2}\lambda\int[dk]G(k,t) , \tag{4.10}$$

the integration over $[dk]=d^3k/(2\pi)^3$ having both quadratic and logarithmic divergences which have to be absorbed by mass and coupling-constant renormalization.

It is fortunate that the large-k behavior of the integral is determined by the WKB approximate Green's function. To see this we can think of solving the equation for $\Omega(t)$ iteratively in terms of higher time derivatives of $\omega(t)$:

$$\Omega^2=\omega^2-\ddot{\omega}/(2\omega)+\tfrac{3}{4}(\dot{\omega}/\omega)^2+\cdots , \tag{4.11}$$

where what is left out has higher derivatives and more powers of ω in the denominator. When we neglect the time derivatives of ω on the right-hand side (RHS) of Eq. (4.11) we obtain the usual WKB approximation. At large k,

$$\dot{\omega}\sim(\dot{m}^2)/k,\quad \ddot{\omega}\sim(\ddot{m}^2)/k . \tag{4.12}$$

Thus, the derivative terms on the RHS of Eq. (4.11) do not affect the large-momentum behavior of Ω, which is solely determined by ω.

Since we have an explicit expression for $G(k,t)$, we can determine the counterterms needed to all orders in λ.

First let us discuss renormalization in the vacuum sector, which is the time-independent solution to the Ω equation, where

$$\Omega^2=\omega^2=k^2+m_R{}^2=k^2+\mu^2+\tfrac{1}{2}\lambda\int[dk]/(2\omega) . \tag{4.13}$$

The renormalized mass is determined by the gap equation

$$m_R{}^2=\mu^2+\tfrac{1}{2}\lambda\int[dk]/[2(k^2+m_R{}^2)^{1/2}] . \tag{4.14}$$

We also need to determine the renormalized coupling constant.

A direct way of determining λ_R is from the scattering amplitude in the large-N approximation, where the sum of bubbles gives

$$\lambda_R = \lambda/[1+\tfrac{1}{2}\lambda\bar{\Sigma}(0)] , \tag{4.15}$$

and

$$\bar{\Sigma}(q)=\frac{1}{(2\pi)^4}\int D_F(q-k)D_F(k)d^4k ,$$
$$\bar{\Sigma}(0)=\int [dk]/(4\omega_k{}^3) , \tag{4.16}$$

and D_F is the usual covariant Feynman propagator with mass m_R, so that $\omega_k{}^2=k^2+m_R{}^2$. If we write λ as $\lambda_R+\delta\tilde{\lambda}$, where $\delta\tilde{\lambda}$ has a power series in λ_R starting with $\lambda_R{}^2$, then in large N we have that this series is totally determined from the geometric series

$$\lambda=\lambda_R/(1-\lambda_R\delta\lambda) , \tag{4.17}$$

where

$$\delta\lambda=\int [dk]/(8\omega_k{}^3) . \tag{4.18}$$

If we write

$$\mu^2=m^2+\delta m^2 ,$$

then

$$\delta m^2=-\tfrac{1}{2}\lambda\int [dk]/(2\omega_k)$$
$$=-\tfrac{1}{2}[\lambda_R/(1-\lambda_R\delta\lambda)]\int [dk]/(2\omega_k) , \tag{4.19}$$

which on expanding in λ_R gives the counterterms needed order by order at large N.

Having summarized what happens in the vacuum sector, let us now look at the initial-value problem for which there is a time-dependent mass. We write this time-dependent mass in terms of counterterms in analogy with the vacuum case. However, now δm^2 and $\delta\lambda$ are unknowns to be determined. We have

$$m^2(t)=m^2+\delta m^2+\tfrac{1}{2}[\lambda_R/(1-\lambda_R\delta\lambda)]\int [dk]G(k,t) . \tag{4.20}$$

If we now choose our finite mass parameter $m^2=m^2(0)$, then we have

$$\delta m^2=-\tfrac{1}{2}[\lambda_R/(1-\lambda_R\delta\lambda)]\int [dk]G(k,0) , \tag{4.21a}$$

$$m^2(t)=m^2(0)+\tfrac{1}{2}\lambda\int [dk][G(k,t)-G(k,0)] \tag{4.21b}$$

$$\equiv m^2(0)+R(t) . \tag{4.21c}$$

The subtraction of $G(k,0)$ from $G(k,t)$ in the integrand removes the quadratic divergence. In terms of $R(t)$ it is easy to investigate the remaining logarithmic divergence in (4.21b) explicitly.

The logarithmic divergence in (4.21b) is coming from the term

$$\tfrac{1}{2}\lambda\int [dk]\{1/[2\omega(t)]-1/[2\omega(0)]\} , \tag{4.22}$$

since $\Omega(0)$ was chosen to be $\omega(0)$ and the large-k behavior of $\Omega(t)$ is determined by $\omega(t)$.

Writing $\omega(t)=[k^2+m^2(t)]^{1/2}$ and simplifying, we obtain that the divergence is

$$\tfrac{1}{2}\lambda[m^2(0)-m^2(t)]\int [dk]/[4\omega^3(0)] . \tag{4.23}$$

We have

$$R(t)=m^2(t)-m^2(0)=\lambda\int [dk][G(k,t)-G(k,0)]/2 . \tag{4.24}$$

So if we set, as in (4.17),

$$\lambda(1-\lambda_R\delta\lambda)=\lambda_R \tag{4.25}$$

with

$$\delta\lambda=\tfrac{1}{2}\int [dk]/[4\omega^3(0)] \tag{4.26}$$

and multiply both sides of (4.24) by $(1-\lambda_R\delta\lambda)$ and simplify, we obtain the *finite* equation

$$R(t)=\tfrac{1}{2}\lambda_R\int [dk]\{G(k,t)-G(k,0)$$
$$+R(t)/[4\omega(0)^3]\} . \tag{4.27}$$

To verify that this is indeed finite, it is sufficient to utilize the lowest-order WKB (or adiabatic) approximation, where $\Omega=\omega$. By substituting (4.9) with $\Omega=\omega$ and (4.10) into (4.27) we may perform the integrations over k explicitly for the $n=F=0$ terms in G. The result is

$$R(t)=\tfrac{1}{2}\lambda_R\int [dk](2\,\mathrm{Re}F(k)\{\cos 2y(t)/[2\omega(t)]-1/[2\omega(0)]\}$$
$$+2n(k)\{1/[2\omega(t)]-1/[2\omega(0)]\}+2\,\mathrm{Im}F(k)\sin 2y(t)/[2\omega(t)])$$
$$+\lambda_R(m^2(t)\{\ln[m^2(t)/m^2(0)]-1\}+m^2(0))/(32\pi^2) . \tag{4.28}$$

This is explicitly finite as long as the number density and density of pairs satisfy Eq. (3.10). However, it is still a difficult equation to work with numerically.

Since $G(k,t)$ depends on $R(t)$ through

$$\omega(t)=[k^2+m^2(0)+R(t)]^{1/2} , \tag{4.29}$$

one can convert the integral equation (4.28) for $R(t)$ into a first-order *differential* equation for $R(t)$ with initial data $R(0)=0$ and a first-order differential equation for the angles $y_k(t)$ which depend on k:

$$y_k(t)=\int_0^t [k^2+m^2(0)+R(x)]^{1/2}dx \tag{4.30}$$

and satisfy

$$dy_k/dt=[k^2+m^2(0)+R(t)]^{1/2},\quad y_k(0)=0\ . \tag{4.31}$$

In this form the equations should be quite easy to solve numerically on the computer because one has converted the integral equation for the self-consistent mass into a set of coupled first-order differential equations for $R(t)$ and $y_k(t)$, which after being solved can be plugged into the explicit expression for $G(k,t)$ [Eq. (4.7)]. The numerical solution of Eqs. (4.7), (4.28), and (4.31) and a comparison with the numerical solution of the exact equation for $G(k,t)$ in two and four dimensions will be presented elsewhere.

The form of G is the same for the exact problem and the WKB approximate solution, except that ω gets replaced by Ω which is itself the solution of a second-order differential equation, and y is given by (4.4). For the exact large-N Green's function we obtain, for $R(t)$,

$$\begin{aligned}R(t)=\tfrac{1}{2}\lambda_R\int[dk]&(2\,\mathrm{Re}F(k)\{\cos 2y(t)/[2\Omega(t)]-1/[2\omega(0)]\}\\&+2n(k)\{1/[2\Omega(t)]-1/[2\omega(0)]\}\\&+2\,\mathrm{Im}F(k)\{\sin 2y(t)/[2\Omega(t)]+1/[2\Omega(t)]-1/[2\omega(0)]+R(t)/[4\omega^3(0)]\})\ .\end{aligned} \tag{4.32}$$

The last three terms can be rewritten using the equations of motion for $\Omega(t)$ as

$$\lambda_R(m^2(t)\{\ln[m^2(t)/m^2(0)]-1\}+m^2(0))/(32\pi^2)+\tfrac{1}{2}\lambda_R\left\{\int[dk][\ddot{\Omega}/(2\Omega)-\tfrac{3}{4}(\dot{\Omega}/\Omega)^2]/\{2\Omega(t)\omega(t)[\Omega(t)+\omega(t)]\}\right\}, \tag{4.33}$$

which is explicitly finite. Thus, we verify that the analysis of divergences in the WKB approximation suffice to renormalize the large-N approximation. Summing things up, we obtain in the Heisenberg picture that at large N the renormalized equations are

$$\begin{aligned}G(k,t)=&[2\Omega(t)]^{-1}\\&\times\{[1+2n(k)]+[2\,\mathrm{Re}F(k)]\cos 2y(t)\\&+[2\,\mathrm{Im}F(k)]\sin 2y(t)\}\ ,\end{aligned} \tag{4.34}$$

where Ω satisfies the second-order differential equation:

$$\ddot{\Omega}/(2\Omega)-\tfrac{3}{4}(\dot{\Omega}/\Omega)^2+\Omega^2=\omega^2(t)=k^2+m^2(0)+R(t) \tag{4.35}$$

with boundary conditions $\Omega(0)=\omega(0)$, $\dot{\Omega}(0)=0$; y satisfies the first-order differential equation

$$dy/dt=\Omega(t),\quad y(0)=0\ ; \tag{4.36}$$

$R(t)$ satisfies the finite self-consistency condition (4.32) or equivalently,

$$\begin{aligned}R(t)=\tfrac{1}{2}\lambda_R\int[dk]\{&G(k,t)-G(k,0)\\&+R(t)/[4\omega(0)^3]\}\ ,\end{aligned} \tag{4.37}$$

which can also be converted to a first-order differential equation.

After doing this renormalization of R with $\langle\phi\rangle=0$, it is easy to look at what happens when $\langle\phi\rangle$ is not zero. For simplicity we consider only the case where the expectation value of ϕ is independent of x. In order for the expectation value of ϕ to be independent of x,

$$\langle i\mid\phi(x,t)\mid i\rangle=\bar{\phi}(t)\ , \tag{4.38}$$

one needs after Fourier decomposition of ϕ that

$$\langle i\mid a_k\mid i\rangle=\tfrac{1}{2}C\delta^3(k)/(2\pi)^3\ , \tag{4.39}$$

with C a constant independent of k. Thus, for nonzero $\bar{\phi}$ one needs C as well as $n(k)$ and $F(k)$ to specify the initial state. In terms of C and f_0 (the $k=0$ mode of f_k) one has

$$\bar{\phi}(t)=Cf_0(t)/2+C^*f_0^*(t)/2\ . \tag{4.40}$$

We can rewrite this as

$$\phi(t)=[2\Omega_0(t)]^{-1/2}(\mathrm{Re}C\cos y_0-\mathrm{Im}C\sin y_0)\ ; \tag{4.41}$$

here $\Omega_0(t)$ is determined from $\omega_0(t)=[m^2(t)]^{1/2}$ using Eq. (4.5).

The Heisenberg parameter C is related to the initial data as

$$\bar{\phi}(0)=\mathrm{Re}C/[2\omega_0(0)]^{1/2}\ , \tag{4.42}$$

$$d\bar{\phi}(0)/dt=-\mathrm{Im}C[\omega_0(0)/2]^{1/2}\ . \tag{4.43}$$

The change that takes place in the equation of motion for ϕ when $\langle\phi\rangle$ is not zero is that the definition of $R(t)$ changes. One has now at large N that

$$m^2(t)=\mu^2+\tfrac{1}{2}\lambda\phi(t)^2+\tfrac{1}{2}\lambda\int[dk]G(k,t)\ . \tag{4.44}$$

Thus, we again define $R(t)$ as $m^2(t)-m^2(0)$ and we obtain

$$\begin{aligned}m^2(t)-m^2(0)=&R(t)\\&=\tfrac{1}{2}\lambda[\phi^2(t)-\phi^2(0)]\\&\quad+\tfrac{1}{2}\lambda\int[dk][G(k,t)-G(k,0)]\ .\end{aligned} \tag{4.45}$$

Again the remaining logarithmic divergence is removed by the coupling constant renormalization which now must be performed with the new mass parameter $m^2(0)$ determined from (4.44). With this implicit change in the definition of $\delta\lambda$ we obtain the renormalized field equation for $\langle\phi\rangle$ [and $f_0(t)$]:

$$\ddot{\phi}(t)+[m^2(0)+R(t)]\phi=0 , \tag{4.46}$$

where the renormalized equation for $R(t)$ is now

$$R(t)=\tfrac{1}{2}\lambda_R[\phi^2(t)-\phi^2(0)] +\tfrac{1}{2}\lambda_R\int[dk]\{G(k,t)-G(k,0) +R(t)/[4\omega^3(0)]\} . \tag{4.47}$$

V. RENORMALIZATION OF THE TDHF EQUATIONS

Now let us turn our attention to the TDHF equations obtained at large N and their renormalization. We have that the equations are

$$2G\ddot{G}-\dot{G}^2+4\Gamma G^2-1=0 , \tag{5.1}$$

$$\Gamma=k^2+m^2(t)=k^2+\mu^2+\tfrac{1}{2}\lambda\int[dk]G(k,t) . \tag{5.2}$$

In the time-independent case we see that

$$G_0(k)=1/(2\sqrt{\Gamma}) . \tag{5.3}$$

Thus, the usual renormalized mass is given by the same gap equation as before:

$$m_R^2=\mu^2+\tfrac{1}{2}\lambda\int[dk]G_0(k) , \tag{5.4}$$

$$G_0=1/[2(k^2+m_R^2)^{1/2}] . \tag{5.5}$$

We expect that the coupling and mass renormalization will be the same as we discovered in the Heisenberg picture, but now we will have to study these nonlinear equations in a perturbation expansion in λ to verify that this is so.

To do this we again expect the counterterms for λ to be a geometric series

$$\lambda_R=\lambda/(1+\lambda\delta\lambda) , \tag{5.6}$$

where $\delta\lambda$ is logarithmically divergent, and we again choose our renormalized mass parameter to be, to $m^2(0)$,

$$m^2(0)=\mu^2+\tfrac{1}{2}\lambda\int[dk]G(k,0) , \tag{5.7}$$

where $[dk]=d^3k/(2\pi)^3$.

To identify the counterterms it is only necessary to calculate to order λ^2. We rewrite (5.2) as

$$\Gamma=k^2+m^2+\delta m^2+(\lambda_R+\tilde{\delta}\lambda)\int[dk]G(k,t)/2 . \tag{5.8}$$

We assume

$$\delta m^2=\lambda_R\delta m_1^2+\lambda_R^2\delta m_2^2+\cdots , \quad \lambda=\lambda_R/(1-\lambda_R\delta\lambda)=\lambda_R+\delta\tilde{\lambda} . \tag{5.9}$$

Thus,

$$\delta\tilde{\lambda}=\lambda_R^2\delta\lambda+\cdots . \tag{5.10}$$

Now, from (5.8) and (5.2), if we choose $m^2=m^2(0)$, we see we will need

$$\delta m_1^2=-\tfrac{1}{2}\int[dk]G(k,0) . \tag{5.11}$$

Since, in general for $m^2=m^2(0)$,

$$\delta m^2=-\tfrac{1}{2}\lambda\int[dk]G(k,0)=(\lambda_R+\delta\tilde{\lambda})\delta m_1^2 , \tag{5.12}$$

we expect

$$\delta m_2^2=\delta\lambda\delta m_1^2, \text{ etc.} \tag{5.13}$$

So at large N we need only to calculate to order λ^2 to identify the counterterms. (We will explicitly verify that our renormalization prescription then works to all orders.) In order to systematically study the λ_R expansion of (5.1) and (5.8), we set

$$G(k,t)=G_0(k,t)+\sum_{n=1}G_n(k,t)\lambda_R^n , \tag{5.14}$$

$$\Gamma=k^2+m^2+\sum_{n=1}\Gamma_n(t)\lambda_R^n . \tag{5.15}$$

Since this is an initial-value problem,

$$G(k,0)=G_0(k,0) \text{ and } \dot{G}(k,0)=\dot{G}_0(k,0) \tag{5.16}$$

completely specify the problem. It is convenient to choose the free-field-theory Green's function $G_0(k,t)$ to satisfy the initial data. We see from Eqs. (2.49) and (2.16) that unless we choose $m^2=m^2(0)$, then $\ddot{G}(k,0)$ will depend explicitly on λ_R and will change order by order in a perturbation expansion in λ. Thus, the most convenient choice for the renormalized mass parameter is $m^2(0)$, the effective mass at $t=0$, since if we consider that parameter as given at $t=0$ the reexpansion of these equations in λ simplifies dramatically. Since we can easily determine the connection between $m^2(0)$ and m_R^2, the pole in the vacuum Green's function, one can always at the end reexpress everything in terms of m_R^2. Choosing $m^2(0)$ to be a given value, $\Gamma_n(k,0)=0$, and for $n\geq 1$ we have

$$G_n(k,0)=\dot{G}_n(k,0)=\ddot{G}_n(k,0)=0 . \tag{5.17}$$

This initial condition on the perturbation expansion will eliminate homogeneous solutions to the differential equations for $G_n(k,t)$.

(a) Order λ_R^0. At zeroth order in λ_R, we obtain

$$2G_0\ddot{G}_0-\dot{G}_0^2+4(k^2+m^2)G_0^2=1 , \tag{5.18}$$

which is finite. The solution satisfying the initial data is given by Eq. (3.8).

(b) Order λ_R^1. Using G_0, we can calculate $\Gamma_1(t)$. We have

$$\Gamma_1(t)=\delta m_1^2+\tfrac{1}{2}\int[dk]G_0(k,t) . \tag{5.19}$$

Requiring $\Gamma_1(0)=0$, so that the initial data does not change order by order in perturbation theory, yields

$$\delta m_1^2=-\tfrac{1}{2}\int[dk]G_0(k,0) , \tag{5.20}$$

so that

$$\Gamma_1(t)=\tfrac{1}{2}\int[dk][G_0(k,t)-G_0(k,0)] . \tag{5.21}$$

Using (3.8) we find

$$\Gamma_1(t)=\tfrac{1}{2}\int[dk](2\omega_k)^{-1} \times[2\,\mathrm{Re}F(k)(\cos 2\omega_k t-1) +2\,\mathrm{Im}F(k)\sin 2\omega_k t] . \tag{5.22}$$

From this result we see that not all initial data will be

rendered finite by the order-λ mass renormalization. Instead we require that

$$\int F(k)[dk]/(2\omega_k)<\infty \ , \tag{5.23}$$

so that $\Gamma_1(t)$ is finite. If we do not choose $m^2(0)$ as the renormalization mass, but another finite mass such as $m_R{}^2$ defined by (5.4), then we would discover that we would also require [see (5.52)]

$$\int n(k)[dk]/(2\omega_k)<\infty \ . \tag{5.24}$$

So the previous conditions on F and n are sufficient conditions here also. Now that we have rendered $\Gamma_1(t)$ finite we can determine $G_1(k,t)$. We have that G_1 obeys the equation

$$G_0\ddot{G}_1+G_1\ddot{G}_0-\dot{G}_0\dot{G}_1+4(k^2+m^2)G_1G_0 +2\Gamma_1(t){G_0}^2=0 \ . \tag{5.25}$$

To determine the coupling-constant counterterm it is only necessary to study the ultraviolet behavior of this equation. Thus we need the leading behavior of G_0, $\dot{G}_0$, and $\ddot{G}_0$ at large k. We have from Eq. (3.8) that, in general,

$$G_0\sim 1/(2\omega_k) \ , \tag{5.26}$$

$$\dot{G}_0\sim 2\,\mathrm{Im}F(k)\cos 2\omega_k t-2\,\mathrm{Re}F(k)\sin 2\omega_k t \ , \tag{5.27}$$

$$\ddot{G}_0\sim -4\omega_k[\mathrm{Im}F(k)\sin 2\omega_k t+\mathrm{Re}F(k)\cos 2\omega_k t] \ . \tag{5.28}$$

We see that we can ignore the $\dot{G}_0$ and $\ddot{G}_0$ contribution at large k as long as $\mathrm{Re}F(k)$ and $\mathrm{Im}F(k)$ fall at least as fast as $1/k^3$. So asymptotically we need to solve

$$\ddot{G}_1+4{\omega_k}^2G_1=-\Gamma_1/\omega_k \tag{5.29}$$

with the initial data

$$G_1(k,0)=\dot{G}_1(k,0)=0 \ . \tag{5.30}$$

Using Green's theorem we obtain, for large k,

$$G_1(k,t)=-\int_0^t dt'\sin 2\omega_k(t-t')\Gamma_1(t')/(4{\omega_k}^2) \ . \tag{5.31}$$

To obtain the ultraviolet behavior of this integral one expands Γ_1,

$$\Gamma_1(t')=\Gamma_1(t)+(t'-t)d\Gamma_1/dt\mid_{t'=t}+\cdots \ , \tag{5.32}$$

to obtain

$$G_1(k,t)=\Gamma_1(t)(\cos 2\omega_k t-1)/(4{\omega_k}^3)+O(1/{\omega_k}^4) \ , \tag{5.33}$$

which is finite.

(c) Order ${\lambda_R}^2$. At order ${\lambda_R}^2$ we have that

$$\Gamma_2=\delta{m_2}^2+\tfrac{1}{2}\int[dk]G_1(k,t)+\delta\lambda\int[dk]G_0(k,t)/2 \tag{5.34}$$

$$=\delta{m_2}^2+\tfrac{1}{2}\int[dk]G_1(k,t)+\delta\lambda[\Gamma_1(t)-\delta{m_1}^2] \ . \tag{5.35}$$

Using the asymptotic form of G_1,

$$G_1(k,t)=\Gamma_1(t)(\cos 2\omega_k t-1)/(4{\omega_k}^3) \ , \tag{5.36}$$

we identify

$$\delta{m_2}^2=\delta\lambda\delta{m_1}^2, \quad \delta\lambda=\int[dk]/(8{\omega_k}^3) \ . \tag{5.37}$$

Thus, we see the coupling-constant renormalization is the same as for the vacuum sector [with m_R replaced by $m(0)$]. We notice here that it is easy to disentangle mass renormalization from coupling-constant renormalization without having to resort to a Feynman graph analysis because the coupling constant multiplies a time-dependent term, whereas the mass-renormalization terms are time independent.

The momentum-space integral of $\Gamma_1(t)\cos 2\omega_k t/(4{\omega_k}^3)$ is finite because $\Gamma_1(t)$ is zero at $t=0$, the only dangerous place.

(d) All orders in λ_R. The explicit results of our calculations up to ${\lambda_R}^2$ are consistent with the vacuum result for coupling-constant renormalization

$$\lambda=\lambda_R/(1-\lambda_R\delta\lambda) \tag{5.38}$$

with $\delta\lambda=\int[dk]/(8{\omega_k}^3)$; however, now we use as our renormalized mass parameter, not the pole in the two-point function but the effective mass at $t=0$:

$$\omega_k=(k^2+m^2)^{1/2}=[k^2+m(0)^2]^{1/2} \ , \qquad m^2(t)=m^2(0)+R(t) \ , \tag{5.39}$$

where

$$R(t)=\tfrac{1}{2}\lambda\int[dk][G(k,t)-G(k,0)] \ . \tag{5.40}$$

We still need to make $R(t)$ manifestly finite. Using

$$\lambda_R=\lambda(1-\lambda_R\delta\lambda) \ , \tag{5.41}$$

we obtain, after multiplying both sides of Eq. (5.40) by $(1-\lambda_R\delta\lambda)$, that

$$R(t)=\tfrac{1}{2}\lambda_R\int[dk][G(k,t)-G(k,0)+R(t)/(4{\omega_k}^3)] \ . \tag{5.42}$$

Since

$$R(t)=\lambda_R R_1(t)+{\lambda_R}^2R_2(t)+\cdots$$

and $G_n(k,0)=0$ for $n\geq 1$, we have that for (5.42) to be finite to all orders in λ_R we need, for $n\geq 1$,

$$R_n(t)=\tfrac{1}{2}\lambda_R\int[dk][G_{n-1}(k,t)+R_{n-1}(t)/(4{\omega_k}^3)] \tag{5.43}$$

to be finite. This requires that, at large k,

$$G_{n-1}(k,t)\sim -R_{n-1}(t)/(4{\omega_k}^3) \ . \tag{5.44}$$

To show that this is the case, we write down the ${\lambda_R}^r$ term in the power series in the λ_R expansion of (5.1) (for $r\geq 1$):

$$\sum_{k=0}\left[2G_k\ddot{G}_{r-k}-\dot{G}_k\dot{G}_{r-k}+4\sum_{m=0}\Gamma_mG_kG_{r-k-m}\right]=0 \ . \tag{5.45}$$

By studying the large-k behavior of the individual terms, taking into account the rapid falloff in k of $n(k)$ and $F(k)$, we find that at large k one needs to solve for all r the same equation as for $r=1$ (5.29):

$$\ddot{G}_r+4(k^2+m^2)G_r=-\Gamma_r/\omega_k \ . \tag{5.46}$$

For $r\geq 1$, $\Gamma_r=R_r$ and so because of the initial conditions on G_r (5.17) we have, at large k [following (5.29)–(5.36)],

$$G_r(k,t)=\Gamma_r(t)(\cos 2\omega_k t-1)/(4\omega_k{}^3) \ , \tag{5.47}$$

which shows that our renormalization prescription works to all orders in λ_R. In summary we have that the finite renormalized equations are

$$2G\ddot{G}-\dot{G}^2+4\Gamma G^2-1=0 \ , \tag{5.48a}$$

$$\Gamma=k^2+m^2(t)=k^2+m^2(0)+R(t) \ , \tag{5.48b}$$

$$R(t)=\tfrac{1}{2}\lambda_R\int[dk]\{G(k,t)-G(k,0)+R(t)/[4\omega(0)^3]\} \ . \tag{5.48c}$$

The equation we obtain for $R(t)$ is exactly the same one that we obtained in the Heisenberg picture using a WKB analysis [Eq. (4.37)].

We have seen that to keep the initial data independent of explicit dependence on λ it was convenient to use $m^2(0)$, the effective mass at $t=0$, as our renormalized mass. [Also in WKB approximation it is $m^2(0)$ that is the natural mass parameter.] But we could have used $m_R{}^2$, the position of the pole in the propagator in the vacuum sector (in the same large-N approximation), equally as well as our renormalized mass parameter. In the latter case our initial data would have explicitly depended on λ. Using the definition of $m_R{}^2$ given in (5.4) and the definition of $m^2(0)$ from (5.7) and (3.18), it is quite easy to determine the relationship between the two finite mass parameters.

We have, from (5.4), that we can write the renormalized mass found in covariant treatments of large N as

$$m_R{}^2=\mu^2+\tfrac{1}{2}\lambda\int[dk]/(2\omega_k), \quad \omega_k{}^2=k^2+m_R{}^2 \tag{5.49}$$

and, from (5.7),

$$m^2(0)=\mu^2+\tfrac{1}{2}\lambda\int[dk][1+2n(k)+2\,\mathrm{Re}F(k)]/[2\omega(0)] \ ,$$

$$\omega(0)^2=k^2+m(0)^2 \ . \tag{5.50}$$

Thus,

$$m^2(0)-m_R{}^2=-\tfrac{1}{2}\lambda[m^2(0)-m_R{}^2]\times\int[dk]/\{2\omega_k\omega(0)[\omega(0)+\omega_k]\}+\lambda\int[dk][n(k)+\mathrm{Re}F(k)]/[2\omega(0)] \ . \tag{5.51}$$

Using the relationship $\lambda_R=\lambda(1-\lambda_R\delta\lambda)$ with the vacuum value of $\delta\lambda$,

$$\delta\lambda=\int[dk]/(8\omega_k{}^3) \ ,$$

we obtain

$$m^2(0)-m_R{}^2=\lambda_R(m^2(0)\{\ln[m^2(0)/m_R{}^2]-1\}+m_R{}^2)/(32\pi^2)+\lambda_R\int[dk][n(k)+\mathrm{Re}F(k)]/[2\omega(0)] \ , \tag{5.52}$$

which is a manifestly finite connection between the two masses as long as $n(k)$ and $\mathrm{Re}F(k)$ are integrable.

Now that we have renormalized the theory for $\langle\phi\rangle=0$, it is easy to go back to the unrenormalized equations for $\langle\phi\rangle\neq 0$:

$$\ddot{\phi}(t)+\left[\mu^2+\tfrac{1}{2}\lambda\phi^2+\tfrac{1}{2}\lambda\int[dk]G(k,t)\right]\phi=0 \ , \tag{5.53}$$

$$2\ddot{G}(k,t)G(k,t)-\dot{G}^2(k,t)+4\Gamma G^2(k,t)-1=0 \ , \tag{5.54}$$

where

$$\Gamma=k^2+\mu^2+\tfrac{1}{2}\lambda\phi^2+\tfrac{1}{2}\lambda\int[dk]G(k,t) \ , \tag{5.55}$$

and we renormalize them by a slight change in the definition of $R(t)$. When ϕ is not zero we have

$$m^2(t)=\mu^2+\tfrac{1}{2}\lambda\phi^2+\tfrac{1}{2}\lambda\int[dk]G(k,t) \ . \tag{5.56}$$

Thus, we again define $R(t)$ as $m^2(t)-m^2(0)$ and we obtain

$$m^2(t)-m^2(0)=R(t)=\tfrac{1}{2}\lambda[\phi^2(t)-\phi^2(0)]+\tfrac{1}{2}\lambda\int[dk][G(k,t)-G(k,0)] \ . \tag{5.57}$$

This equation is identical to the one we obtain in the Heisenberg analysis, and we again use coupling-constant renormalization to obtain the finite equations

$$\ddot{\phi}(t)+[m^2(0)+R(t)]\phi=0 \ , \tag{5.58}$$

$$2\ddot{G}(k,t)G(k,t)-\dot{G}^2(k,t)+4\Gamma G^2(k,t)-1=0 \ , \tag{5.59}$$

where

$$\Gamma=k^2+m^2(0)+R(t) \tag{5.60}$$

and

$$R(t)=\tfrac{1}{2}\lambda_R[\phi^2(t)-\phi^2(0)]+\tfrac{1}{2}\lambda_R\int[dk]\{G(k,t)-G(k,0)+R(t)/[4\omega^3(0)]\} \ ,$$
$$\omega^2(0)=k^2+m^2(0) \ . \tag{5.61}$$

VI. CONCLUSIONS

We have shown in this paper how to make sense of the divergent equations one obtains from the functional Schrödinger equation in the time-dependent Hartree-Fock approximation and from the Heisenberg equations in the large-N approximation by performing mass and

coupling-constant renormalization. For the TDHF approximation at large N we proved that our renormalization prescription, found by looking at the divergences found at order λ and λ^2, works to all orders in λ because of the geometric structure of the coupling-constant renormalization in Hartree-Fock (or large-N) approximation. In the Heisenberg-equation approach we were able to study the divergences to all orders in λ using a WKB type of ansatz. The renormalization prescriptions found in both approaches were identical. We hope that this simple approach to the renormalization problem permits those without extensive field-theory backgrounds to derive and solve the semiclassical equations relevant to a wide variety of problems. We have indicated that our approach appears also to be practical for numerical methods of solution.

This work presents a starting point for a systematic analysis of the more interesting problem of what happens in a curved background space. The WKB method of attacking the renormalization problem is easily extended to the curved-background-space problem and we feel it is easier to work with than the functional Schrödinger equation.

We also found in this study that the large-N approximation as formulated in the Heisenberg picture allows for a more general class of initial data than the TDHF approximation at large N. That is because at large N one does not have to restrict oneself to initial wave functionals that are Gaussian. Thus, for example, one can study near-thermal-equilibrium initial states in the Heisenberg formalism. Such states are not Gaussian initial states and cannot be studied in the TDHF approximation. This clarifies some confusion about the relationship of the two approaches in the literature. Also, previously in the literature, the only configurations studied had been either the vacuum or an equilibrium initial configuration. Our work shows that the condition necessary for renormalization of an arbitrary configuration are the integrability conditions on the initial density and correlated pair density. These conditions are automatically satisfied by a thermal distribution which falls exponentially fast at a large momentum.

In trying to understand the properties of the single or two-time Green's functions, we found it quite useful to look at the WKB approximate solutions to our equations. We hope to present detailed numerical studies of the range of validity of the WKB approximation and its well-defined corrections in another paper.[17]

ACKNOWLEDGMENTS

This work was begun in collaboration with So-Young Pi and M. Samiullah and we thank them for all their contributions. They have independently, using a different approach, worked out the renormalization of the TDHF equations and have reached conclusions similar to ours.[18] One of us (F.C.) has benefited greatly from conversations with Harvey Rose, Gerald Guralnik, and Roman Jackiw.

APPENDIX: RENORMALIZATION OF THE EFFECTIVE POTENTIAL OF THE TDHF APPROXIMATION

For completeness we will show how to do the renormalization of the effective potential. In order to simplify some integrals, it is convenient to use the subtraction procedure of Refs. 11 and 14. That is, we let

$$\mu^2/\lambda = \pm \mu_R{}^2/\lambda_R - A, \quad A = \tfrac{1}{2}\int [dk]\frac{1}{2k} , \tag{A1}$$

where the plus (minus) pertains when μ^2 is positive (negative). And we define

$$\begin{aligned} &1/\lambda_R = 1/\lambda + \Sigma(\tilde{M}) , \\ &\Sigma(\tilde{M}) = \tfrac{1}{2}\int [dk](2\tilde{M}^2)^{-1}[1/k - (k^2+\tilde{M}^2)^{-1/2}] , \end{aligned} \tag{A2}$$

where $\tilde{M}$ is an arbitrary subtraction mass ($\tilde{M}^2 > 0$) which will be chosen conveniently later depending on the sign of μ^2.

This latter renormalization procedure makes it quite easy to render the effective potential finite.

$V_{\text{eff}}(\phi)$ is obtained in the usual fashion from $V_{\text{eff}}[\phi, G]$ [see (2.42)] by first determining $G(\phi)$ via

$$\partial V_{\text{eff}}[\phi, G]/\partial G = 0 \Longrightarrow G = G(\phi) \tag{A3}$$

and defining

$$V_{\text{eff}}(\phi) = V_{\text{eff}}[\phi, G = G(\phi)] . \tag{A4}$$

The effective potential $V_{\text{eff}}[\phi, G]$ is given by Eqs. (2.41) and (2.42). Differentiating V with respect to G yields the gap equation, which defines G as a function of ϕ:

$$m^2[\phi] = \mu^2 + \lambda\phi^2/2 + \frac{\lambda}{2}\int [dk]G(k) , \tag{A5}$$

where

$$G(k) = \tfrac{1}{2}(k^2 + m^2[\phi])^{-1/2} . \tag{A6}$$

Using this gap equation allows us to rewrite $V[\phi, G]$ as

$$\begin{aligned} V[\phi, G[\phi]] = {} & \tfrac{1}{4}G^{-1}(x,x) - \lambda G^2(x,x)/8 \\ & + \lambda\phi^4/8 + \mu^2\phi^2/2 . \end{aligned} \tag{A7}$$

If we choose the arbitrary subtraction mass $\tilde{M}$ to be m_R, the self-consistent mass when $\phi = 0$, then we obtain the renormalized equation for $m^2[\phi]$:

$$\begin{aligned} m^2[\phi] = {} & m_R{}^2 + \lambda_R\phi^2/2 \\ & + \tfrac{1}{2}\lambda_R \int [dk] \left\{ [G(k) - 1/(2k)] \right. \\ & \left. + \frac{m^2[\phi]}{m_R{}^2}[1/(2k) - 1/(2\omega_k)] \right\} , \end{aligned} \tag{A8}$$

where

$$G(k) = \tfrac{1}{2}(k^2 + m^2[\phi])^{-1/2}, \quad \omega_k = (k^2 + m_R{}^2)^{1/2} .$$

Performing the integrations we obtain the finite gap

equation:

$$m^2[\phi]=m_R{}^2+\lambda_R\phi^2/2 +\lambda_R m^2[\phi]\ln(m^2[\phi]/m_R{}^2)/(32\pi^2)\ . \quad \text{(A9)}$$

We can rewrite Eq. (2.42) using the definition of $m^2[\phi]$ given in (A5) in the form

$$V_{\rm eff}[\phi]=\frac{m^4[\phi]}{2\lambda}+\int [dk]\left[\frac{G^{-1}(k)}{8}+\tfrac{1}{2}k^2G(k)\right]-\frac{\mu^4}{2\lambda}\ . \quad \text{(A10)}$$

The integral contains a quartic divergence which is ϕ independent as is the last term. Discarding these ϕ-independent terms and using (A2) to renormalize λ we are left with a finite integration to perform. The resulting renormalized effective potential is

$$V_{\rm eff}[\phi]=\frac{m^4[\phi]}{2\lambda_R}\left[1+\frac{\lambda_R}{32\pi^2}\ln\left[\frac{m_R{}^2}{m^2[\phi]}\right]\right], \quad \text{(A11)}$$

which is the standard result.

[1]A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[2]A. D. Linde, Phys. Lett. **114B**, 431 (1982); A. Albrecht and P. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[3]A. Guth and S.-Y. Pi, Phys. Rev. D **32**, 1899 (1985).

[4]P. Bouche, S. Koonin, and J. W. Negele, Phys. Rev. C **13**, 1226 (1976); A. Kerman and S. Koonin, Ann. Phys. (N.Y.) **100**, 332 (1976).

[5]S. Epstein, *The Variational Method in Quantum Chemistry* (Academic, New York, 1974).

[6]F. Cooper and S.-Y. Pi, in *Current Trends in Physics,* edited by A. Khare and T. Pradhan (World Scientific, Singapore, 1986); F. Cooper, S.-Y. Pi, and P. Stancioff, Phys. Rev. D **34**, 3831 (1986).

[7]P. A. M. Dirac, Proc. Cambridge Philos. Soc. **26**, 376 (1930).

[8]R. Jackiw and A. Kerman, Phys. Lett. **71A**, 158 (1979).

[9]G. F. Mazenko, Phys. Rev. Lett. **54**, 2163 (1985); Phys. Rev. D **34**, 2223 (1986).

[10]J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).

[11]S. Coleman, R. Jackiw, and H. D. Politzer, Phys. Rev. D **10**, 2491 (1974).

[12]F. Cooper, G. S. Guralnik, and S. H. Kasdan, Phys. Rev. D **14**, 1607 (1976).

[13]T. Barnes and G. Ghandour, Phys. Rev. D **22**, 924 (1980).

[14]W. A. Bardeen and M. Moshe, Phys. Rev. D **28**, 1372 (1983).

[15]G. Semenoff and N. Weiss, Phys. Rev. D **31**, 699 (1985).

[16]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982), and references therein.

[17]F. Cooper and E. Mottola (in preparation).

[18]S.-Y. Pi and M. Samiullah, Phys. Rev. D (to be published).