Topological defects at finite temperature

D. Bazeia
Departamento de Física, CCEN, Universidade da Paraíba, CEP 58000, João Pessoa, Paraíba, Brazil

O. J. P. Éboli*
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

J. M. Guerra, Jr. and G. C. Marques
Departamento de Física Matemática, Instituto de Física da Universidade de São Paulo, C.P. 20516, CEP 01498, São Paulo, São Paulo, Brazil
(Received 3 April 1987)

We obtain the phase diagram of gauge theories by studying the influence of topologically nontrivial boundary conditions. For this reason, we develop a scheme for computing the free energy of topological defects at finite temperature. As an application, the free energy of topological defects for the minimal SU(5) model are evaluated in the semiclassical approximation.

I. INTRODUCTION

One of the most important problems one faces in field theory is the determination of the phase diagram of gauge theories. The difficulty in this context relies upon the question of how to distinguish the different phases of gauge theories when coupled to matter fields. Bricmont and Fröhlich[1] have proposed that the distinction (between different phases in gauge theories) is achieved by analyzing the free energies of "topological defects."

The defect free-energy approach relies upon the study of the change in the free energy that takes place when one forces a topological defect to appear in the system through the use of convenient boundary conditions. Comparing the free energies of the system, when one imposes different boundary conditions, one can learn about its phase diagram: when the free energy becomes insensitive to certain boundary conditions the system has reached a new "phase."

One can have a better picture of this method by applying it to the Ising model: the analogue of the free-energy defect is the surface tension. The surface tension is nothing but the free energy of the topological defect of the Ising model (domain walls). The temperature at which the surface tension vanishes is the critical temperature of the model, since one can prove that the spontaneous magnetization vanishes at the same temperature.[2]

Now suppose that the system, originally at a given phase at zero temperature, is in contact with a heat bath at temperature $T=\beta^{-1}$. At low temperatures, defects with positive free energy are rare or nonexistent (those which require an infinite amount of energy). However, for sufficiently high temperatures, quantum (entropy) effects come into play in such a way that the free energy of a given topological defect vanishes.[3,4] Hence, there is no energy cost to introduce an extra topological defect into the system, which implies that one has reached another phase of the theory—that is, the one in which the condensation of defects takes place.

Topologically nontrivial structures (defects) emerge in field theories whose symmetry are spontaneously broken. At the classical level, these defects correspond to topologically nontrivial solutions of the Euler-Lagrange equations.

In this paper we shall be concerned with the computation of defect free energy for non-Abelian gauge theories at finite temperature. Explicit results are derived in the one-loop approximation. We have illustrated how the scheme works by considering the minimal SU(5) model, which exhibits two types of topological defects: domain walls and magnetic monopoles. The extension of this method to other models and different defects is straightforward.

This paper is organized as follows. Section II deals with the general framework and especially with a formal expansion which allows us to implement the semiclassical approximation and an explicit separation of the zero- and finite-temperature terms of the free energy. In Sec. III we apply the scheme to obtain the free energy of the topological defects of the minimal SU(5) model. In the high-temperature limit we obtain closed expressions for the free energy of domain walls and magnetic monopoles. We end this paper with conclusions in Sec. IV. This paper is supplemented by two Appendixes.

II. FORMAL EXPRESSION FOR DEFECT FREE ENERGY

The partition function for a given gauge theory, whose Euclidean Lagrangian density is L, may be expressed as a functional integral[5]

$$Z(\beta)=N^{-1}(\beta)\oint[D\phi]\exp\left[-\int_0^\beta d\tau\int d^3\mathbf{x}[L-J(x)\phi(x)]\right]\times\text{gauge-fixing terms}\ , \tag{2.1}$$

where τ is the Euclidean time, ϕ stands for all fields in the theory, and the integral over the fields is subject to the following boundary condition on ϕ: $\phi(\mathbf{x},0)=\phi(\mathbf{x},\beta)$ for bosonic fields and $\phi(\mathbf{x},0)=-\phi(\mathbf{x},\beta)$ for fermionic fields.

N is a normalization constant which may be chosen such that $Z(\infty)=1$.

The free energy of the system is defined through the equations[6]

$$F(\beta,J)=-\beta^{-1}\ln Z\ , \tag{2.2}$$

$$M(x,J)\equiv M_J(x)=-\frac{\delta(\beta F)}{\delta J(x)}\ , \tag{2.3}$$

$$\Gamma(\beta,M_J)=F(\beta,J)+\beta^{-1}\int_0^\beta d\tau\int d^3\mathbf{x}\,M_J(x)J(x)\ . \tag{2.4}$$

$\Gamma(\beta,M_J)$ is the generating functional of one-particle-irreducible Green's functions and is the free energy of the field configuration M_J. The effective potential method analyzes Γ for constant field configurations M_J in order to obtain the phase diagram of the model.[6]

One can define the free energies of the different types of topological defects[1,4,7] by

$$F_M=-\beta^{-1}\ln\left[\frac{Z_M}{Z_V}\right]\ , \tag{2.5}$$

$$F_S=-\frac{\beta^{-1}}{L}\ln\left[\frac{Z_S}{Z_V}\right]\ , \tag{2.6}$$

and

$$F_W=-\frac{\beta^{-1}}{L^2}\ln\left[\frac{Z_W}{Z_V}\right]\ , \tag{2.7}$$

where F_W, F_S, and F_M are, respectively, the free energy for domain walls, strings, and magnetic monopoles. Usually a given model does not exhibit all the three different topological defects, so one must consider only the relevant ones. Z_M, Z_S, and Z_W stands for the partition function of the system evaluated when one imposes boundary conditions that force the existence of a magnetic monopole, string, and domain-wall defect in the system, while Z_{vac} is the partition function obtained using topologically trivial boundary conditions (vacuum sector). L is the size of the system.

The various thermodynamical functions can be written, in the one-loop approximation, as shown in Appendix A, as differences of the effective action of the theory evaluated at certain field configurations. Let $\Gamma(\phi)$ be the effective action of the theory and ϕ_V be the constant field configuration associated to the vacuum of the theory. In terms of the effective action one can write the effective potential

$$V_{\text{eff}}\equiv\frac{1}{L^3\beta}[\Gamma(\bar{\phi})-\Gamma(\phi_V)]\ , \tag{2.8}$$

where the overbar stands for constant field configurations.

Whereas for the defects that we are concerned with in grand unified theories (monopole, string, and wall) one has

$$F_M=[\Gamma(\phi_M)-\Gamma(\phi_V)]\ , \tag{2.9}$$

$$F_S=\frac{1}{L}[\Gamma(\phi_S)-\Gamma(\phi_V)]\ , \tag{2.10}$$

and

$$F_W=\frac{1}{L^2}[\Gamma(\phi_W)-\Gamma(\phi_V)]\ ; \tag{2.11}$$

that is, all thermodynamical parameters can be written as differences between the effective action computed at some special field-theoretical configurations and those associated to the vacuum of the theory. These special field-theoretical configurations, within the semiclassical scheme, are the defects associated to the classic solutions to the Euler-Lagrange equations of the model.

The general structure of $\Gamma[\beta,\phi_D(x)]$ is

$$\Gamma[\beta,\phi_D(x)]=\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\left[\int_0^\beta d\tau_j\int d^3x_j\phi_D(x_j)\right]\Gamma^{(n)}(\tau_1\mathbf{x},\ldots,\tau_n\mathbf{x}_n)\ , \tag{2.12}$$

where $\Gamma^{(n)}(\tau_1\mathbf{x}_1,\ldots,\tau_n\mathbf{x}_n)$ are the one-particle-irreducible Green's functions, ϕ_D stands for the fields associated to the defect. If one uses the Fourier transform of $\Gamma^{(n)}$, given by

$$\Gamma^{(n)}(\tau_1\mathbf{x}_1,\ldots,\tau_n\mathbf{x}_n)=\beta^{-n}\prod_{j=1}^{n}\sum_{n_j=-\infty}^{\infty}\int\frac{d^3\mathbf{k}_j}{(2\pi)^3}\tilde{\Gamma}^{(n)}(\omega_1\mathbf{k}_1,\ldots,\omega_n\mathbf{k}_n)\exp\left[-i\sum_{l=1}^{n}(\omega_l\tau_l+\mathbf{k}_l\cdot\mathbf{x}_l)\right]\ , \tag{2.13}$$

where $\omega_l=2\pi l\beta^{-1}$ and remembering that translational symmetry allows us to set

$$\tilde{\Gamma}^{(n)}(\{\omega_i\mathbf{k}_i\})=\beta(2\pi)^3\delta\left[\sum\omega_i\right]\delta^3\left[\sum\mathbf{k}_i\right]\bar{\Gamma}^{(n)}(\{\omega_i\mathbf{k}_i\})\ , \tag{2.14}$$

then, for static field configurations (those with which we will be concerned with in this paper), the general structure of $\Gamma(\beta,\phi_D)$ is

$$\Gamma(\beta,\phi_D)=\beta\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int d^3\mathbf{k}_j\tilde{\phi}_D(-\mathbf{k}_j)\overline{\Gamma}^{(n)}(\{\mathbf{k}_j,\omega_j=0\})\delta^3\left[\sum\mathbf{k}_j\right] . \tag{2.15}$$

The graphs that contribute to $\overline{\Gamma}^{(n)}$ will involve sums over the discrete ω_j which, once performed, yield a term independent of temperature plus one which has the full T dependence. This separation can always be implemented if one uses identities of the form

$$\beta^{-1}\sum_{n=-\infty}^{\infty}\frac{1}{\left[\dfrac{2n\pi}{\beta}\right]^2+z^2}=\frac{1}{2z}+\frac{1}{z(e^{\beta z}-1)} . \tag{2.16}$$

One can then split $\overline{\Gamma}^{(n)}$ into two parts:

$$\overline{\Gamma}^{(n)}(\{\mathbf{k}_i,\omega_i=0\})=\overline{\Gamma}_0^{(n)}(\{k_i\})+\overline{\Gamma}_T^{(n)}(\{\mathbf{k}_i\},\omega_i=0) , \tag{2.17}$$

where the second term contains all the T dependence. The general structure of this dependence can be inferred by making a change in all internal momenta integration variables. This change is just a replacement $\mathbf{p}\to\mathbf{p}'=\mathbf{p}\beta$. After this scaling in the internal momenta one can predict, from pure dimensional analysis, that $\overline{\Gamma}_T^{(n)}(\{\mathbf{k}_i,\omega_i=0\})$ have the structures[6]

$$\overline{\Gamma}_T^{(n)}(\{\mathbf{k}_i\},\omega_i=0)=\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[\frac{\mathbf{k}_i}{T},\frac{m}{T}\right] , \tag{2.18}$$

where $d(\gamma_n)$ is the superficial degree of divergence of a graph γ_n contributing to $\overline{\Gamma}$ and G_{γ_n} is dimensionless. Putting (2.15), (2.17), and (2.18) together, we have

$$\Gamma(\beta,\phi_D)=\Gamma_0(\phi_D)+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int d^3\mathbf{k}_j\tilde{\phi}_D(-\mathbf{k}_j)\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[\frac{\mathbf{k}_j}{T},\frac{m}{T}\right]\delta^3\left[\sum\mathbf{k}_j\right] , \tag{2.19}$$

where $\Gamma_0(\phi_D)$ is the effective action computed at the background field ϕ_D at zero temperature.

Using (2.9)–(2.11) and (2.19), the free energies of the various topological defects can then be written as

$$\begin{aligned}F^D(\beta)&=[\Gamma_0(\phi_D)-\Gamma_0(\phi_V)]\frac{1}{L^\alpha}\\&\quad+\frac{1}{L^\alpha}\left[\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{\infty}\int d^3\mathbf{k}_j\tilde{\phi}_D(-\mathbf{k}_j)\delta^3\left[\sum\mathbf{k}_j\right]\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[\frac{\mathbf{k}_j}{T},\frac{m}{T}\right]\right.\\&\quad\left.-\sum_n\frac{1}{n!}\phi_V^n\sum_{\gamma_n}T^{d(\gamma^n)}G_{\gamma_n}\left[0,\frac{m}{T}\right]L^3\right] ,\end{aligned} \tag{2.20}$$

where α is an index that, in accordance with (2.9)–(2.11), runs from 0 to 2.

To get a formal series for the free energy from any solution associated to a particular defect, we just introduce it in (2.20). Just for the sake of completeness we write the expression for the effective potential. From (2.8) and (2.19) it follows that

$$\begin{aligned}V_{\text{eff}}(\bar{\phi})&=\frac{1}{V}[\Gamma_0(\bar{\phi})-\Gamma_0(\phi_V)]\\&\quad+\sum_{n=1}^{\infty}\frac{1}{n!}(\bar{\phi}^n-\phi_V^n)\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[0,\frac{m}{T}\right] .\end{aligned} \tag{2.21}$$

Once the general formalism is set we shall apply it to our specific model: the minimal SU(5) grand unified theory (GUT).

III. PHASE DIAGRAM FOR THE MINIMAL SU(5) GUT[8]

We shall consider the minimal SU(5) GUT at finite temperature. Its Euclidean Lagrangian density is (in our calculations, we are assuming that the coupling constants are such that the phase transitions are expected to be of second order)

$$L=-\tfrac{1}{4}\text{Tr}(G_{\mu\nu}G_{\mu\nu})+\tfrac{1}{2}\text{Tr}[(D_\mu\Phi)^2]+V(\Phi) , \tag{3.1}$$

where Φ is the Higgs multiplet belonging to the adjoint representation,

$$\begin{aligned}V(\Phi)&=-\frac{\mu^2}{2}\text{Tr}(\Phi^2)+\frac{a}{4}[\text{Tr}(\Phi^2)]^2+\frac{b}{2}\text{Tr}(\Phi^4) ,\\G_{\mu\nu}&=\sum_{i=1}^{24}G_{\mu\nu}^i\frac{\lambda^i}{\sqrt{2}},\quad W_\mu=\sum_{i=1}^{24}W_\mu^i\frac{\lambda^i}{\sqrt{2}} ,\\\Phi&=\sum_{i=1}^{24}\phi^i\frac{\lambda^i}{\sqrt{2}},\quad D_\mu\Phi=\partial_\mu\Phi-\frac{ig}{\sqrt{2}}\text{Tr}[W_\mu,\Phi] ,\end{aligned} \tag{3.2}$$

and λ^i ($i=1,\ldots,24$) are the generators of SU(5) in the fundamental representation [normalized so that $\mathrm{Tr}(\lambda^i\lambda^j)=2\delta^{ij}$]. We also impose that $b>0$ and $a>-\frac{7}{15}b$. The notation used is the one in Ref. 9.

This model exhibits two different topological defects: domain walls and magnetic monopoles. The background field describing a domain wall is

$$\bar{\Phi}_W=\frac{\mu}{\sqrt{\lambda}}\tanh\left[\frac{\mu}{\sqrt{2}}x\right]\frac{\lambda_{24}}{\sqrt{2}},\quad \bar{W}^{\,\mathrm{cl}}_{\mu}=0\ , \tag{3.3}$$

with $\lambda=a+\frac{7}{15}b$. Note that this solution depends only on one spatial coordinate, which we choose to be the x one. The classical field configuration associated to a magnetic monopole satisfies the ansatz[10]

$$\begin{aligned}
&\bar{W}^{a}_{4}=0 \quad \text{for } +a=1,\ldots,24\ ,\\
&\bar{W}^{a}_{\mu}=0=\bar{\phi}^{a} \quad \text{for } a=1,2,\ldots,20,24\ ,\\
&\bar{W}_k^{\,2a}=\frac{1}{g}\epsilon^{akj}\frac{x^j}{r^2}F(r) \quad \text{for } a,k=1,2,3\ ,\\
&\bar{\phi}^{\,2a}=\frac{x^a}{r}\eta(r)\phi_V \quad \text{for } a=1,2,3\ ,
\end{aligned} \tag{3.4}$$

and the boundary conditions

$$F(r)\xrightarrow[r\to\infty]{}1,\quad \eta(r)\xrightarrow[r\to\infty]{}1\ .$$

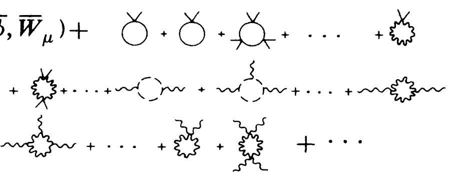
Let us exhibit the structure of the free energies of the system under these background fields in the one-loop approximation. In the zero-loop approximation one has, from (2.20),

$$F^{D}_{(0)}(\beta)=\frac{T[S(\phi_D)-S(\phi_V)]}{L^{\alpha}}\equiv\Delta\epsilon_{(0)}\ . \tag{3.5}$$

That is, in the zero-loop approximation, the free energy of the topological defect is just the difference between the classical action associated to the defect and the energy of vacuum. For the monopole, $\Delta\epsilon_{(0)}$, defined in (3.5), is its mass whereas for the domain wall $\Delta\epsilon_{(0)}$ is the mass per unit area.

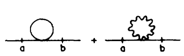
Within the one-loop approximation $\Gamma(\bar{\phi},\bar{W}_{\mu})$ will have the structure predicted from (2.12) which, for the example that we are considering, has the structure

$\Gamma(\bar{\phi},\bar{W}_{\mu})=S_{\mathrm{cl}}(\bar{\phi},\bar{W}_{\mu})+$

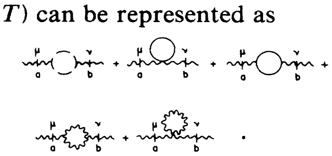


$$=S_{\mathrm{cl}}(\bar{\phi},\bar{W}_{\mu})-\frac{1}{2!}\Sigma^{ab}(T)\int_0^{\beta}d\tau\int d^3\mathbf{x}\,\bar{\phi}^{a}\bar{\phi}^{b}-\frac{1}{2!}\Pi^{ab}_{\mu\nu}(T)\int_0^{\beta}d\tau\int d^3\mathbf{x}\,\bar{W}^{a}_{\mu}\bar{W}^{b}_{\nu}+\cdots\ . \tag{3.6}$$

S_{cl} is the classical action associated with the background field, and $\Sigma^{ab}(T)$ can be represented graphically as

$\Sigma^{ab}(T)=$  , (3.7)

whereas $\Pi^{\mu\nu}_{ab}(T)$ can be represented as

$\Pi^{ab}_{\nu\mu}(T)=$  . (3.8)

The wavy, solid, and dotted lines stand, respectively, for the gauge bosons, Higgs, and ghost fields (for the fluctuations we are working in the Landau gauge). $\Pi^{ab}_{\mu\nu}$ can be identified as the polarization tensor for zero external momenta.[11] Following our earlier prescription (2.17) we can also write

$$\Sigma^{ab}(T)=\Sigma^{ab}_0+\bar{\Sigma}^{\,ab(n)}_{T}(\{\mathbf{k}_i\},\omega_i=0) \tag{3.9}$$

$$\Pi^{ab}_{\mu\nu}(T)=\Pi^{ab}_{\mu\nu 0}+\bar{\Pi}^{\,ab}_{\mu\nu}(T)\ . \tag{3.10}$$

First of all one notes, looking at (3.6), the appearance of ultraviolet divergences. These, however, can be treated, as usual, by adding appropriate renormalization counterterms which are just the usual ones at zero temperature.[6] This means that the zero-temperature renormalization scheme suffices for getting a finite expression to free energies of topological defects. Substituting (3.6) into (2.15), one can obtain the topological defect free energies of the SU(5) model. For a wall one has

$$F_{\mathrm{wall}}(T)=\Delta\epsilon_W-\frac{1}{2!}\frac{\bar{\Sigma}^{\,24,24}(T)}{L^2}\int_0^{\beta}d\tau\int d^3\mathbf{x}[\bar{\phi}^{\,W}_{24}(x)\bar{\phi}^{\,W}_{24}(x)-\bar{\phi}_V^{\,2}]+\cdots\ , \tag{3.11}$$

where $\Delta\epsilon_W$ is the energy density of the wall taking into account quantum corrections at zero temperature up to one loop; $\bar{\phi}^{W}_{24}(x)$ is given in (3.3), $\bar{\phi}_V=(\mu/\sqrt{2\lambda})\lambda_{24}$, $\bar{\Sigma}^{24,24}(T)$ is given by (3.9), and the dots represent one-loop contributions not included explicitly in (3.11).

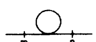
On the other hand, for the magnetic monopole one obtains

$$F_M(T)=M-\frac{1}{2!}\bar{\Sigma}^{ab}(T)\int_0^\beta d\tau\int d^3\mathbf{x}[\bar{\phi}^a(\mathbf{x})\bar{\phi}^b(\mathbf{x})-\phi_V{}^2\delta_{a24}\delta_{b24}]$$
$$-\frac{1}{2!}\bar{\Pi}^{ab}_{\mu\nu}(T)\int_0^\beta d\tau\int d^3\mathbf{x}\,\overline{W}^a_\mu(\mathbf{x})\overline{W}^b_\nu(\mathbf{x})+\cdots\ , \tag{3.12}$$

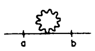
where now M stands for the renormalized mass of the monopole at the one-loop level, $\bar{\Sigma}^{ab}(T)$ and $\Pi^{ab}_{\mu\nu}(T)$ are given in (3.9) and (3.10), the fields $\bar{\phi}^a$ and $\overline{W}^a_\mu$ are defined in (3.4), and the dots represent contributions that are not shown in (3.12).

One could go further and write down a similar expression for all the one-loop graphs for the topological structures of the SU(5) model. However, instead of doing this explicitly, we will just analyze the high-temperature limit of the free energy. In this limit, the form (2.20) is particularly useful, since the leading power in T of series (2.20) is easily obtained. Property (2.18) permits us to identify these contributions, which are the ones with higher superficial degrees of divergence. These contributions are precisely the ones we have written explicitly.

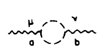
In the high-temperature limit, the graphs appearing in (3.7) and (3.8) yield



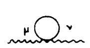
$$=-(26a+\tfrac{282}{15}b)\frac{T^2}{12}\delta^{mn}\ , \tag{3.13}$$



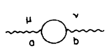
$$=-\tfrac{5}{4}g^2T^2\delta^{ab}\ , \tag{3.14}$$



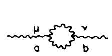
$$=\begin{cases}-\tfrac{5}{24}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2,3,\\ \tfrac{5}{24}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu \text{ and/or } \nu=4,\end{cases} \tag{3.15}$$



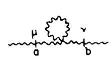
$$=-\tfrac{5}{12}g^2T^2\delta^{ab}\delta_{\mu\nu}\ , \tag{3.16}$$



$$=\begin{cases}\tfrac{5}{12}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2,3,\\ -\tfrac{5}{12}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu \text{ and/or } \nu=4,\end{cases} \tag{3.17}$$



$$=\begin{cases}\tfrac{5}{4}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2,3,\\ -\tfrac{5}{4}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu \text{ and/or } \nu=4,\end{cases} \tag{3.18}$$



$$=-\tfrac{25}{24}g^2T^2\delta^{ab}\delta_{\mu\nu}\ . \tag{3.19}$$

From (3.13)–(3.19), (3.7), and (3.8) we have the asymptotic expressions for $\Sigma^{cd}(T)$ and $\Pi^{cd}_{\mu\nu}(T)$:

$$\bar{\Sigma}^{cd}(T)=-\frac{T^2}{4}[5g^2+\tfrac{1}{3}(26a+\tfrac{282}{15}b)]\delta^{cd}\ , \tag{3.20}$$

$$\bar{\Pi}^{cd}_{\mu\nu}(T)=-\tfrac{35}{12}g^2T^2\delta^{cd}\delta_{\mu4}\delta\nu4\ . \tag{3.21}$$

One obtains from (3.11)–(3.21) the high-temperature behavior

$$F_{\text{wall}}(T)=\Delta\epsilon_w+\frac{T^2}{8}[5g^2+\tfrac{1}{3}(26a+\tfrac{282}{15}b)]\int dx[\phi_W{}^2(x)-\phi_V{}^2]\ , \tag{3.22}$$

$$F_M(T)=M+\frac{T^2}{8}[5g^2+\tfrac{1}{3}(26a+\tfrac{282}{15}b)]\int d^3\mathbf{x}\left[\sum_{a=1}^{24}\bar{\phi}^a\bar{\phi}^a-\phi_V{}^2\right]+\tfrac{35}{24}g^2T^2\int d^3\mathbf{x}\sum_{a=1}^{24}\overline{W}^a_4\overline{W}^a_4\ . \tag{3.23}$$

At first sight, the appearance of the term $\int d^3\mathbf{x}(\overline{W}^a_4)^2$ in the last expression could seem to be a problem: (3.23) is not explicitly gauge invariant with respect to gauge transformations of the background fields (we have just fixed the gauge for the fluctuations[7]). At this point we are forced to adopt a "physical" gauge with respect to the magnetic monopole degrees of freedom or generalize our calculation to include a Jacobian for the ghostlike degrees of freedom associated with the magnetic monopole.[12] When considering the background (3.4) for the magnetic monopole we have decided for the former strategy, since this background satisfies $\overline{W}_4^a=0$. Therefore, in this gauge we have for the monopole

$$F_M(T)=M+\frac{T^2}{8}[5g^2+\tfrac{1}{3}(26a+\tfrac{282}{15}b)]\times\int d^3\mathbf{x}\left[\sum_{a=1}^{24}\bar{\phi}^a\bar{\phi}^a-\phi_V{}^2\right]. \quad (3.24)$$

The substitution of (3.3) into (3.22) leads to

$$F_{\text{wall}}(T)=\Delta\epsilon_W-\frac{T^2}{12}\frac{\mu\sqrt{2}}{\lambda}(26a+\tfrac{282}{15}b+15g^2)\ , \quad (3.25)$$

whereas the substitution of (3.4) into (3.24) implies

$$F_M(T)=M-\frac{T^2}{8}[5g^2+\tfrac{1}{3}(26a+\tfrac{282}{15}b)]\times 4\pi\int_0^\infty r^2dr\,\phi_V{}^2[1-\eta^2(r)]\ . \quad (3.26)$$

IV. CONCLUSIONS

As is well known, one way of characterizing the phase transitions in the Ising model is by studying the spontaneous magnetization as a function of the temperature. At temperatures smaller than the critical one (T_c) the spontaneous magnetization is nonvanishing, indicating that the symmetry spin up $\rightleftarrows$ down is spontaneously broken. For temperatures above T_c, this symmetry is restored since the magnetization vanishes.

The effective-potential method in field theory implements an approach similar to the one above. We evaluate the thermal expectation value of the field $\langle\phi\rangle_T$ which allow us to classify the phases of the system since $\langle\phi\rangle_T$ is nonzero for spontaneously broken symmetries, while it vanishes when the symmetry is restored.

On the other hand we can also draw the phase diagram of the Ising model by analyzing the influence of boundary conditions on the properties of the system.[2] At high temperatures there is no long-range order so that the boundary conditions are irrelevant. However, at low-temperatures (below T_c), the existence of long-range order implies that the properties of the system will depend upon the choice of boundary conditions.

This procedure for characterizing the existence of phase transitions can also be applied to field theory.[1,3,4,7] This is accomplished by studying the free energy of the topological defects that can occur in the system. As we have shown, the computation of this thermodynamical parameter is related to the effective action for the field configuration associated to the topological defect.

We exemplified how this method works by computing the free energy of domain walls and magnetic monopoles, in the high-temperature limit, for the minimal SU(5) model. The temperatures in which these free energies vanish indicate the occurrence of phase transitions. These critical temperatures, for the minimal SU(5) model, are

$$T_W{}^2=\frac{60\mu^2}{\frac{225}{2}g^2+13(15a+7b)+50b}\ , \quad (4.1)$$

$$T_M{}^2=\frac{8M}{[5g^2+\frac{1}{3}(26a+\frac{282}{15}b)]4\pi\phi_V{}^2\int_0^\infty dr\,r^2[1-\eta^2(r)]}\ , \quad (4.2)$$

where we have taken for $\Delta\epsilon_W$ in (3.25) its classical values $\Delta\epsilon_W\simeq\Delta\epsilon_0=(2\sqrt{2}/3)\mu^2/\lambda$.

Since we evaluate the topological defect-free energies [(3.25) and (3.26)] in the one-loop approximation we should pay attention to the reliability of this perturbative skin. As is well known,[6] perturbation theory breaks down at high temperatures. However, from the study of the effective potential, we know that the one-loop approximation is a good indication of the underlying physics of the problem. Certainly, higher-order corrections modify the results [(4.1) and (4.2)] but corrections are expected to be of order[6] λT_c, where T_c is the critical temperature obtained by using the effective potential method. We used only the one-loop approximation in our examples since the evaluation of higher-order corrections is beyond the scope of this paper.

T_W has a simple interpretation, as pointed out in Ref. 4. The point is, since the minima of the effective potentials at T_W enter into the region in which the effective potential develops an imaginary part (see Appendix B for this), this temperature is just the highest one for which the description of the system in terms of perturbative constant field configuration makes sense (and consequently the perturbative effective potential). In Appendix B we show further that if the phase transition is of second order, then one cannot avoid the perturbative effective potential becoming complex at the minimum for temperatures sufficiently high.

Moreover, at T_W the functional integral for Z starts to be dominated by the configurations associated to the domain walls (or bubblelike configurations) since they have the least free energy. Therefore at temperatures greater or equal to T_W the restoration of symmetry occurs since

$$\langle\phi\rangle=\int dx\,\phi_D(x)=0\ .$$

It would be interesting to know if T_W and T_M are equal or which one is smallest. If these temperatures are not equal it is important to know how to evaluate the free energy of a given topological defect in the presence of a condensate of the other. (We met before this kind of problem in the determination of the phase transition for the Z_n-symmetry spin and gauge theories.[13])

From the study of the defect-free energy, we can draw the following physical picture. At low temperatures, topological defects are scarce since the free energy changes significantly when a defect is forced into the system by imposing appropriate boundary conditions. On the other hand, at high temperatures, there is an abundance of defects since the introduction of an additional defect does not much affect the system.

Therefore, for sufficiently high temperatures we expect the system to be in a new phase which is characterized by a condensate of topological defects. The existence of this phase may have far-reaching consequences to cosmology. For instance, the existence of a magnetic monopole condensate is usually accompanied by electrical charge confinement which radically changes our view of the early Universe. Moreover, relics from a primordial condensate of topological defects might be responsible for the generator of the contrast density, and consequently, the large structure of the Universe.[14]

ACKNOWLEDGMENTS

We would like to thank C. Schwartz for a critical reading of the manuscript. O.J.P.E. is grateful to the Departamento de Física Matemática—IFUSP, for their kind hospitality.

APPENDIX A

In this appendix we will justify expressions (2.9)–(2.11) that give the (Gibbs) free energies with respect to three different backgrounds. Although, within the one-loop approximation, our expressions given results that are by now standard and can be found in textbooks,[15] we present this derivation due to the fact that it is fairly general and is just an extension, to finite temperature, of the background-field method.[16]

Assume that ϕ_0 is a generic field configuration and let us compute the thermodynamical properties of the system in the presence of such a background field. This should be inferred from the functional $Z[J,\phi_0]$ defined by

$$Z^0[J,\phi_0]=\int D[\phi]\exp\left[-S[\phi-\phi_0]+\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\phi(x)\right] . \tag{A1}$$

By means of a change of variables one can write

$$Z^0[J,\phi_0]=Z[J]\exp\left[\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\phi_0(x)\right] . \tag{A2}$$

From (A2) it follows that

$$W^0[J,\phi_0]=W[J]-\beta^{-1}\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\phi_0(x) , \tag{A3}$$

where $W^0(W)$, $Z^0(Z)$ stands for the thermodynamical functions evaluated with (without) the background field.

By using the definition (2.4) it follows that the Gibbs free energy in the presence of the background field (ϕ_0) is given by

$$\Gamma^0[\bar\phi_0,\phi_0]\equiv W^0[J,\bar\phi_0] -\beta^{-1}\int_0^\beta d\tau\int d^3\mathbf{x}\frac{\delta(\beta W^0)}{\delta J(x)}J(x) , \tag{A4}$$

where $\bar\phi_0=\delta(\beta W_0)/\delta J$.

By substituting (A3) into (A4) it follows that

$$\Gamma^0(\bar\phi_0,\phi_0)=W[J] -\beta^{-1}\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)[\bar\phi_0(x)+\phi_0] , \tag{A5}$$

consequently if one derives (A5) with regard to J one obtains

$$\frac{\delta W}{\delta J}=\bar\phi_0+\phi_0 . \tag{A6}$$

Being $W[J]$ and $\Gamma[\bar\phi]$ the generating functionals in the absence of the background field one gets, from (A6), the relationship

$$\bar\phi_0=\bar\phi-\phi_0 . \tag{A7}$$

If one substitutes (A7) into (A5) one then obtains

$$\Gamma^0(\bar\phi_0,\phi_0)=W[J]-\beta^{-1}\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\bar\phi(x) \equiv\Gamma(\bar\phi)\equiv\Gamma(\bar\phi_0+\phi_0) . \tag{A8}$$

Expression (A8) is well known within the context of the background-field method; that is, the generating function for the theory in the presence of the background can be obtained from the generating functional without the background field computed just making the replacement $\bar\phi\to\bar\phi_0+\phi_0$.

The free energy in the presence of the background

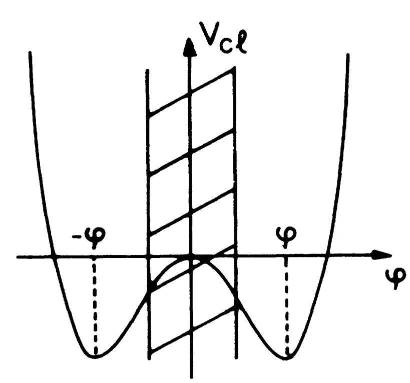


FIG. 1. $V_{\rm cl}(\phi)$. The region between the dashed lines is the one where $V''_{\rm cl}<0$.

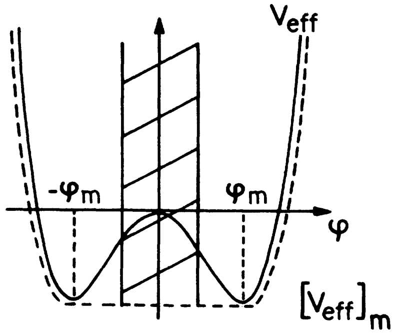


FIG. 2. The solid (dashed) line stands for $V_{\rm eff}$ $[(V_{\rm eff})_M]$. The region between the vertical lines is the one for which $V''_{\rm cl}<0$.

field is

$$F^0(\beta)\equiv \lim_{J\to 0} W^0[J,\phi_V]$$
$$\equiv \lim_{J\to 0}\left[\Gamma^0(\bar\phi^0,\phi^0)+\beta^{-1}\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\bar\phi_0\right] . \tag{A9}$$

Finally, one notes that if ϕ_0 is a particular solution of the classical equation

$$\left.\frac{\delta\Gamma}{\delta\phi}\right|_{\phi=\phi_c}=0 , \tag{A10}$$

that is,

$$\phi_0=\phi_c=\bar\phi , \tag{A11}$$

then in the limit $J\to 0$ (A11) leads to $\bar\phi_0=0$. Under this circumstance it follows from (A9) and (A8) that

$$F(\beta,\phi_c)=\Gamma[\phi_c] ; \tag{A12}$$

i.e., the free energy of the system in the presence of the background field ϕ_c satisfying the classical equation (A10) is given by the effective action computed at this configuration. If Γ is computed at the zero-loop level, (A10) corresponds to the classical Euler-Lagrange equations. This is precisely the situation that we are interested in the semiclassical approximation.

APPENDIX B

Usually one evaluates perturbatively the effective potential in order to know the different phases of the model. However, the perturbative effective potential exhibits some problems such as nonconvexity (and imaginary parts).[4,17] Although these problems can be solved at zero temperature by means of a Maxwell construction,[4,17] this is not always true when one works at finite temperature, as we shall show. If the phase transition is expected to be of second order or very weak first order, then one cannot avoid the effective potential becoming complex at the minimum for sufficiently high temperatures.

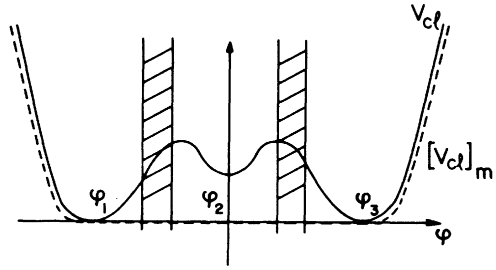


FIG. 3. The solid (dashed) line stands for V $[(V_{\rm eff})_M]$. The regions between the vertical lines are the ones for which $V''<0$.

Our starting point is

$$Z(J)=\int D[\phi]\exp\left[-\int_0^\beta d\tau\int d^3\mathbf{x}(L-J\phi)\right] , \tag{B1}$$

where L is the effective Lagrangian for the field ϕ; that is, we have already integrated all the other degrees of freedom. For high temperatures L can be written as[18]

$$L=\tfrac{1}{2}(\partial_\mu\phi)^2+V(\phi) . \tag{B2}$$

Let us, initially, analyze the case in which the phase transition is expected to be of second order. In this situation, $V(\phi)$ is well described by $V_{\rm cl}(\phi)$. Without loss of generality we are going to consider the minimal SU(5) model and evaluate the effective potential for fields $\hat\Phi=\phi{\rm diag}(1,1,1,-\frac{3}{2},-\frac{3}{2})$. In this case $V_{\rm cl}(\phi)$ is shown in Fig. 1.

The Maxwell construction for $V_{\rm eff}(\phi)$ is obtained by considering the contribution of all the local minima of $V_{\rm cl}-J\phi$ to $Z(J)$ (Ref. 17). This procedure yields

$$(V_{\rm eff})_M=\begin{cases}V_{\rm eff}(\phi) & \text{for } |\phi|\ge\phi_M,\\ V_{\rm eff}(\phi_M) & \text{for } |\phi|\le\phi_M,\end{cases} \tag{B3}$$

where $(V_{\rm eff})_M$ is the Maxwell construction for the effective potential, $V_{\rm eff}$ is the result that one obtains when just the global minimum of $V_{\rm cl}-J\phi$ is considered, and ϕ_M is the positive of the minimum of $V_{\rm eff}$ (see Fig. 2). At temperatures low enough ϕ_M is outside the region where $V''_{\rm cl}<0$ and $(V_{\rm eff})_M$ is real. However, for sufficiently high temperatures, ϕ_M lies in the region of $V''_{\rm cl}<0$ and $(V_{\rm eff})_M$ is complex. Since ϕ_M goes to zero continuously, $(V_{\rm eff})_M$ becomes complex before ϕ_M vanishes. Thus, one cannot trust the perturbative effective potential when the phase transition is expected to be of second order.

For strong first-order phase transitions, the situation is completely different since V differs a lot from $V_{\rm cl}$. In this situation $\phi_2=0$ is a local minimum (if T is high enough) as shown in Fig. 3. The Maxwell construction for the effective potential yields[17]

$$(V_{\rm eff})_M=\begin{cases}V_{\rm eff}(\phi) & \text{for } |\phi|\ge\phi_M,\\ V_{\rm eff}(\phi_M) & \text{for } |\phi|\le\phi_M.\end{cases} \tag{B4}$$

As the temperature is raised, ϕ_M jumps to zero without passing the region of $V''<0$. Therefore $(V_{\rm eff})_M$ is always real, and so, reliable.

*Permanent address: Departamento de Física, Universidade Estadual Paulista Júlio de Mesquita Filho, C.P., 178, CEP 13500, Rio Claro, São Paulo, Brazil.

[1]G. Mack and H. Moyer, Nucl. Phys. **B200**, 249 (1982); J. Bricmont and J. Fröhlich, *ibid.* **B230**, 407 (1984).
[2]J. L. Lebowitz and C. E. Pfister, Phys. Rev. Lett. **46**, 1031 (1981).
[3]I. Ventura, Phys. Rev. B **24**, 2812 (1981).
[4]R. J. Rivers, Z. Phys. C **22**, 137 (1984); C. A. A. Carvalho, G. C. Marques, A. J. Silva, and I. Ventura, Nucl. Phys. **B265**, 45 (1986); D. Bazeia, C. A. A. Carvalho, O. J. P. Éboli, G. C. Marques, A. J. Silva, and I. Ventura, Phys. Rev. D **31**, 1411 (1985).
[5]C. Bernard, Phys. Rev. D **9**, 3312 (1974).
[6]S. Weinberg, Phys. Rev. D **9**, 3357 (1974); L. Dolan and R. Jackiw, *ibid.* **9**, 3320 (1974).
[7]D. Bazeia, C. A. A. Carvalho, O. J. P. Éboli, and G. C. Marques, Phys. Rev. D **32**, 3256 (1985).
[8]H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32**, 938 (1974).
[9]M. Daniel and C. E. Vayonakis, Nucl. Phys. **B180**, 301 (1981).
[10]For a review, see J. Preskill, Annu. Rev. Nucl. Part. Sci. 461 (1984).
[11]D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[12]G. 't Hooft, Nucl. Phys. **B190**, 455 (1981).
[13]M. B. Einhorn, R. Savit, and E. Rabinovici, Nucl. Phys. **B170**, 16 (1980).
[14]J. M. Guerra, Jr., G. C. Marques, I. Ventura, and S. Rodrigues (unpublished).
[15]D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).
[16]L. F. Abbott, Acta Phys. Pol. **B13**, 33 (1982).
[17]Y. Fujimoto, L. O'Raifeartaigh, and G. Parravicini, Nucl. Phys. **B212**, 268 (1983); C. M. Bender and F. Cooper, *ibid.* **B224**, 403 (1983); F. Cooper and B. Freedman, *ibid.* **B239**, 459 (1984).
[18]C. M. Bender, F. Cooper, B. Freedman, and R. W. Haymaker, Nucl. Phys. **B256**, 653 (1985).