Casimir effect around a cone

J. S. Dowker
Department of Theoretical Physics, The University, Manchester, England
(Received 4 May 1987)

The vacuum average of the energy density of a free, massless scalar field around a conical flux-tube singularity in $d+1$ space-time dimensions is calculated. A complex contour method is employed and the results involve higher-order Bernoulli polynomials. The formulas might have applications to cosmic strings.

I. INTRODUCTION

The explicit calculation of the vacuum polarization caused by particular external fields has played an important part in the development of quantum field theory.[1]

In gravitational situations, one aspect of this polarization is a nonvanishing vacuum expectation value $\langle T_{\mu\nu}\rangle$ of the energy-momentum tensor.[2] This Casimir effect has been analyzed over a period of years (some references are given in Birrell and Davies[3]) and this paper is a small but specific calculation in this general area. The calculation concerns space-times that possess conical singularities, in fact, just one. Some aspects of this have been considered before[4,5] and I now wish to enlarge on previous work.

II. THE GEOMETRY

We assume that space-time has $n=d+1$ dimensions and is static. Its metric, in cylindrical coordinates, is

$$d\sigma^2=dt^2-dr^2-r^2d\phi^2-d\mathbf{z}^2\ , \tag{1}$$

where the spatial points $(r,\phi,\mathbf{z})$ and $(r,\phi+\beta,\mathbf{z})$ are to be identified.[6] $d\mathbf{z}^2$ is the metric on a flat $(d-2)$-dimensional space and we can assume $\mathbf{z}$ to be Cartesian coordinates. If $d=3$, (1) might be thought of as the metric exterior to a cosmic string.[6,7]

In addition to the conical singularity, we assume that there is a magnetic flux running through the singularity axis. This can be accommodated by inserting phase factors, as described elsewhere.[4,8]

III. THE CALCULATION

The formula[2,3,9] for $\langle T_{\mu\nu}\rangle$ that we use involves the coincidence limit of a differential operator acting on the real part G^1 of the Feynman Green's function G_F. The essence of the calculation will be exhibited if we restrict our attention to $\langle T_{00}\rangle$ which, for massless scalar fields, is given by

$$\langle T_{00}\rangle=i\{[(2\xi+\tfrac{1}{2})\partial_0\partial_0+(2\xi-\tfrac{1}{2})\Delta_p]G_F(x,x')\}\ , \tag{2}$$

where $\xi=(d-1)/4d$ for conformally invariant fields. Δ_p is the polarized Laplacian on the spatial section and the curly brackets stand for the coincidence limit $x'\to x$, where $x=(t,r,\phi,\mathbf{z})$ and $x'=(t',r',\phi',\mathbf{z}')$.

We now note that, in order to allow for the conical structure, the Green's functions, etc., are obtained simply by inserting the flat-space forms into contour integrals. Thus, $\langle T_{00}\rangle$ will also be given by such an integral.

The standard expression for the massless Green's function in $(1+d)$-dimensional flat space-time is

$$G_F=-i\tfrac{1}{4}(-\pi)^{-(d+1)/2}\Gamma\left[\frac{d-1}{2}\right](\sigma^2)^{-(d-1)/2}\ , \tag{3}$$

considered as a distribution. σ^2 is the space-time interval between x' and x:

$$\sigma^2=(t-t')^2-r'^2-r^2+2rr'\cos(\phi'-\phi)-|\mathbf{z}'-\mathbf{z}|^2$$
$$=-2rr'[\cosh\alpha_1-\cos(\phi'-\phi)]\ .$$

Following Sommerfeld[10] and Carslaw[11] (see also Rubinowicz[12]) the Green's function with period β is[5]

$$G_{\beta,\delta'}(x',x)=\frac{1}{2\beta i}\int_A d\alpha\, G_F(\phi'-\phi-\alpha)\frac{\exp[\pi i\alpha(2\delta'-1)/\beta]}{\sin(\pi\alpha/\beta)}\quad (0<\delta'\le 1)\ . \tag{4}$$

A phase factor δ' has been included to allow for the effect of the flux.[4] Perhaps I should remark now that the field is a complex scalar one and that the Green's function in (2) is really $G_{\beta,\delta'}+G_{\beta,-\delta'}$. (To effect this sign reversal, the range of δ' has to be extended by periodicity from that indicated.) Ultimately this produces a factor of 2 in the coincidence-limit quantities.

As described in the references, the contour A consists of two parts. The section in the upper-half plane runs from $i\infty$ to $i\infty$ in the range $-\pi+\phi'-\phi<\mathrm{Re}\alpha<\pi+\phi'-\phi$ and passes below the singularity at $\alpha=i\alpha_1+\phi'-\phi$. The lower section is obtained by reflection in the point $\alpha=\phi'-\phi$.

If d is odd, the singularity is a pole and $G_{\beta,\delta'}$ can be obtained in closed form. If d is even, or if the field is massive, there is a branch point and we are usually reduced to numerical evaluation.

The obvious procedure is to substitute these forms into (2), perform the differentiations, and then take the coincidence limit. Since the latter diverges, the Minkowski value has to be subtracted. This is not difficult and was the method employed earlier,[5] but it is actually more convenient to remove the Minkowski Green's function at an earlier stage. To this end, the contour A is deformed [in fact, by reversing the process by which Sommerfeld and Carslaw motivated their forms corresponding to (4)]. Replace A by an anticlockwise loop around the pole at $\alpha=0$, and the two vertical lines, Γ, $\alpha=c+iy$ and $\alpha=-c+iy$ $(-\infty<y<\infty)$ with c a constant. The loop is easily seen to give the standard Minkowski expression (3), $G_{2\pi,1}$, which can be forthwith discarded as part of the renormalization recipe. It is the remainder that we put into (2). The constant c is chosen so that Γ never crosses any of the other poles occurring at $\alpha=s\beta$ $(s=\pm1,\pm2,\ldots)$, i.e., $c<\beta$.

It is easy to carry through the operations in (2) onto the integrand and, after some simplification, we find

$$\langle T_{00}\rangle=-\beta^{-1}(-4\pi r^2)^{-(d+1)/2}2^{(d-1)/2}\Gamma\left[\frac{d+1}{2}\right]\times[W_{d+1}+(2\xi-\tfrac{1}{2})(1-d)W_{d-1}]\,, \tag{5}$$

where

$$W_N=\int_\Gamma(-1+\cos\alpha)^{-N/2}\frac{\cos(2\pi\alpha\delta/\beta)}{i\sin(\pi\alpha/\beta)}d\alpha \tag{6}$$

($\delta=\delta'-\frac{1}{2}$). The symmetry of the contour has been used to replace the exponential by a cosine.

At this point we distinguish between odd and even d. For odd d the integrand in (6) has only poles. In particular, it has a multiple pole at the origin. Furthermore, the two vertical lines making up Γ can be recombined into a clockwise loop around $\alpha=0$, so that

$$W_N=-i(-2)^{-N/2}\oint\frac{\cos(2\pi\alpha\delta/\beta)}{[\sin(\alpha/2)]^N\sin(\pi\alpha/\beta)}\alpha\alpha\,. \tag{7}$$

Hence, W_N is real and equals $-2\pi(-2)^{-N/2}$ times the coefficient of $1/\alpha$ in the power-series expansion of the integrand.

We will now determine this coefficient. Consider the quantity

$$I=\left[\frac{z}{\sin z}\right]^N\left[\frac{zT}{\sin zT}\right]\cos(2\delta zT)$$

that we are required to expand. This can be done in one step in terms of higher-order Bernoulli polynomials (see the Appendix), but the structure is more clearly brought out if the parts are treated separately. In terms of ordinary Bernoulli polynomials $B_n(x)$, we have, quite generally,

$$\left[\frac{zT}{\sin zT}\right]\cos(2\delta zT)=\sum_{n=0}^{\infty}(-1)^nB_{2n}(\delta')\frac{(2zT)^{2n}}{(2n)!}\,. \tag{8}$$

Also, we have the expansion

$$\left[\frac{z}{\sin z}\right]^N=\sum_{\nu=0}^{\infty}(-1)^\nu D_{2\nu}^{(N)}\frac{z^{2\nu}}{(2\nu)!}\,, \tag{9}$$

where the coefficients D are given in terms of the higher-order Bernoulli polynomials $B_n^{(N)}(x)$ by (Nörlund, Ref. 13, p. 130)

$$D_{2\nu}^{(N)}=2^{2\nu}B_{2\nu}^{(N)}(N/2)\,.$$

The coefficient C_N of z^N in the expansion of I will be a polynomial in T whose coefficients are polynomials in δ. Write

$$C_N=\sum_{p=0}^{\infty}C_{N,2p}D_{2p}(\delta)(T/2)^{2p}\,,$$

where $D_{2p}(\delta)=2^{2p}B_{2p}(\delta')$ is an even polynomial.

Using (8) and (9), we find

$$C_{N,2p}=(-1)^{N/2}2^{2p}[(2p)!(N-2p)!]^{-1}D_{N-2p}^{(N)}\,,\quad 0\le p\le N/2$$

$$=0,\quad p>N/2\,.$$

For convenience we list several of the $D_n(x)$:

$$D_2(x)=4x^2-\tfrac{1}{3}\,,$$

$$D_3(x)=x(2x^2-\tfrac{1}{2})\,,$$

$$D_4(x)=16x^4-8x^2+\tfrac{7}{15}\,,$$

$$D_5(x)=4x(2x^2-12)(4x^2-\tfrac{7}{3})\,,$$

$$D_6(x)=64x^6-80x^4+28x^2-\tfrac{31}{21}\,,$$

$$D_8(x)=256x^8-1793/3x^6+1568/3x^4-4963x^2+\tfrac{127}{15}\,,$$

$$D_{10}(x)=1024x^{10}-3840x^8+6272x^6-4960x^4+1524x^2-\tfrac{2555}{33}\,,$$

$$D_{12}(x)=4096x^{12}-22\,528x^{10}+59\,136x^8-87\,296x^6+67\,056x^4-20\,440x^2+\tfrac{1414477}{1365}\,.$$

The numbers $D_\nu^{(n)}$ can be evaluated using symbolic manipulation from the basic equations (Nörlund, Ref. 13, p. 147)

$$B_\nu^{(n+1)}(x)=\frac{\nu!}{n!}\left[\frac{d}{dx}\right]^{n-\nu}(x-1)(x-2)\cdots(x-n),\quad n\ge\nu\,,$$

$$B_n^{(n)}(x)=\int_x^{x+1}(t-1)\cdots(t-n)dt\,.$$

Alternatively, Tables 6 and 11 in Ref. 13 may be consulted.

$\langle T_{00}\rangle$ now follows from (5) as

$$\langle T_{00}\rangle=(4\pi r^2)^{-(d+1)/2}\Gamma\left[\frac{d+1}{2}\right]\left[C_{d+1,d+1}D_{d+1}(\delta)(T/2)^{d+1}+\sum_{p=0}^{(d-1)/2}[C_{d+1,2p}+(4\xi-1)(d-1)C_{d-1,2p}]D_{2p}(\delta)(T/2)^{2p}\right].$$

We next give some specific expressions in the conformally invariant case, writing the general form in $d+1$ dimensions as $(T=2\pi/\beta)$

$$\langle T_{00}\rangle=(4\pi r^2)^{-(d+1)/2}\times[A_0+A_2D_2(\delta)T^2+A_4D_4(\delta)T^4+\cdots],\tag{10}$$

where the constants A_n are

$$\begin{aligned}&d=3\quad(1/45,0,1/24),\\&d=5\quad(62/4725,0,1/60,-1/360),\\&d=7\quad(578/33\,075,0,2/105,-1/252,1/6720),\\&d=9\quad(2536/66\,825,0,4/105,-7/810,1/2160,\\&\qquad-1/151\,200).\end{aligned}\tag{11}$$

All these forms must be extended by periodicity if δ lies outside the range $-\frac{1}{2}$ to $\frac{1}{2}$. When $\delta=\frac{1}{2}$ (i.e., no flux) the $d=3$ result of (11) agrees with the known expression.

For the special case when $d=1$, a simple scaling of the result in Ref. 14 gives the energy density on a "circle" of angle β to be

$$-(T^2/8\pi)[D_2(\delta)+\tfrac{2}{3}].$$

The coefficient A_0, which determines the large β limit, is given by

$$A_0=(-1)^{(d+1)/2}[D_{d+1}^{(d+1)}+(d^2-1)D_{d-1}^{(d-1)}]\times\frac{2^{-d-1}(4\pi)^{1/2}}{(d+1)\left[\frac{d}{2}\right]!}.$$

It is interesting to note that, in the conformal case, there is no term proportional to $1/\beta^2$, a result valid for any d since, using the general form (Nörlund, Ref. 13, p. 147)

$$D_n^{(n+2)}=\frac{(-1)^{n/2}}{(n+1)}2^n\left[\left[\frac{n}{2}\right]!\right]^2\quad(n\text{ even}),$$

it follows that

$$dC_{d+1,2}=(d-1)C_{d-1,2}.$$

We denote by δ_n the modulus of that root of $D_n(x)=0$ lying between $-\frac{1}{2}$ and $\frac{1}{2}$ (e.g., $\delta_4{}^2=\frac{1}{4}-1/\sqrt{30}$). Then $\langle T_{00}\rangle$ will attain its limiting value, as $\beta\to\infty$, from above (below) if $\delta<\delta_4$ ($\delta>\delta_4$) and will tend to $+\infty$ $(-\infty)$ as $\beta\to 0$ if $\delta\gtrless\delta_{d+1}$ ($\delta\lessgtr\delta_{d+1}$) for

$$d=\begin{cases}5+4n\\3+4n,\quad n=0,1,\ldots.\end{cases}$$

As n increases, δ_n tends downward to $\frac{1}{4}$. We list a few values:

$$\delta_4\approx 0.259\,664\,811,\quad \delta_6\approx 0.252\,459\,284,$$
$$\delta_8\approx 0.250\,619\,617,\quad \delta_{10}\approx 0.250\,155\,282.$$

Further comments on the form of the results (10) and (11) will be found in Sec. IV.

When there is no conical singularity, i.e., when $\beta=2\pi$, the form (10) is a factorizable polynomial in δ. In particular, for the quantity $(4\pi r^2)^{(d+1)/2}\langle T_{00}\rangle$, we have

$$\begin{aligned}&d=3,\quad \tfrac{2}{3}(\delta^2-\tfrac{1}{4})^2,\\&d=5,\quad -\tfrac{8}{45}(\delta^2-\tfrac{1}{4})^2(\delta^2-\tfrac{9}{4}),\\&d=7,\quad \tfrac{4}{105}(\delta^2-\tfrac{1}{4})^2(\delta^2-\tfrac{9}{4})(\delta^2-\tfrac{25}{4}),\\&d=9,\quad -\tfrac{32}{4725}(\delta^2-\tfrac{1}{4})^2(\delta^2-\tfrac{9}{4})(\delta^2-\tfrac{25}{4})(\delta^2-\tfrac{49}{4}).\end{aligned}$$

These results are not surprising. They can be derived directly in terms of Γ functions from the original integral. The $d=3$ expression is plotted in Fig. 1.

For even d we return to the integral W_N in (6). Since N is now odd, the integrand has a root branch point at the origin and we cannot recombine Γ into a loop, as was done in the odd d case. (The cut structure of the integrand in (4) is described in detail by Rubinowicz.[12])

To produce a computable expression, some technicalities are necessary. First, a choice of branch has to be made. Ours is such that $(-1+\cos\alpha)^{1/2}$ is positive imaginary on the real α axis. If R and Θ are defined by

$$(-1+\cos\alpha)^{1/2}=\sqrt{2}R\exp(i\Theta),\tag{12}$$

Θ is $\pi/2$ on the real axis, 0 on the left-hand side of the cut $\alpha=iy$ $(0\le y\le\infty)$, and π on the right-hand side. Θ is also $\pi/2$ for $\alpha=\pm\pi+iy$ $(-\infty\le y\le\infty)$ (cf. Garnir, Ref. 15, Fig. 1).

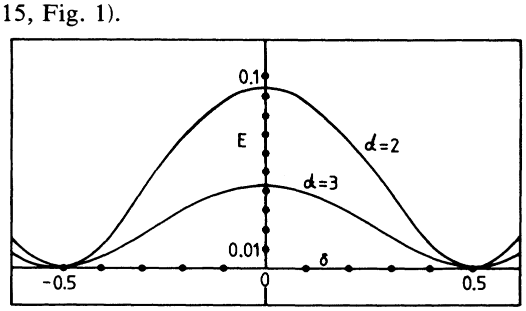

FIG. 1. The vacuum energy density $E\equiv(4\pi r^2)^{(d+1)/2}\langle T_{00}\rangle$ in the case of no conical singularity ($\beta=2\pi$) as a function of the flux parameter δ for spatial dimension d equal to 2 and 3.

The location of Γ, i.e., the value of c, that leads to a computable expression, depends on the size of the angle β. Two cases are distinguished: $\beta \geq 2\pi$ and $\beta \leq 2\pi$. In the latter, there can be poles in the region $-\pi \leq \alpha \leq \pi$ other than the Minkowski one at $\alpha = 0$ and so, to avoid crossing problems, we choose $c = \beta/2$. If $\beta \geq 2\pi$, the value $c = \pi$ is the most convenient one and we obtain, after some minor manipulation,

$$W_N = 2^{(4-N)/2} \exp(-i\pi N/2) \int_0^\infty \frac{dy}{(\cosh y/2)^N} \frac{\cosh(T\delta' y)\sin[\pi T(\delta'-1)] - \cosh[T(\delta'-1)y]\sin(\pi T\delta')}{\cosh(Ty) - \cos(\pi T)} . \tag{13}$$

On the other hand, if $\beta \leq 2\pi$ we rearrange the integral to give

$$W_N = 4i 2^{-N/2} \int_0^\infty \frac{R^{-N} dy}{\cosh(Ty/2)} [\cos N\Theta \sin\pi\delta \sinh(Ty\delta) + \sin N\Theta \cos\pi\delta \cosh(Ty\delta)] , \tag{14}$$

where R and Θ are defined by Eq. (12) with Θ lying between 0 and π. Equations (13) and (14) agree in the region around $\beta = 2\pi$.

When there is no conical singularity ($\beta = 2\pi$), the integrals can be evaluated and we find

$$W_N = -\exp(-i\pi N/2) 2^{(3+N/2)} (p^2 - \delta^2) \cdots (1-\delta^2) \pi \cot(\pi\delta) / \Gamma(2p+2) , \tag{15}$$

where $p = (N-1)/2$.

The nonpolynomial form of this result reflects the existence of the branch point.

Our choice of branch implies that $(-1)^{(d+1)/2} = \exp[i\pi(d+1)/2]$ and, since W_N is imaginary, Eq. (5) gives a real $\langle T_{00} \rangle$. Then, for the special value $\beta = 2\pi$, we have the closed form, in the conformal case,

$$\langle T_{00} \rangle = (4\pi r^2)^{-(d+1)/2} \frac{(4\pi)^{1/2}}{(d+1)\left(\frac{d}{2}\right)!} \delta \cot\pi\delta \left(\tfrac{1}{4} - \delta^2\right) \left[\left(\frac{d}{2} - 1\right)^2 - \delta^2\right] \cdots (1-\delta^2) . \tag{16}$$

When $d = 2$, this result can be compared with the expression for the effective Lagrangian given in Ref. 16.

Apart from its intrinsic interest, Eq. (16) provides a useful check of the numerical integrations necessary when $\beta \neq 2\pi$.

A plot of Eq. (16) for $d = 2$ is given in Fig. 1. The curves for other dimensions are very similar, differing only in vertical scaling.

As $|\delta|$ exceeds $\frac{1}{2}$, the curve is extended by periodicity. At the joining points there is a discontinuity in the odd derivatives of order 3 and higher.

IV. RESULTS AND DISCUSSION

A discussion of the form of the above expressions is now presented, always restricting the numerical evaluations to the conformally invariant case, which is the most interesting one.

We consider $\langle T_{00} \rangle$ as a function of β $(=2\pi/T)$ for fixed values of δ and look first at odd d. For the case $d = 3$ a brief discussion has been already given. If $d > 3$, the general shape of the curve is independent of d. A typical example is given in Fig. 2(a).

Application of the fact that the roots δ_n decrease as n, the order of the Bernoulli polynomial, increases shows that the coefficients of the derivative (with respect to T^2) of the $\langle T_{00} \rangle$ general form, (10), are such that, if δ lies between δ_4 and δ_{d+1}, there is only one maximum as β varies. If δ lies outside this range there are no turning points.

The physical significance of these results is problematic, but, if we assume that the system adjusts itself so as to minimize the vacuum energy, we see that the angle β will either increase to infinity or decrease to zero depending on the value of the flux. If δ lies between the above stated values, the limiting value of β depends on which side of the maximum β starts. A more interesting behavior occurs for even d where numerical integration is necessary (except for $\beta = 2\pi n$).

Figures 2(b) and 2(c) plot the $d = 2$ and $d = 4$ results. Various features can be noted. For large β, $\langle T_{00} \rangle$ again tends to a constant positive value. The relevant asymptotic expansion of (14) gives, for $\langle T_{00} \rangle$, an infinite power series of terms $\approx T^{2n} D_n(\delta)$, $n = 0, 2, 3, \ldots$, all with positive coefficients. Hence, if $\delta > \delta_4$, the asymptotic value is approached from below, while if $\delta < \delta_4$, it is approached from above.

As β tends to zero, analysis of the integral (14) shows that $\langle T_{00} \rangle$ diverges, the leading term of the asymptotic expansion being equal to

$$\beta^{-d-1} 2^{d+1} \cos\left(\frac{\pi d}{2}\right) (\pi)^{1/2} \Gamma\left(\frac{d+1}{2}\right) \int_0^{\pi/2} [\cos\pi\delta \cosh(\pi\delta \cot\tau) \sin(d+1)\tau + \sin\pi\delta \sinh(\pi\delta \cot\tau) \cos(d+1)\tau] (\sin\tau)^{(d-1)/2} d\tau . \tag{17}$$

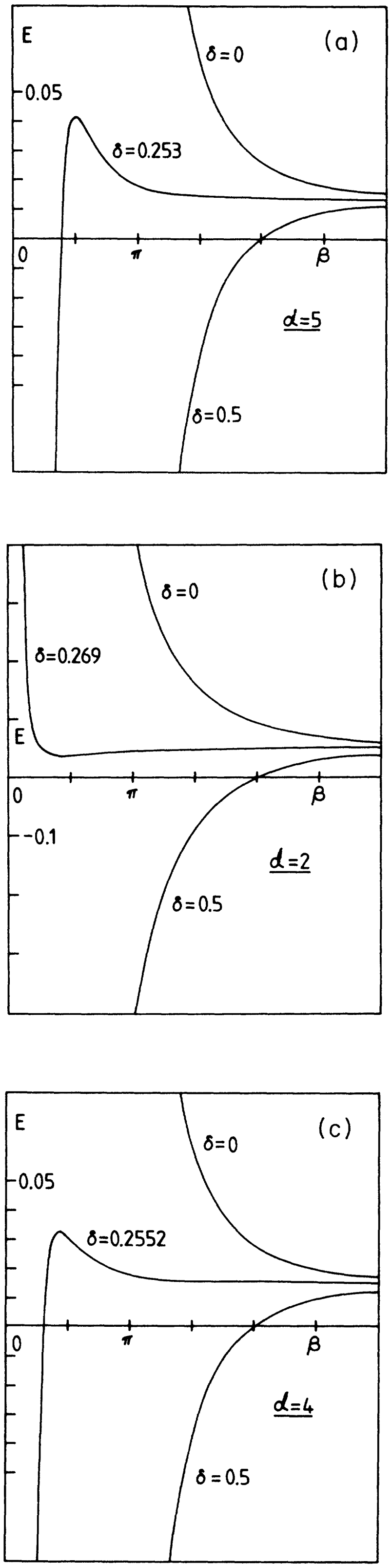

FIG. 2. Vacuum energy density E as a function of cone angle β for various values of the flux parameter δ and dimension d.

The (lengthy) expressions for the other terms will not be given here.

The coefficient of $\beta^{-(d+1)}$ in (17) vanishes for a specific value of δ, δ_A, and, when δ equals this value, $\langle T_{00}\rangle$ diverges as $\beta^{-(d-1)}$.

If $d=2$ then $\delta_A \approx 0.269\,170$, which is larger than δ_4. While if $d=4$, $\delta_A \approx 0.254\,881$, which is less than δ_4. This explains, numerically, the different shapes in these two, lowest even dimensions. For $d=2$, if δ lies between δ_4 and δ_A, there is a minimum which can be made to occur at any value of β from $+\infty$ to $-\infty$. If δ is outside the range mentioned, $\langle T_{00}\rangle$ has no turning points. The behavior is, thus, quite sensitive to the value of the flux parameter over this small range.

The same remarks that were made in the odd-d case regarding physical significance apply here equally, although there is now the possibility of an "equilibrium" value of β which is neither zero nor infinity.

I have concentrated on the analysis in this paper and hope to return with physical applications and also a discussion of other fields and finite temperatures in a future paper.

APPENDIX

Here we outline some mathematical points of incidental interest.

Regarding the higher-order Bernoulli polynomials mentioned in Sec. III, Barnes[17] defines a multiple ζ function by

$$\zeta_r(s,a\mid\omega)=\frac{i\Gamma(1-s)}{2\pi}\int_L \frac{e^{-az}(-z)^{s-1}}{\prod_{k=1}^{r}(1-e^{-\omega_k z})}dz$$

$$=\sum_{\mathbf{m}=0}^{\infty}(a+\mathbf{m}\cdot\omega)^{-s} \text{ if } \operatorname{Re} s>r\;, \qquad \text{(A1)}$$

where $\omega\equiv(\omega_1,\ldots,\omega_r)$ and $\mathbf{m}\equiv(m_1,\ldots,m_r)$. Barnes gives the rules for determining the (infinite) contour L, which loops around the origin.

Setting $a=-\frac{1}{2}\sum_k \omega_k \pm b$, one easily finds

$$\zeta_r\left[s,-\tfrac{1}{2}\sum\omega+b\mid\omega\right]+\zeta_r\left[s,-\tfrac{1}{2}\sum\omega-b\mid\omega\right]$$

$$=\frac{i\Gamma(1-s)}{4\pi}\int_L\frac{\cosh bz(-z)^{s-1}}{\prod_{k=1}^{r}\sinh(\frac{1}{2}\omega_k z)}dz\;.$$

If $s=1$ we see essentially the same integral as in Eq. (7) for W_N. Thus, W_N is the residue of $\zeta_{N+1}(s,a\mid\omega)$ at $s=1$ (up to a factor). Barnes evaluated this residue as the higher-order Bernoulli polynomial ${}_{N+1}S_1^{(2)}$. (His notation has not become the standard one.) Either from this result or from the standard generating function definition of these polynomials[13] we have, for the coefficient C_N,

$$C_N=\frac{(-1)^N}{N!}D_N^{(N+1)}(T\delta\mid\omega)\ ,$$

where the D are defined in terms of the B by

$$D_N^{(N+1)}(T\delta\mid\omega)=2^N B_N^{(N+1)}\left[\frac{N+T}{2}+T\delta\,\middle|\,\omega\right]$$

and the $(N+1)$-vector ω equals $(1,\ldots,1,T)$.

Barnes also gives the value of $\zeta_r(s,a\mid\omega)$ for all integral s. There are poles, when $s=1,\ldots,r$, whose residues are higher-order Bernoulli polynomials, as might be expected.

This ζ function has a number of interesting properties. For example, it satisfies the finite difference equation

$$\zeta_r(s,a+\omega_1\mid\omega)-\zeta_r(s,a\mid\omega)=-\zeta_{r-1}(s,a\mid\omega_2,\ldots,\omega_r)\ ,$$

where ω_1 stands for any component of ω. Iteration of this equation yields the result

$$\zeta_r(s,a+\omega_1+\cdots+\omega_r)-\sum_{*=1}^{r}\zeta_r(s,a+\omega_1+\cdots+*+\cdots+\omega_r)$$
$$+\sum_{*=1}^{r}\sum_{*=1}^{r}\zeta_r(s,a+\omega_1+\cdots+*+\cdots+*+\cdots+\omega_r)-\cdots+(-1)^{r-1}\sum_k\zeta_r(s,a+\omega_k)+(-1)^r\zeta_r(s,a)=a^{-s}\ .$$

In the first summation, the asterisk denotes that one of the ω's is to be omitted; in the second summation every two different pairs of ω's must be successively omitted, and so on. It is easily shown, conversely, that this result leads to (A1).

Another basic result is the multiplication theorem (or the transformation formula[17]). This is a generalization of the one for ordinary Bernoulli polynomials[13] which is the one we now give since it is of relevance. In terms of the $D_\nu(x)$ polynomials we have

$$D_\nu(x)=m^{\nu-1}\sum_{s=0}^{m-1}D_\nu\left[\frac{2x+2s+1-m}{2m}\right]\ . \qquad \text{(A2)}$$

From (10) this leads to a "sum rule" for $\langle T_{00}\rangle$ for odd d. It is convenient to write $E(\beta,\delta)=\langle T_{00}\rangle$. Then, changing the notation slightly,

$$E(q\beta,\delta)=\frac{1}{q}\sum_{p=1}^{q}E\left[\beta,\frac{2\delta+2p+1-q}{2q}\right],\quad q\in\mathbb{Z}\ . \qquad \text{(A3)}$$

In fact this equation is true for even d as well, the real reason being more fundamental than (A2).

It is easily shown from the contour integral (4) (or from the expression in terms of a sum over homotopy classes of paths given in Ref. 4) that

$$G_{q\beta,q\delta'}=\frac{1}{q}\sum_{p=1}^{q}G_{\beta,\delta'+p/q} \qquad \text{(A4)}$$

which immediately yields (A3).

Simple examples of formula (A3) are

$$E(2\beta,\delta)=\frac{1}{2}\left[E\left[\beta,\frac{2\delta+1}{4}\right]+E\left[\beta,\frac{2\delta+3}{4}\right]\right]\ ,$$

which can be used to check the numerical results, and the "zero" sum rule,

$$\sum_{p=1}^{q}E\left[\frac{2\pi}{q},\frac{p+1}{q}-\frac{1}{2}\right]=0\ .$$

Also, since the $E(2\pi,\delta)$ are known in closed form, we can find similar such forms when β is any integral multiple of 2π. Other applications of (A3) and (A4) will not be presented here.

Some considerations relevant to Eq. (A4) have been given by A. Y. Shiekh (Ph.D. thesis, Imperial College London, 1986 and Ref. 18).

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