Reduction formulas for coherent states

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The Lehmann-Symanzik-Zimmermann reduction formulas consistent with the use of asymptotic coherent states to model external fields are developed for the case that the in-state external field differs from the out-state external field. It is shown for the specific choice of coherent state considered that the reduction formulas deviate from the standard forms presented in textbooks. The coherent states are not reduced directly but instead are incorporated into the perturbative representation of the time-ordered product. An effective potential created by the coherent state is found which self-consistently determines the form of the asymptotic field. An application is made to the case of nonrelativistic quantum electrodynamics in a laser pulse.

I. INTRODUCTION

In the Lehmann-Symanzik-Zimmermann[1] (LSZ) reduction technique the spectrum of the scattering matrix is constructed from the large-time or asymptotic form of the interacting field Ψ^α. In textbook presentations[2] it is assumed that the asymptotic field obeys, at least in the weak limit, a free-field equation. Particle states are then defined by smearing the asymptotic field with a suitable wave packet and allowing this operator to act on the Fock vacuum associated with the free field. Once reduced, the time-ordered product of fields is then given a perturbative representation in terms of the same free-field spectrum.

It is another standard assumption that the asymptotic states being reduced possess a definite number of particles. It is the intent of this paper to examine the case that the asymptotic states possess an indeterminate number of particles and to formulate the reduction formulas for such a case. To be specific, it will be assumed that the asymptotic states are coherent states.[3] These states are realized by a (pseudo)unitary operator acting on the Fock vacuum to create a Poisson distribution in the particle number. The standard form for such a state is

$$
\begin{aligned}
| f,t \rangle &= V^{-1}(t) \, | 0 \rangle \\
&= \exp \left\{ -i \int d^3x [\dot{f}_\alpha(\mathbf{x},t)\phi^\alpha(\mathbf{x},t) - f_\alpha(\mathbf{x},t)\dot{\phi}^\alpha(\mathbf{x},t)] \right\} | 0 \rangle \ , \qquad (1.1)
\end{aligned}
$$

where

$$[\phi_\alpha(\mathbf{x},t),\dot{\phi}_\beta(\mathbf{y},t)] = i\delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}) \qquad (1.2)$$

and $| 0 \rangle$ is cyclic with respect to the algebra of the field ϕ. The exact nature of the field ϕ_α and the restrictions on the function f_α will be made explicit in the next section. The state (1.1) has the property that

$$\langle f,t \mid \phi^\alpha(\mathbf{x},t) \mid f,t \rangle = f^\alpha(\mathbf{x},t) \ . \qquad (1.3)$$

Property (1.3) is ideal for modeling systems where the field develops a nonvanishing expectation value asymptotically, as in the case of either the vector potential of electrodynamics in the presence of an external field[4] or a nonlinear field in the presence of a soliton.[5] In addition, squeezed states,[6] a variant of coherent states, are of current interest in quantum optics.

It is also possible to define states with a lower bound on particle number by applying the operator V to a standard Fock state of fixed particle number. Such a state takes the form

$$| f,p_1, \ldots ,p_n \rangle = V^{-1}(t) \, | p_1, \ldots ,p_n \rangle \ , \qquad (1.4)$$

so that the Poisson distribution of particles is shifted by n of the field quanta associated with the fields of the theory.

Asymptotic states of the form (1.4) will be given an exact definition and reduced in the next section. Only coherent states of Bose fields will be considered. Matrix elements of such states will be given a perturbative representation and it will be shown how such a perturbative representation leads to a natural self-consistency condition for the spectral decomposition of the asymptotic fields in the presence of the coherent-state operator. In Sec. III an application of the results of Sec. II will be made to nonrelativistic electron scattering in the presence of a laser pulse to calculate both the cross section of laser-electron scattering and the change in amplitude induced on the incoming field by the scattering.

II. REDUCTION FORMULAS

For the sake of clarifying notation and reminding the reader of the salient features of scattering theory the LSZ formalism will be briefly reviewed. The interpolating or full interacting field Ψ^α is assumed to obey the equation of motion

$$\dot{\Psi}^\alpha = i[H[\Psi],\Psi^\alpha] \ , \qquad (2.1)$$

where H is the full Hamiltonian written in terms of the interacting field. The weak asymptotic limit of the interacting field is written

$$\underset{t\to t_\pm}{\text{w-lim}}\,(\Psi^\alpha - \sqrt{Z}\,\phi_{(\text{in})(\text{out})}) = 0 \;, \qquad (2.2)$$

where the subscripts in and out correspond to the respective time limits t_- and t_+, which lie in the remote past and future, while Z is the wave-function renormalization constant. The in and out fields are used to construct asymptotic particle states. These states are then reduced by one of the standard LSZ techniques, the choice of which depends on the generic type of field quantum being considered. The in and out fields are related by the S matrix

$$\phi^\alpha_{\text{in}} = S\phi^\alpha_{\text{out}}S^{-1} \;, \qquad (2.3)$$

so that ϕ^α_{in} and ϕ^α_{out} coincide only in a trivial theory.

It is a standard textbook assumption that ϕ^α_{in} possesses a known spectrum, so that ϕ^α_{in} can be related to some field whose structure is known. The standard assumption, discussed in the Introduction, is that ϕ^α_{in} obeys a free-field equation. It will be assumed in this paper that the asymptotic field satisfies a linear homogeneous equation of motion which determines the modes of the field. For a relativistic Bose field ϕ_α this takes the general form

$$\partial^2\phi^\alpha_{\text{in}}(x) + V_{\alpha\beta}(x)\phi^\beta_{\text{in}}(x) = 0 \;, \qquad (2.4)$$

where $V_{\alpha\beta}(x)$ is some self-adjoint potential. It is an assumption of this paper that $V_{\alpha\beta}$ is such that the solutions of (2.4) constitute a complete basis of the space of functions which are integrable and at least piecewise continuous, and that these possess nondamped time dependence. A similar statement may be made for the Fermi fields of the theory. In the nonrelativistic case this corresponds to solving the Schrödinger equation for the modes of the field, while in the relativistic case an equation similar to (2.4) may be obtained by iterating a Dirac equation with a spinor-valued potential. Normally, in order to yield a sensible theory, $V_{\alpha\beta}$ is assumed to be a static self-adjoint potential so that (2.4) may be reduced to a Sturm-Liouville problem for the eigenmodes of the theory. However, some time-dependent potentials may yield a sensible spectrum (see Sec. III), and so such a possibility will not be *a priori* excluded.

Once found, the complete set of functions can be used to expand the fields in a manner which allows the canonical equal-time (anti)commutation relations to be satisfied in the same way the plane waves are used in the absence of a potential. For the sake of simplicity the complete set of eigenfunctions in the scalar case will be denoted $\{u(\mathbf{k},x)\}$, so that it is assumed that no bound states are present. A discrete (perhaps denumerably infinite) set of bound-state solutions to (2.4) could be incorporated just as easily if they are found to exist. These functions are assumed to be orthonormal. In the relativistic case this means that

$$\int d^3x[\dot{u}^*(\mathbf{k},x)u(\mathbf{p},x) - u^*(\mathbf{k},x)\dot{u}(\mathbf{p},x)] = i\delta^3(\mathbf{k}-\mathbf{p}) \;. \qquad (2.5)$$

The scalar field is then expanded as

$$\phi_{\text{in}}(x) = \int d^3k[u(\mathbf{k},x)b_{\text{in}}(\mathbf{k}) + u^*(\mathbf{k},x)b^\dagger_{\text{in}}(\mathbf{k})] \;. \qquad (2.6)$$

The limit (2.2) allows the definition of incoming and outgoing coherent-state operators. From now on subscripts on the fields will be suppressed for notational simplicity. The coherent-state operators are written

$$V_{\text{in}}(t) = \exp\left[i\int d^3x(\dot{f}\phi_{\text{in}} - f\dot{\phi}_{\text{in}})\right] \qquad (2.7\text{a})$$

and

$$V_{\text{out}}(t) = \exp\left[i\int d^3x(\dot{f}\phi_{\text{out}} - f\dot{\phi}_{\text{out}})\right] \;, \qquad (2.7\text{b})$$

where, for simplicity, π_{in}, the momentum canonically conjugate to ϕ_{in}, has been assumed to be $\dot{\phi}_{\text{in}}$. In the case of the photon field in electrodynamics there are gauges where this will not be true. In that case the proper form for π_{in} should be substituted for $\dot{\phi}_{\text{in}}$. In terms of the normal modes of the theory the operator of (2.7) takes the form

$$V_{\text{in}}(t) = \exp\left[\int d^3k[f(\mathbf{k},t)b_{\text{in}}(\mathbf{k}) - f^*(\mathbf{k},t)b^\dagger_{\text{in}}(\mathbf{k})]\right] \;, \qquad (2.8)$$

where

$$f(\mathbf{k},t) = i\int d^3x(\dot{f}u_{\mathbf{k}} - f\dot{u}_{\mathbf{k}}) \;. \qquad (2.9)$$

The $f(\mathbf{k},t)$ constitute the "Fourier" transform of the function f with respect to the complete set of functions solving (2.4). For the remainder of this paper it will be assumed that the function $f(x)$ is such that

$$\frac{\partial}{\partial t}f(\mathbf{k},t) = 0 \;, \qquad (2.10)$$

so that the function may be expanded in terms of the eigenfunctions of the field ϕ_{in} in such a way that V_{in} is manifestly time independent. This is guaranteed if $f(x)$ is chosen to satisfy the same linear equation as ϕ_{in}. The spectrum of the in field ϕ_{in} will still be referred to as a Fock spectrum.

The asymptotic particle states of the theory, as discussed in the Introduction, are given by applying the coherent-state operator $V_{\text{in}}{}^{-1}$ to the Fock representation associated with ϕ_{in}. The in and out states are thus written

$$\begin{aligned} |\alpha\rangle_{\text{in}} &= V_{\text{in}}{}^{-1}[f]\,|\alpha\rangle^{\text{in}}_{\text{Fock}} \;, \\ |\alpha\rangle_{\text{out}} &= V_{\text{out}}{}^{-1}[g]\,|\alpha\rangle^{\text{out}}_{\text{Fock}} \;. \end{aligned} \qquad (2.11)$$

The in and out states are constructed from different functions, f and g, to allow the incoming external field and outgoing external field to differ. Such a situation allows a calculation of the probability of beam scattering in the case of electrodynamics or the decay rate of an unstable vacuum structure. The sequence of operators in (2.11), i.e., that the coherent-state operator lies to the left of the Fock operators, is essential to maintaining a complete set of states. The state $|\alpha\rangle^{\text{in}}_{\text{Fock}}$ is understood to be a state of fixed particle number constructed from the spectrum of the in field ϕ_{in} by acting on the Fock vacuum, and therefore has no manifest time dependence. It is important to remember that the asymptotic vacuum (no particle state as opposed to ground state) $|0\rangle$ of the theory is the Fock vacuum associated with $\phi_{\rm in}$, and is therefore cyclic with respect to the algebra of the in field and the out field.

The LSZ reduction procedures consistent with (2.11) can now be developed. A possible approach would be to expand the coherent-state operator in a functional power series, and reduce it term by term. However, the more straightforward approach is to reduce only the Fock state appearing in (2.11) and to leave the operator V in place on the Fock vacuum $|0\rangle$, later incorporating its effects into the perturbation series representation of the S-matrix element.

It is clear that only those particles which do not commute with the coherent-state operator $V_{\rm in}$ could differ in their reduction formula. As a result, only Bose particles of the type constituting the coherent state will be affected. For the sake of simplicity the case of a single species of scalar particle will be considered. For that case the creation operator $b_{\rm in}^{\dagger}$ takes the form

$$b_{\rm in}^{\dagger}(\mathbf{k})=-i\int d^3x[u(\mathbf{k},\mathbf{x},t_-)\dot{\phi}_{\rm in}(\mathbf{x},t_-) -\dot{u}(\mathbf{k},\mathbf{x},t_-)\phi_{\rm in}(\mathbf{x},t_-)]\ , \tag{2.12}$$

where $u(\mathbf{k},x)$ is a solution to (2.4). Using the standard LSZ trick

$$\lim_{t\to t_-}=\lim_{t\to t_+}-\int_{t_-}^{t_+}dt\frac{\partial}{\partial t} \tag{2.13}$$

and the assumption that $f(x)$ satisfies

$$\partial^2 f+V(x)f=0\ , \tag{2.14}$$

it follows that

$${}_{\rm out}\langle A\,|\,T[\Psi(x_1)\cdots]\,|\,\mathbf{k},B\rangle_{\rm in}=\frac{i}{\sqrt{Z}}\int d^4x\,u(\mathbf{k},x)[\partial^2+V(x)]_{\rm out}\langle A\,|\,T[\Psi(x)\Psi(x_1)\cdots]\,|\,B\rangle_{\rm in} +{}_{\rm out}\langle A-\mathbf{k}\,|\,T[\Psi(x_1)\cdots]\,|\,B\rangle_{\rm in}+i\,{}_{\rm out}\langle A\,|\,T[\Psi(x_1)\cdots]\,|\,B\rangle_{\rm in} \times\int d^3x[u_{\mathbf{k}}(\dot{f}-\dot{g})-\dot{u}_{\mathbf{k}}(f-g)]. \tag{2.15}$$

Clearly, the first term on the right-hand side is the usual reduction formula for a scalar particle, while the second particle is the standard forward-scattering term, which is usually suppressed. The third term is new, and in effect represents a forward-scattering term created by the presence of the coherent states. It appears because the $b_{\rm in}^{\dagger}$ must be commuted past $V_{\rm in}{}^{-1}[f]$ and $V_{\rm out}[g]$. It is obvious that if f and g coincide the third term will vanish. However, the case where f and g differ is precisely the central feature of this paper. It is also clear, for the assumptions of this paper, that the spatial integral of the third term yields a time-independent quantity. It is also worth noting in passing that had $f(x)$ been chosen such that $V_{\rm in}$ had manifest time dependence the reduction formulas would have been altered even further. Such a possibility is not being considered in this paper, but would be of interest to pursue elsewhere.

As a result of the reduction process, the S-matrix element is expressed as a sum of time-ordered products of the interacting fields of the form (2.15). Before giving a perturbative representation to the time-ordered product it will be assumed that all additional particles beyond those of the coherent state have been reduced. This leaves a Green's function of the form

$$G\equiv\langle 0\,|\,V_{\rm out}[g]T[\Psi(x_1)\cdots]V_{\rm in}{}^{-1}[f]\,|\,0\rangle\ . \tag{2.16}$$

The standard prescription for generating a perturbative representation of (2.16) is the assumption that the bare interacting field Ψ can be related to the in field by the unitary transformation

$$U(t)\Psi(\mathbf{x},t)U^{-1}(t)=\sqrt{Z}\,\phi_{\rm in}(\mathbf{x},t)\ . \tag{2.17}$$

Coupling assumption (2.17) with (2.2) implies

$$\operatorname*{w-lim}_{t\to t_-}U(t)\Psi(\mathbf{x},t)U^{-1}(t)=\sqrt{Z}\,U(t_-)\phi_{\rm in}(\mathbf{x},t_-)U^{-1}(t_-) =\sqrt{Z}\,\phi_{\rm in}(\mathbf{x},t_-)\ , \tag{2.18}$$

so that, in the weak sense,

$$U(t_-)b_{\rm in}^{(\dagger)}(\mathbf{k})=b_{\rm in}^{(\dagger)}(\mathbf{k})U(t_-)\ . \tag{2.19}$$

Likewise,

$$\operatorname*{w-lim}_{t\to t_+}U(t)\Psi(\mathbf{x},t)U^{-1}(t)=\sqrt{Z}\,U(t_+)\phi_{\rm out}(\mathbf{x},t_+)U^{-1}(t_+) =\sqrt{Z}\,\phi_{\rm in}(\mathbf{x},t_+)\ , \tag{2.20}$$

so that

$$b_{\rm out}^{(\dagger)}U^{-1}(t_+)=U^{-1}(t_+)b_{\rm in}^{(\dagger)}\ . \tag{2.21}$$

Inserting a factor of unity in the form $U^{-1}(t_+)U(t_+)$ and $U^{-1}(t_-)U(t_-)$ and using relations (2.19) and (2.21) gives

$$G=\langle 0\,|\,U^{-1}(t_+)V_{\rm in}[g]T[U(t_+)\Psi(x_1)\cdots U^{-1}(t_-)] \times V_{\rm in}{}^{-1}[f]U(t_-)\,|\,0\rangle\ . \tag{2.22}$$

Next, the coherent-state operators are split into two factors. Defining the operators

$$Z_{\rm in}[f]=\exp\left[\int d^3k\,f(\mathbf{k})b_{\rm in}(\mathbf{k})\right] \tag{2.23a}$$

and

$$Z_{\rm in}{}^{-1}[f]=\exp\left[-\int d^3k\, f(\mathbf{k})b_{\rm in}(\mathbf{k})\right], \tag{2.23b}$$

and using the Baker-Campbell-Hausdorf relation it is straightforward to show that

$$V_{\rm in}[f]=(Z_{\rm in}[f])^\dagger Z_{\rm in}{}^{-1}[f]e^{-O(f)}, \tag{2.24}$$

where

$$O(f)=\tfrac{1}{2}\int d^3k\, f^*(\mathbf{k})f(\mathbf{k}). \tag{2.25}$$

The $Z_{\rm in}$ are not unitary. Instead they have the property that

$$Z_{\rm in}[f]\,|\,0\rangle=Z_{\rm in}{}^{-1}[f]\,|\,0\rangle=|\,0\rangle \tag{2.26a}$$

and

$$\langle 0\,|\,(Z_{\rm out}[g])^\dagger=\langle 0\,|\,(Z_{\rm out}{}^{-1}[g])^\dagger=\langle 0\,|\,. \tag{2.26b}$$

Using results (2.24)–(2.26) and the nature of time-ordering allows (2.22) to be rewritten as (ignoring factors of the wave-function renormalization $\sqrt{Z}$)

$$G=\langle 0\,|\,U^{-1}(t_+)Z_{\rm in}{}^{-1}[g]T[\phi_{\rm in}(x_1)\cdots U(t_+)U^{-1}(t_-)](Z_{\rm in}{}^{-1}[f])^\dagger U(t_-)\,|\,0\rangle e^{-O(f)-O(g)}. \tag{2.27}$$

It has been shown[7] that $|\,0\rangle$ is an eigenstate of $U^{-1}(t_+)$ and $U(t_-)$. Denoting the eigenvalues as λ_+ and λ_-, respectively, it follows from (2.26) that

$$(\lambda_+^*\lambda_-)^{-1}G=\langle 0\,|\,(Z_{\rm in}[f])^\dagger Z_{\rm in}{}^{-1}[g]T[\phi_{\rm in}(x_1)\cdots U(t_+)U^{-1}(t_-)](Z_{\rm in}{}^{-1}[f])^\dagger Z_{\rm in}[g]\,|\,0\rangle e^{-O(f)-O(g)}. \tag{2.28}$$

Denoting

$$W=(Z_{\rm in}[f])^\dagger Z_{\rm in}{}^{-1}[g] \tag{2.29}$$

and using the Baker-Campbell-Hausdorf relation again allows (2.28) to be written

$$G=\lambda_-\lambda_+^*\langle 0\,|\,WT[U(t_+)U^{-1}(t_-)\cdots\phi_{\rm in}(x_1)\cdots]W^{-1}\,|\,0\rangle e^{X(f,g)}, \tag{2.30}$$

where

$$X(f,g)=-\tfrac{1}{2}\int d^3k[f^*(\mathbf{k})f(\mathbf{k})+g^*(\mathbf{k})g(\mathbf{k})-2f^*(\mathbf{k})g(\mathbf{k})], \tag{2.31}$$

so that X measures the functional "overlap" of the two coherent states. The operator W may be moved inside the time-ordered product to yield

$$(\lambda_+^*\lambda_-)^{-1}G=\langle 0\,|\,T\{[\phi_{\rm in}(x_1)+\alpha(x_1)]\cdots WU(t_+)U^{-1}(t_-)W^{-1}\}\,|\,0\rangle e^{X(f,g)}, \tag{2.32}$$

where the function $\alpha(x)$ is given by

$$\alpha(x)=\int d^3k[g^*(\mathbf{k})u(\mathbf{k},x)+f(\mathbf{k})u^*(\mathbf{k},x)]. \tag{2.33}$$

The function α is induced by the operator W in the sense that

$$W\phi_{\rm in}(x)W^{-1}=\phi_{\rm in}(x)+\alpha(x), \tag{2.34}$$

so that the field $\phi_{\rm in}$ is everywhere translated by a function which is the algebraic sum of the negative-frequency part of f and the positive-frequency part of g. It is obvious from the form (2.33) that α is not necessarily a real function, so that real fields may be translated by a complex function. It is also clear from (2.32) that the evolution operator for this system is effectively given by

$$E(t_+,t_-)=WU(t_+)U^{-1}(t_-)W^{-1}. \tag{2.35}$$

The evolution operator can be given a time-ordered representation by noting that it solves the differential equation

$$\frac{\partial}{\partial t}E(t,t')=[W\dot{U}(t)U^{-1}(t')W^{-1}]E(t,t'). \tag{2.36}$$

Using (2.1) and denoting H_0 as the Hamiltonian which drives the time development of $\phi_{\rm in}$, it follow that

$$\begin{aligned}W\dot{U}(t)U^{-1}(t')W^{-1}&=-i(H[\phi_{\rm in}+\alpha,\pi_{\rm in}+\dot{\alpha}]\\&\quad-H_0[\phi_{\rm in}+\alpha,\pi_{\rm in}+\dot{\alpha}])\\&\equiv -iH_I[\phi_{\rm in},\alpha],\end{aligned} \tag{2.37}$$

where factors of the wave-function renormalization constant have been suppressed. Iterating Eq. (2.36) gives the result

$$E(t_+,t_-)=T\left[\exp\left[-i\int_{t_-}^{t_+}dt\,H_I(t)\right]\right]. \tag{2.38}$$

The final result is that the time-ordered product of fields from the reduction process has the perturbative representation

$$\begin{aligned}G=\lambda_+^*\lambda_-\Big\langle 0\Big|T\Big[[\phi_{\rm in}(x_1)+\alpha(x_1)]\cdots\\ \times\exp\left[-i\int_{t_-}^{t_+}dt\,H_I(t)\right]\Big]\Big|0\Big\rangle\\ \times e^{X(f,g)}.\end{aligned} \tag{2.39}$$

Clearly, the Dyson-Wick contraction scheme may be applied to (2.39) to generate a perturbative representation of scattering processes in the presence of the coherent states. Furthermore, the form of H_I gives a simple self-consistency condition for the form of H_0, and hence for the form of the differential equation (2.4) which $\phi_{\rm in}$ must solve. Perturbative stability of the spectrum of the fields $\phi_{\rm in}$ requires at least no terms quadratic or linear appearing in H_I. H_0 must therefore be chosen to remove any and all terms which are quadratic in the $\phi_{\rm in}$. If terms linear in the fields are necessary in the definition of H_0, then the analysis of this paper breaks down. This is because a term linear in $\phi_{\rm in}$ appearing in H_0 will perforce cause $\phi_{\rm in}$ to obey an inhomogenous linear equation. Under such circumstances it is easy to see that the coherent-state operator $V_{\rm in}$ will no longer be time independent and result (2.15) would no longer be correct.

In the next section an application of this section's technique to laser-electron scattering will be made.

III. NONRELATIVISTIC QUANTUM ELECTRODYNAMICS IN A LASER FIELD

The behavior of charged particles in a laser field is a problem which continues to draw a great deal of attention.[8] Nonrelativistic quantum electrodynamics is described by the Lagrangian density

$$\begin{aligned} L &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2m}D_i^\dagger\Psi_a^* D_i\Psi_a \\ &\quad -eA_0\Psi_a^*\Psi_a - i\Psi_a^*\dot{\Psi}_a\ , \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\ , \\ D_j &= \partial_j + ieA_j\ . \end{aligned} \tag{3.1}$$

A_μ is the vector potential and Ψ_a is a spinor electron field. The theory will be quantized in the Coulomb gauge, so that

$$\partial_i A_i = 0 \tag{3.2}$$

and the equal-time (anti)commutation relations

$$\begin{aligned} \{\Psi_a(\mathbf{x},t),\Psi_b^*(\mathbf{y},t)\}_+ &= \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y})\ , \\ [A_j(\mathbf{x},t),F_{k0}(\mathbf{y},t)]_- &= i\delta_{jk}^{\rm TR}(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{3.3}$$

are assumed to hold, where $\delta_{jk}^{\rm TR}$ is the transverse δ function. A_0 is removed from the theory by demanding that Gauss's law be satisfied:

$$A_0(\mathbf{x},t) = \frac{e}{4\pi}\int d^3y \frac{1}{|\mathbf{x}-\mathbf{y}|}[\Psi_a^*(\mathbf{y},t)\Psi_a(\mathbf{y},t)]\ . \tag{3.4}$$

This form for Gauss's law is consistent with the use of asymptotic coherent states only if the external vector potential is sourceless, i.e., that $\partial^2 A_0^{\rm ext}=0$. The vector potential for the in and out states will be that of a linearly polarized plane wave. The external field is taken to have the form

$$\mathbf{A}_{\rm (in)(out)}(x) = \mathbf{M}_{\rm (in)(out)}\cos(\boldsymbol{l}\cdot\mathbf{x}-\omega t),\quad \omega=|\boldsymbol{l}|,\quad \boldsymbol{l}\cdot\mathbf{M}_{\rm (in)(out)}=0\ . \tag{3.5}$$

The interaction picture/asymptotic fields ϕ_a and a_i must be constructed to satisfy relations (3.3), and will be chosen to obey the equations of motion

$$\partial^2 a_i = 0,\quad \left[-\frac{1}{2m}\partial_j{}^2 + i\frac{e}{m}A_j^{\rm avg}\partial_j + \frac{e^2}{2m}A_j^{\rm avg}A_j^{\rm avg}\right]\phi_a = i\dot{\phi}_a\ , \tag{3.6}$$

where $A_j^{\rm avg}$ is the average of the incoming and outgoing electromagnetic fields, so that

$$A_{\rm avg}^j \equiv \tfrac{1}{2}(A_{\rm in}^j + A_{\rm out}^j) = \tfrac{1}{2}(M_{\rm out}^j + M_{\rm in}^j)\cos(\boldsymbol{l}\cdot\mathbf{x}-\omega t) \equiv M_{\rm avg}^j\cos(\boldsymbol{l}\cdot\mathbf{x}-\omega t)\ . \tag{3.7}$$

The reason for selecting forms (3.6) and (3.7) will become apparent when the effective interaction Hamiltonian, given by the general expression (2.37), is derived for this case. The forms for a_j and ϕ_a which satisfy (3.3) and (3.6) are

$$a_j(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\sum_{\lambda=1}^{2}[\epsilon_j^\lambda(\mathbf{k})a_\lambda(\mathbf{k})e^{-ikx} + \epsilon_j^\lambda a_\lambda^\dagger(\mathbf{k})e^{ikx}] \tag{3.8a}$$

and

$$\phi_a(x) = \int d^3p\, b_a(\mathbf{p})h_p(x)\ , \tag{3.8b}$$

where $\omega_k = |\mathbf{k}|$. The commutation relations are satisfied by the conditions

$$[a_\sigma(\mathbf{k}),a_\rho^\dagger(\mathbf{p})]_- = \delta_{\sigma\rho}\delta^3(\mathbf{k}-\mathbf{p})\ , \tag{3.9a}$$

$$\sum_{\lambda=1}^{2}\epsilon_\lambda^i(\mathbf{k})\epsilon_\lambda^j(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}\cdot\mathbf{k}}\ , \tag{3.9b}$$

$$\mathbf{k}\cdot\boldsymbol{\epsilon}_\lambda(\mathbf{k}) = 0\ , \tag{3.9c}$$

$$\{b_a(\mathbf{k}),b_c^\dagger(\mathbf{p})\}_+ = \delta_{ac}\delta^3(\mathbf{k}-\mathbf{p})\ . \tag{3.9d}$$

The equations of motion are approximately satisfied by selecting the form[9]

$$h_p(x)=\frac{1}{(2\pi)^{3/2}}e^{ip\cdot x-iE_pt}\exp\left[\frac{im}{m\omega-p\cdot l}\left[\frac{e}{m}p\cdot M_{\text{avg}}\sin(l\cdot x-\omega t)-\frac{e^2}{4m}M_{\text{avg}}{}^2[2(l\cdot x-\omega t)+\sin 2(l\cdot x-\omega t)]\right]\right], \tag{3.10}$$

where $E_p=p^2/2m$. The function h_p may be expanded in terms of generalized Bessel functions. However, in the limit that the terms in (3.10) which are proportional to $M_{\text{avg}}{}^2$ may be ignored, h_p may be written

$$h_p(x)=\frac{1}{(2\pi)^{3/2}}\sum_{n=-\infty}^{\infty}J_n\left[\frac{ep\cdot M_{\text{avg}}}{m\omega-l\cdot p}\right]\exp[i(p-nl)\cdot\mathbf{x}-i(E_p-n\omega)t], \tag{3.11}$$

where J_n is the ordinary Bessel function of integer order. Approximation (3.11) will be employed in the remainder of this section.

In this work the incoming state of interest contains a single electron of momentum **p**, while the outgoing state will possess an electron of momentum **k** and an additional photon of momentum **q** and polarization λ. The momentum of this additional photon is assumed to be different from the momentum l of the individual photons making up the incoming and outgoing laser beam. The matrix element of interest then takes the form

$$S={}_{\text{out}}\langle \mathbf{k},b,\mathbf{q},\lambda,\mathbf{A}_{\text{out}}\mid \mathbf{p},a,\mathbf{A}_{\text{in}}\rangle_{\text{in}}, \tag{3.12}$$

and is readily reduced to give

$$S=i\int d^4x_1d^4x_2d^4x_3h_k^*(x_1)h_p(x_2)f_{q,\lambda}^j(x_3)L_{x_1}L_{x_2}^*\partial_{x_3}{}^2 \times\langle 0\mid V_{\text{out}}T[\Psi_b(x_1)\Psi_a^*(x_2)A_j(x_3)]V_{\text{in}}{}^{-1}\mid 0\rangle, \tag{3.13}$$

where

$$f_{q,\lambda}^j=\frac{1}{\sqrt{16\pi^3\omega_q}}\epsilon_\lambda^j(q)e^{-iq\cdot x+i\omega_qt} \tag{3.14}$$

and

$$L_x=-\frac{1}{2m}\partial_j\partial_j+i\frac{e}{m}A_{\text{avg}}^j\partial_j+\frac{e^2}{2m}A_{\text{avg}}\cdot A_{\text{avg}}-i\frac{\partial}{\partial t}. \tag{3.15}$$

Expression (3.13) may be given a perturbative representation by the techniques of the previous section. It follows that the function α^j of (2.33) is a vector-valued function and is given by

$$\alpha^j(x)=M_{\text{avg}}^j\cos(l\cdot x-\omega t)+\tfrac{1}{2}i(M_{\text{out}}^j-M_{\text{in}}^j)\sin(l\cdot x-\omega t). \tag{3.16}$$

The imaginary piece of α will be suppressed since it will be shown to be small compared to the real first term. This will be true in the limit of a large photon number in the beam, but not so large that the approximation (3.11) is invalid. The physics represented by the imaginary piece of α will be analyzed elsewhere. It is straightforward to show, by virtue of having chosen ϕ_a to obey (3.6), that the effective interaction Hamiltonian is given by

$$H_I(t)=\int d^3x\left[\frac{ie}{2m}a^j(\partial_j\phi_a^*\phi_a-\phi_a^*\partial_j\phi_a)+\frac{e^2}{2m}a\cdot a\phi_a^*\phi_a+\frac{e^2}{m}a\cdot A_{\text{avg}}\phi_a^*\phi_a\right] +\frac{e^2}{8\pi}\int d^3x\,d^3y\left[\phi_a^*(\mathbf{x},t)\phi_a(\mathbf{x},t)\frac{1}{|\mathbf{x}-\mathbf{y}|}\phi_b^*(\mathbf{y},t)\phi_b(\mathbf{y},t)\right], \tag{3.17}$$

so that all terms linear and quadratic in the fields (again, suppressing the imaginary piece of α) have been removed by the choice of equation of motion for the ϕ.

The way is now clear to evaluate the matrix element (3.13). Using the Dyson-Wick contraction scheme the lowest-order form for the matrix element reduces to

$$S=e^{X(A_{\text{in}},A_{\text{out}})}i\delta_{ab}\int d^4x\left[\frac{ie}{2m}f_{q,\lambda}^j(x)[\partial_jh_k^*(x)h_p(x)-h_k^*(x)\partial_jh_p(x)]+\frac{e^2}{m}A_{\text{avg}}(x)\cdot f_{q,\lambda}(x)h_k^*(x)h_p(x)\right], \tag{3.18}$$

where, for the moment, the evaluation of the overlap integral $X(A_{\text{in}},A_{\text{out}})$ will be deferred.

The evaluation of (3.18) is greatly simplified by choosing **p** to be zero. In that case the first term in (3.18) vanishes. This is seen by substituting (3.11) into (3.18) and performing the integration. The first term becomes

$$\frac{ie}{m}\delta_{ab}\sum_{n,n'=-\infty}^{\infty}J_n(\gamma_k)J_{n'}(\gamma_p)[k+p-(n-n')l]\cdot\epsilon_\lambda(\mathbf{q}) \times\frac{1}{\sqrt{4\pi\omega_k}}\delta^3(q-p+k-(n-n')l) \times\delta(\omega_q+E_k-E_p-(n-n')\omega), \tag{3.19}$$

where

$$\gamma_k = \frac{e k\cdot M_{\rm avg}}{m\omega - k\cdot l} \ . \tag{3.20}$$

When $\mathbf{p}=0$ only the $n'=0$ term is nonzero in the sum, and (3.19) becomes

$$\delta_{ab}\frac{ie}{m}\sum_{n=-\infty}^{\infty} J_n(\gamma_k)\frac{(k-nl)\cdot\epsilon_\lambda(q)}{\sqrt{4\pi\omega_k}}\delta^3(q+k-nl) \times\delta(\omega_q+E_k-n\omega) \ . \tag{3.21}$$

The δ function in (3.21) yields the result that

$$(k-nl)\cdot\epsilon_\lambda(q)\delta^3(q+k-nl) = -q\cdot\epsilon_\lambda(q)\delta^3(q+k-nl) \ , \tag{3.22}$$

which vanishes from (3.9c). Thus, the first term may be ignored for the case $\mathbf{p}=0$.

Substituting (3.11) into the second term of (3.18) and performing the integration gives

$$S = e^{X(A_{\rm in},A_{\rm out})} i\delta_{ab}\sum_{n,n'=-\infty}^{\infty} J_{n'}(\gamma_p)J_n(\gamma_k)\frac{M_{\rm avg}\cdot\epsilon_\lambda(q)}{\sqrt{4\pi\omega_k}}\Delta \ , \tag{3.23a}$$

where

$$\begin{aligned}\Delta = {} & \delta^3(p-k-q-(n'-n-1)l) \\ & \times\delta(E_k+\omega_q-E_p-(n-n'+1)\omega) \\ & +\delta^3(p-k-q-(n'-n+1)l) \\ & \times\delta(E_k+\omega_q-E_p-(n-n'-1)\omega) \ .\end{aligned} \tag{3.23b}$$

For the case that $\mathbf{p}=0$ expression (3.23) becomes

$$\begin{aligned}S = {} & e^{X(A_{\rm in},A_{\rm out})} \\ & \times i\delta_{ab}\frac{e^2}{2m}\sum_{n=-\infty}^{\infty}\frac{M_{\rm avg}\cdot\epsilon_\lambda(q)}{\sqrt{4\pi\omega_q}}[J_{n+1}(\gamma_k)+J_{n-1}(\gamma_k)] \\ & \times\delta^3(k+q-nl)\delta(E_k+\omega_q-n\omega) \ .\end{aligned} \tag{3.24}$$

Expression (3.24) is simplified further by using the property of Bessel functions that

$$J_{n+1}(\gamma_k)+J_{n-1}(\gamma_k) = \frac{2n}{\gamma_k}J_n(\gamma_k) \ , \tag{3.25}$$

so that

$$\begin{aligned}S = {} & i\delta_{ab}\frac{e^2}{m\gamma_k}\frac{M_{\rm avg}\cdot\epsilon_\lambda(q)}{\sqrt{4\pi\omega_q}} \\ & \times\sum_{n=-\infty}^{\infty} nJ_n(\gamma_k)\delta^3(k+q-nl) \\ & \times\delta(E_k+\omega_q-n\omega)\exp[X(A_{\rm in},A_{\rm out})] \ .\end{aligned} \tag{3.26}$$

It is instructive to examine $|S|^2$, interpreting it as a probability, and find what values of $M_{\rm in}$ and $M_{\rm out}$ maximize the probability. Evaluating the overlap integral is straightforward. It follows from (2.8) that the average number of photons in the incoming and outgoing beams is given by

$$N_{\rm in} \equiv \langle N\rangle_{\rm in} = 4\pi^3\omega M_{\rm in}\cdot M_{\rm in}\delta^3(0) \tag{3.27a}$$

and

$$N_{\rm out} \equiv \langle N\rangle_{\rm out} = 4\pi^3\omega M_{\rm out}\cdot M_{\rm out}\delta^3(0) \ , \tag{3.27b}$$

where $\delta^3(0)$ has the units of $(\text{length})^3$. It is obvious from (3.27) that $N_{\rm in}$ and/or $N_{\rm out}$ can be large but finite and remain consistent with the assumption that $M_{\rm in}$ and $M_{\rm out}\to 0$. On the other hand, nonzero $M_{\rm in}$ or $M_{\rm out}$ leads to an infinite photon number. The overlap integral is readily evaluated to obtain

$$X(A_{\rm in},A_{\rm out}) = -2\pi^3\omega\,|\mathbf{M}_{\rm in}-\mathbf{M}_{\rm out}|^2\delta^3(0) \ . \tag{3.28}$$

By virtue of assuming the incoming and outgoing beams possess the same plane of polarization, it follows from (3.27) that

$$X(A_{\rm in},A_{\rm out}) = -\tfrac{1}{2}(\sqrt{N_{\rm in}}-\sqrt{N_{\rm out}})^2 \equiv -\tfrac{1}{2}\eta \ . \tag{3.29}$$

It is obvious from the form of (3.26) that, to lowest order in $N_{\rm in}$ and $N_{\rm out}$,

$$|S|^2 \propto (\sqrt{N_{\rm in}}+\sqrt{N_{\rm out}})^2 e^{-\eta} \ . \tag{3.30}$$

It is clear from (3.28) and (3.30) that $|S|^2$ would vanish for the finite amplitude case unless $M_{\rm out}=M_{\rm in}$. For finite $N_{\rm in}$ and $N_{\rm out}$ expression (3.30) is maximized if

$$N_{\rm in} = N_{\rm out}\pm 1 \ , \tag{3.31}$$

where the $\pm$ sign occurs because the time-reversed process, i.e., absorption of an external photon by the electron, is described by the same matrix element. Thus, the additional external photon assumed to be present in the outgoing state leads to a depletion of the outgoing beam's intensity in a self-consistent manner.

Result (3.26) can also be used to calculate the total cross section for laser-electron scattering in the event that the electron is initially stationary. The total cross section is defined

$$\sigma = \int d\Omega_q\frac{d\sigma}{d\Omega_q} = \frac{1}{16\pi^4\delta^4(0)}\sum_{\lambda,b=1}^{2}\int d^3k\,d^3q\frac{|S|^2}{N_e F} \ . \tag{3.32}$$

The integration and summation run over the phase space of the scattered electron and the additional photon as well as all possible spin polarizations. N_ϵ is the density of initial electron states

$$N_e = \frac{1}{8\pi^3} \tag{3.33}$$

and F is the incident flux of photons per unit volume of space. F is obtained by calculating the average power in the beam and dividing by the energy of an individual photon in the beam. It follows that

$$F=\tfrac{1}{2}\omega M_{\text{in}}^2 \ . \tag{3.34}$$

Inserting these into (3.32) the cross section becomes

$$\sigma=\sum_{n=-\infty}^{\infty}\frac{e^4}{m^2}\int d^3k\,d^3q\sum_{\lambda=1}^{2}|M_{\text{avg}}\cdot\epsilon_\lambda(q)|^2 \times\frac{n^2J_n^2(\gamma_k)}{4\pi^2\omega\gamma_k^2\omega_q M_{\text{in}}^2} \times\delta^3(k+q-nl) \times\delta(E_k+\omega_q-n\omega)e^{-\eta} \ . \tag{3.35}$$

Integrating over the δ function in momentum gives

$$\sigma=\sum_{n=-\infty}^{\infty}\frac{e^4}{m^2}\int d^3q\sum_{\lambda=1}^{2}|M_{\text{avg}}\cdot\epsilon_\lambda(q)|^2 \times\frac{n^2J_n^2(\gamma_q)}{4\pi^2\omega\gamma_q^2\omega_q M_{\text{in}}^2} \times\delta(E_{nl-q}+\omega_q-n\omega)e^{-\eta} \ , \tag{3.36}$$

where the identity

$$\gamma_{nl-q}=-\gamma_q \tag{3.37}$$

has been used. There is no loss in generality in picking l to lie in the z direction. The δ function for energy conservation reduces to

$$\delta(E_{nl-q}+\omega_q-n\omega) =\delta(q-q_n)\left[\frac{q_n}{m}+\left[1-\frac{n\omega\cos\theta}{m}\right]\right]^{-1} , \tag{3.38}$$

where θ is the polar angle of $\mathbf{q}$ and

$$q_n=(m-n\omega\cos\theta)\left[\left[1+\frac{2n\omega\left[1-\dfrac{n\omega}{2m}\right]}{m\left[1-\dfrac{n\omega\cos\theta}{m}\right]^2}\right]^{1/2}-1\right] . \tag{3.39}$$

Expression (3.39) demonstrates that n must be a positive number, which simply reflects the fact that stimulated absorption of laser photons is the mechanism which drives the spontaneous radiation of the electron. It also shows that the validity of the nonrelativistic structure of the theory breaks down when

$$\frac{nh\omega}{4\pi c^2}\approx 1 \ . \tag{3.40}$$

This places the restriction on frequencies for which this analysis is valid of

$$\omega \ll 10^{21}\ \text{rad/sec} \ . \tag{3.41}$$

In the limit that $n\omega/m\approx 0$, relation (3.38) reduces to

$$\delta(E_{nl-q}+\omega_q-n\omega)\approx\delta(q-n\omega) \ , \tag{3.42}$$

so that the energy of the radiated photon will be a multiple of the laser photon energy in this approximation.

The sum over photon polarizations yields

$$\sum_{\lambda=1}^{2}|M_{\text{avg}}\cdot\epsilon_\lambda(q)|^2=M_{\text{avg}}^2[1-\sin^2\theta\cos^2(\phi-\phi_l)] \ , \tag{3.43}$$

where ϕ_l is the azimuthal angle of $\mathbf{M}_{\text{avg}}$, while θ and ϕ are the angular coordinates of $\mathbf{q}$. The cross section then becomes

$$\sigma=\int dq\,d\Omega_q\frac{qe^4}{4\pi^2m^2\omega\gamma_q^2}\frac{M_{\text{avg}}^2}{M_{\text{in}}^2}[1-\sin^2\theta\cos^2(\phi-\phi_l)] \times\sum_{n=1}^{\infty}n^2J_n^2(\gamma_q)\delta(q-n\omega)e^{-\eta} \ . \tag{3.44}$$

For the limit in which (3.42) is valid, i.e., low-frequency laser photons,

$$\gamma_q\approx\frac{eqM_{\text{avg}}}{m\omega}\sin\theta\cos(\phi-\phi_l) \ . \tag{3.45}$$

Performing the integral over q gives

$$\sigma=\int d\Omega_q\sum_{n=1}^{\infty}\frac{e^4n^3}{4\pi^2m^2\beta_n^2}\frac{M_{\text{avg}}^2}{M_{\text{in}}^2}J_n^2(\beta_n) \times[1-\sin^2\theta\cos^2(\phi-\phi_l)]e^{-\eta} \ , \tag{3.46}$$

where

$$\beta_n=\frac{neM_{\text{avg}}}{m}\sin\theta\cos(\phi-\phi_l) \ . \tag{3.47}$$

For the case that $\beta_n\approx 0$, i.e., the weak-field case of this paper, it follows that

$$J_n(\beta_n)\approx(\tfrac{1}{2}\beta_n)^n \ , \tag{3.48}$$

so that

$$\sigma=\int d\Omega_q\frac{e^4}{16\pi^2m^2}\frac{M_{\text{avg}}^2}{M_{\text{in}}^2} \times\sum_{n=1}^{\infty}n^3(\tfrac{1}{2}\beta_n)^{2n-2}[1-\sin^2\theta\cos^2(\phi-\phi_l)]e^{-\eta} \ . \tag{3.49}$$

The differential cross section is therefore

$$\frac{d\sigma}{d\Omega_q}=\frac{e^4}{16\pi^2 m^2}\frac{{M_{\text{avg}}}^2}{{M_{\text{in}}}^2}\times\sum_{n=1}^{\infty} n^3(\tfrac{1}{2}\beta_n)^{2n-2}[1-\sin^2\theta\cos^2(\phi-\phi_l)]e^{-\eta} . \tag{3.50}$$

Using (3.27) and (3.29) it follows that

$$\frac{{M_{\text{avg}}}^2}{{M_{\text{in}}}^2}=\frac{(\sqrt{N_{\text{in}}}+\sqrt{N_{\text{out}}})^2}{4N_{\text{in}}} , \tag{3.51}$$

so that, in the limit of large photon number and restriction (3.31),

$$\frac{{M_{\text{avg}}}^2}{{M_{\text{in}}}^2}\approx 1 \tag{3.52a}$$

and

$$\eta\approx 0 . \tag{3.52b}$$

In such a limit the differential cross section becomes

$$\lim_{\substack{M_{\text{avg}}\to 0\\ N\gg 1}}\frac{d\sigma}{d\Omega_q}=\frac{e^4}{16\pi^2 m^2}[1-\sin^2\theta\cos^2(\phi-\phi_l)] , \tag{3.53}$$

which is identical to the result which is obtained by calculating Compton scattering of a polarized photon in the zero-frequency limit. It is now clear that the limit represented by (3.52) is consistent with the suppression of the imaginary term in (3.16). Result (3.49) also reduces to the results first derived by Brown and Kibble[10] for the case that $A_{\text{in}}=A_{\text{out}}$, i.e., the case where there are finite but small field strengths in the incoming and outgoing states. For that case, because a plane-wave representation has been used, the condition (3.31) has no effect on the magnitude of the field strength, and no error is induced by maintaining them the same. In such a case conditions (3.52) are exact, while the higher-order terms in (3.49) correspond to the contribution of multiphoton processes present due to the laser.

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[1]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 1425 (1955); **2**, 425 (1955); **6**, 319 (1957).

[2]See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 129ff; C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 205ff. The conventions followed in this paper are those of the latter.

[3]R. J. Glauber, Phys. Rev. **131**, 2766 (1963); J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968); B. DeFacio and C. Hammer, J. Math. Phys. **17**, 267 (1976); J. R. Klauder and Bo-Sture Skagerstam, *Coherent States* (World Scientific, Singapore, 1985).

[4]Klauder and Sudarshan, *Fundamentals of Quantum Optics* (Ref. 3).

[5]K. Cahill, A. Comtet, and R. J. Glauber, Phys. Lett. **64B**, 283 (1976); M. Swanson, Phys. Rev. D **26**, 1956 (1982); Phys. Rev. B **27**, 4421 (1983).

[6]G. D'Ariano and M. Rasetti, Phys. Rev. D **35**, 1239 (1987); J. Katriel, M. Rasetti, and A. Solomon, *ibid.* **35**, 1248 (1987); F. Hong-Yi, H. Zaidi, and J. R. Klauder, *ibid.* **35**, 1831 (1987).

[7]Bjorken and Drell, *Relativistic Quantum Fields* (Ref. 2), p. 179ff.

[8]The literature in this area is vast. The interested reader is recommended to M. H. Mittelman, *Theory of Laser-Atom Interactions* (Plenum, New York, 1982), and references therein.

[9]C. Leone, P. Cavaliere, and G. Ferrante, J. Phys. B **17**, 1027 (1984).

[10]L. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).