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Two-body Dirac equations for particles interacting through world scalar and vector potentials

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In a recent Letter, we used "two-body Dirac equations" to make the naive quark model fully relativistic. In this paper, we apply Dirac's constraint mechanics and supersymmetry not to a string but instead to a system of two spinning particles to derive the two coupled Dirac equations that govern the quantum mechanics of two spin-$\frac{1}{2}$ particles interacting through world scalar and vector potentials. Along the way, we demonstrate that our equations are compatible, that the scalar and vector interaction structures contribute separately to compatibility when both are present, and that compatibility persists even in the presence of relative-momentum-dependent interactions. We show that extrapolation of supersymmetries associated with the ordinary single-particle Dirac equation to the interacting two-body system eliminates spin complications and reduces the compatibility problem to that of the corresponding spinless system. In addition, supersymmetry introduces physically important nonperturbative recoil effects that contribute to the correct perturbative spectrum while rendering nonsingular certain singular terms of standard semirelativistic approximations. This eliminates the need for singularity-softening parameters or finite particle sizes in phenomenological applications. In order to make clear the underlying structure of our equations, we also derive the corresponding coupled Dirac or Klein-Gordon equations that govern a system of two particles, one or neither of which has spin.

I. INTRODUCTION

In a recent Letter,[1] we used two-body Dirac equations (derived from Dirac's constraint mechanics and supersymmetry[2–4]) to make the naive quark model fully relativistic. For two relativistic quarks interacting through a system of world scalar and vector potentials, these equations are compatible 16-component (or 4×4 matrix) Dirac equations that take the c.m. forms

$$\mathcal{S}_1\psi\equiv\gamma_{51}[\gamma_1\cdot(p_1-\tilde{A}_1)+m_1+\tilde{S}_1]\psi=0\ , \tag{1a}$$

$$\mathcal{S}_2\psi\equiv\gamma_{52}[\gamma_2\cdot(p_2-\tilde{A}_2)+m_2+\tilde{S}_2]\psi=0\ , \tag{1b}$$

where $\tilde{A}_i$ and $\tilde{S}_i$ are (spin-dependent) constituent vector and scalar potentials. The spin dependence of the relativistic interaction between the spinning quarks arises naturally from the relativistic potential structure and the mutual compatibility of these two Dirac equations and is not a patchwork of semirelativistic corrections inspired by field theory. In applications to meson spectroscopy, the relativistic structure of these equations permits a one-parameter fit to the meson spectrum that is surprisingly good.

The resulting wave equations constitute the first fully covariant treatment of the quantum relativistic two-body problem that simultaneously (a) includes constituent spin, (b) regulates the relative time in a covariant manner, (c) provides an exact reduction to four decoupled four-component equations, (d) includes nonperturbative recoil effects in a natural way that eliminates the need for singularity-softening parameters or finite particle size in phenomenological applications, (e) is canonically equivalent in the semirelativistic approximation to the Fermi-Breit approximation to the Bethe-Salpeter equation, (f) (unlike the Bethe-Salpeter equation) has a local momentum structure as simple as that of the nonrelativistic Schrödinger equation, (g) is well defined for zero-mass constituents (hence, permits investigation of the chiral-symmetry limit as well as provides simplified models of bound systems of massless particles such as gluonium), (h) permits two different types of world vector interaction in addition to one type of scalar interaction, (i) possesses spin structure that yields an exact solution for singlet positronium, (j) has static limits that are relativistic, reducing to the ordinary single-particle Dirac equation in the limit that either particle becomes infinitely heavy, (k) possesses a great variety of equivalent forms that are rearrangements of its two coupled Dirac equations (hence, is directly related to many previously known quantum descriptions of the relativistic two-body system), and (l) is the canonically quantized version of a sensible (pseudo)classical mechanics that is described by coupled Lorentz and Bargmann-Michel-Telegdi equations.

Other methods for including relativistic corrections use a number of different approaches that fall roughly into four categories. The first consists of those approaches that rely on semirelativistic corrections to the

Schrödinger equation (e.g., slow-motion weak-potential approximations involving p^4 kinetic energy contributions and Breit-inspired spin-dependent potential corrections[5–8]). The second consists of those methods that treat the kinematics exactly [e.g., through

$$(\mathbf{p}_1{}^2+m_1{}^2)^{1/2}+(\mathbf{p}_2{}^2+m_2{}^2)^{1/2} ,$$

the Breit equation,[9] or truncations of the Bethe-Salpeter equation[10,11]] while employing spin-dependent corrections to the potential inspired by perturbative quantum field theory. Hence, these methods treat the potential as though it were weak since the spin-dependent operators are (usually illegal) quantum operators that can only be treated perturbatively. The third category consists of those methods that treat the kinematics exactly but dress up the potentials abstracted from field theory with smoothing factors so that the operators are quantum-mechanically legal,[12–14] effectively bypassing a weak-potential limitation. The final category consists of those relativistic schemes that treat the quark and antiquark asymmetrically by using a single interacting Dirac equation for the quark while repeatedly treating the antiquark as a free particle.[15]

Our method is distinguished from all of those by the fact that it exactly incorporates Dirac's relativistic spin structure for each spinning particle in an interacting system. Hence, our wave equations preserve an exactly relativistic nonperturbative structure naturally present in the ordinary (one-body) Dirac equation that has been left out of all of the other approaches for one or both spinning particles.

We derive our wave equations by canonically quantizing a fully covariant version of classical mechanics given by Dirac's Hamiltonian constraint technique.[16] Discovered by a number of authors in the mid to late 1970s,[17–21] this form of relativistic dynamics was initially a "toy" of interest primarily to specialists in relativistic quantum mechanics. Subsequently, we and others[20,22] found that it was a useful phenomenological tool for including relativistic two-body kinematical effects in spin-independent treatments of the quark model. Recently, by incorporating realistic interactions (with dynamical recoil effects) into the formalism[4] and by introducing spin for both particles through the use of supersymmetries,[2,3] we have brought this formalism to the stage at which we have been able to use it to make a realistic spectral calculation.[1] At the same time, the outlines of multiple underlying relationships between this method and the spectral methods of quantum field theory have begun to emerge.

Initially, various authors became interested in the constraint method as a route around the Currie-Jordan-Sudarshan (CJS) "non-interaction theorem,"[23] which apparently forbade canonical treatment of the problem of N interacting particles in relativistic mechanics (and thus a simple canonical transition to relativistic quantum mechanics). In spite of this theorem, there existed successful covariant treatments of the relativistic quantum-mechanical two-body problem abstracted from quantum field theory (e.g., the quasipotential equation[24]) for which there were apparently no classical analogs.[25] Relativistic constraint mechanics, constructed in accord with Dirac's Hamiltonian constraint formalism[16] (with one relativistic constraint for each particle), allows a covariant elimination of essentially relativistic variables (such as the c.m. relative time and relative energy in the two-body problem), reducing (covariantly) the number of degrees of freedom to those of the nonrelativistic problem. In its classical version, this dynamics evades the CJS theorem by introducing an extra degree of freedom, the relative time, which it controls in a covariant manner. When quantized, this structure automatically eliminates quantum ghosts (relative-time excitations with negative norms). One no longer has to exorcise them by hand (e.g., as done for the relativistic oscillator by Feynman, Kislinger, and Ravndal[26]) or in an *ad hoc* manner by setting the relative time equal to zero (as done in effect in some treatments of the Bethe-Salpeter equation). The resulting relativistic wave equations reduce for spinless particles to the older quasipotential equation, explaining its existence despite the classical CJS theorem. However, the early versions of the constraint approach were suitable for only the crudest phenomenological applications since they included neither spin nor dynamical recoil effects.

We have found how to extend the constraint formalism to systems containing spin-$\frac{1}{2}$ particles.[2,3] The essential difficulty is that the (pseudo)classical constraints or the resulting quantum-mechanical wave equations (one for each constituent) must be compatible. Our solution to this problem is to insist that a hidden (super)symmetry of the ordinary Dirac equation be preserved by each spinning particle during interaction. This ensures that the resulting "two-body Dirac equations" effectively swallow up two one-body Dirac equations in a consistent way. The structures that such symmetries dictate automatically introduce dynamical recoil effects, permit rearrangement of wave equations into convenient forms, and in one case of physical importance permit an exact 16-component solution to the coupled wave equations.

The special features of the wave equations give the constraint approach a practical advantage over the four classes of alternate approaches for realistic spectral calculations. Most importantly, the fact that our method uses two simultaneous Dirac equations provides us with a natural mechanism for reducing the equations (with no truncations or positive-energy projections) to four decoupled four-component Schrödinger-like forms,[2] in the same manner that the one-body Dirac equation can be reduced to two decoupled two-component forms. Since the two-body equations have incorporated two ordinary Dirac equations, their decoupled Schrödinger-like forms inherit the nonperturbative structure of their one-body counterparts as well. The decoupled upper and lower component equations of the ordinary Dirac equation have potential- (and energy-) dependent denominators in Darwin and $\mathbf{L}\cdot\mathbf{S}$ terms that reduce their effect in regions of strong potential. The two-body equations produce these as well as corresponding denominators for more general spin-dependent recoil terms. If these denominators were not present, certain singular potentials (e.g., Coulomb-like potentials) could only be treated perturbatively. In two-body descriptions that lack such denominators, one is forced to resort to *ad hoc* smoothing procedures if one wishes to use strong potentials. While it may turn out that quarks do have finite extension, one should not be forced to tamper with the short-range behavior of the potential merely through lack of an adequate wave equation.

Our two-body Dirac equations even inherit their momentum structure from Dirac's one-body equation. This means that in the c.m. frame the decoupled forms have momentum dependence as simple as that appearing in the nonrelativistic Schrödinger equation. Their kinetic forms are quadratic rather than square root, while their potentials do not necessarily need complicated momentum structure to represent relativistic interaction. Unlike what happens in many other approaches (e.g., that used in Ref. 14), the potential in this one is not derived from truncations of (16×16) matrix scattering amplitudes. Those truncations generally contain momentum-dependent factors that must be continued off mass shell.[14] The continuation produces operator-ordering ambiguities not necessarily present in the constraint approach. Instead, two-body Dirac equations take the off-shell structure automatically into account through the potential-dependent "matrix square-root" mass-shell constraints that define our method.

Most interestingly, even though our equations are not dependent on a field theory for their dynamical spin structure, they plausibly extend Dirac's one-body spin forces to the interacting two-body case. In fact, one can see that they are canonically related to the familiar Fermi-Breit approximation to the Bethe-Salpeter equation by unravelling their covariant form in a slow-motion weak-potential expansion.[4] This exposes the correct perturbative spin-orbit, Thomas precession, tensor, spin-spin, and Darwin interactions in just the same way that semirelativistic expansion of the ordinary one-body Dirac equation reveals the correct Pauli equation. Viewed from the standpoint of early atomic theorists, the twin successes of the one-body Dirac equation were that it brought order to the patchwork of semirelativistic corrections to the nonrelativistic Schrödinger equation and that it provided the crucial (purely quantum-mechanical) Darwin term producing the "correct" $2S_{1/2}$-$2P_{1/2}$ splitting. The two-body Dirac equations can be regarded as providing the same type of service: coordinating the patchwork of semirelativistic corrections abstracted from field theory, but in the framework of a relativistic quantum-mechanical structure, so that important quantum Darwin-like effects are automatically included. Moreover, their utility is not restricted to the domain of weak potentials and slow motion, the region described by perturbative quantum field theory.

The covariant constraint approach yields a consistent description of particles that interact directly through effective mechanical potentials that are functions of the relativistic particle degrees of freedom. Thus, the resulting dynamical system may be viewed as a purely phenomenological model of relativistic interaction. In turn, the mechanical potentials for a particular interaction structure are given in terms of a set of underlying Poincaré-invariant interaction functions. For example, we shall show that for scalar interactions and vector interactions, the number of independent invariant functions needed is three or less (in our quark-model paper[1] we used just one—a nonunique relativistic extrapolation of Richardson's nonrelativistic potential[27]). In phenomenological applications the forms of these interaction functions need not be tied to a particular field theory, although they may be motivated by one. On the other hand, if one wishes to use the constraint formalism to extract relativistic information from a quantum (or classical) field theory, one needs a systematic method to relate these invariant functions to that field theory. Since the constraint mechanics is initially defined at the classical level, one can first match its relativistic structures or semirelativistic expansions to those of classical fieldlike dynamics such as the electrodynamics of Wheeler and Feynman,[28] then quantize.[29] Or one may carry out the relation of field-theoretic quantities to invariant functions directly at the quantum level through a variety of procedures. One method associates the system wave equation produced by the constraint formalism with the perturbative scattering matrix of quantum field theory found through Todorov's inhomogeneous quasipotential equation[24] (a relativistic Lippmann-Schwinger equation derivable from the constraint approach[30]). For constraints of the form $p_i^2+m_i^2+\Phi=0$, this leads to a perturbative determination of Φ (Ref. 30). On the other hand, Sazdjian has started from the field-theoretic end—the homogeneous Bethe-Salpeter equation—and derived a transformation between the Bethe-Salpeter wave function and the constraint formalism wave function that simultaneously determines Φ in terms of the Bethe-Salpeter kernel.[31]

In this paper, we shall not pursue the issue of how and to what degree reduction of the two-particle sector of field theory to the two-particle constraint approach takes place. Instead, we conjecture that the practical features of the constraint equation listed above will emerge substantially intact. Furthermore, we contend that any such reduction of a gauge or scalar field theory ought to preserve the "memory" of the gauge or scalar structure underlying it by translating that structure into characteristic forms of the effective potentials. Thus, we conjecture that the two-body system for vector or scalar interaction looks just like that for two relativistic "charged" particles interacting with a classical field (except that the field seen by each particle has been replaced by an effective mechanical potential) introduced through minimal substitution on the constituent four-momenta, $p_i\rightarrow p_i-A_i$, and the mass-potential substitution $m_i\rightarrow m_i+S_i$.

The main goal of this paper is to show how the minimal momentum and mass substitutions lead in the constraint approach to the compatible wave equations (1a) and (1b) with spin-dependent potential structure. (For those readers who wish to go directly to the full form of these equations [(170), (171), and (189)], we have provided a summary (following the conclusion) with detailed results and definitions.) As they stand these equations provide a consistent relativistic description of the quantum-mechanical two-body system ready for the phenomenological applications to be detailed in a future paper.[32] However, as becomes apparent in the applications, these equations contain especially important relativistic behavior for strong potentials (e.g., at short distance) and light or zero-mass constituents that distinguishes them from the wave equations of other approaches. Such behavior results directly from the combined effects of the compatibility of the two Dirac equations that govern our system and the parametrization of the invariant interaction functions that determine the effective potentials appearing in the equations. For readers interested in the origin of these structures as well as for readers unfamiliar with relativistic constraint mechanics, we begin with a detour through the corresponding treatment for two spinless particles.

In Sec. II of this paper, we use minimal substitutions to derive two-body Klein-Gordon equations with simultaneous vector and scalar interactions. This section first reviews some standard procedures used in the constraint approach, starting from its derivation from classical constraint mechanics. Along the way, we encounter several new results. First, we generalize our earlier treatment to include relative-momentum-dependent interactions. At the same time, we demonstrate that vector and scalar interactions can be introduced simultaneously in such a way that each separately leads to compatible constituent wave equations, and that it takes no more than two invariant functions to specify the vector interaction and two to specify the scalar interactions (only three of these functions are independent, however). We require that the constituent mass and energy potentials (M_i and E_i) depend on these functions in a way that incorporates the correct heavy-particle and nonrelativistic limits. An important consequence of the covariance of the constraint approach is that these constituent potentials automatically contain the correct semirelativistic limit [including slow-motion and weak-potential effects of $O(1/c^2)$]. Finally, we give the quantum wave equations corresponding to the classical constraints. We review previous applications of these equations, present some new results, and indicate future applications. The primary role of this section is to set up the proper spinless dynamics into which we introduce spin in the next two sections.

The treatment of spin given in those sections starts from the construction of a relativistic "pseudoclassical" mechanics for spinning Dirac particles described by Grassmann variables that are constrained by a "pseudoclassical" version of the ordinary one-body Dirac equation. This approach to "pseudoclassical spin" results from application of the correspondence principle to generalized quantum brackets (involving anticommutators as well as commutators) associated with the ordinary Dirac equation. In that case, canonical quantization of the "pseudoclassical mechanics" leads back to the Dirac equations. In Refs. 2 and 3, we found that we could construct a consistent "pseudoclassical" mechanics for a system of two relativistic point particles either of which may have its own Dirac spin. There, we found that for two particles interacting through a world scalar potential we could construct compatible Dirac-like constraints by exploiting supersymmetries present in the "pseudoclassical" correspondence limit of the ordinary Dirac equation. Simply stated, we introduced those supersymmetries into the two-body spinless constraint formalism by replacing the two generalized mass-shell constraints with two Dirac-like constraints, one for each spinning particle. We found that these constraints became automatically compatible when we required each to remain supersymmetric in the presence of mutual interaction, if the corresponding spinless constraints were already compatible. That is, our use of supersymmetry reduced the problem of compatibility of the spin-dependent dynamics to that of the corresponding spinless system. After reviewing this procedure for scalar interactions of a spin-$\frac{1}{2}$ particle with a spin-zero particle, we show that an analogous treatment for timelike vector interactions also leads to compatible constraints. We also find that when both types of interactions are present simultaneously, the supersymmetries associated with each in isolation are broken. Nevertheless, the structures that enforce each supersymmetry in the limits that either type of interaction vanishes are sufficient to guarantee the compatibility of the two Dirac-like constraints when both types of interaction are simultaneously present. Both of these (broken) supersymmetries dictate a common mechanism for the generation of spin-dependent interactions through $\tilde{S}$ and $\tilde{A}$, which are the same functions of a supersymmetric position variable $(\tilde{x}_1-\tilde{x}_2)_\perp$ as the spinless S and A are of the relative position x_1-x_2. We show how *both* forms are modified when an additional electromagneticlike interaction is introduced. This leads to compatible constraints for a system of scalar, timelike vector, and electromagneticlike vector interactions. Quantization of the resulting constraints leads to a compatible system of wave equations for an interacting spin-$\frac{1}{2}$ spin-zero system.

In Secs. V and VI we derive a system of two coupled but compatible Dirac equations for two spinning particles interacting through simultaneous scalar and vector interactions. We find these "two-body Dirac equations" by first constructing a compatible system of pseudoclassical constraints in which each spinning particle is described by its own set of five Grassmann variables. Again, when either timelike vector or scalar interactions (but not both) are present, we find that the two Dirac-like constraints become compatible when we require that the spinless interaction potentials for the constituent particles S_i and A_i be replaced by supersymmetric versions $\tilde{S}_i$ and $\tilde{A}_i$ of the same functional form but with supersymmetric coordinate arguments. So too, when both types of interaction are present, the broken supersymmetries that become exact when either type of interaction vanishes are sufficient to generate compatible constraints provided that the corresponding spinless system is already compatible. Finally, we generalize once more to arrive at compatible constraints containing simultaneous scalar, timelike vector, and electromagneticlike vector interactions. Because of the isomorphism between pseudosclassical brackets and classical (anti)commutators, canonical quantization automatically turns these constraints into two coupled but compatible Dirac equations. Our use of supersymmetries to introduce spin into a consistent spinless dynamics has important physical consequences linked with its mathematical consequence—compatibility. The extra terms induced by dependence of interactions on supersymmetric position variables turn out to be important spin-dependent recoil terms. Furthermore, supersymmetry introduces this extra spin structure without disrupting the correct heavy-particle, semirelativistic, and nonrelativistic limits that are built into our spinless potentials. The spin structure imposed by supersymmetry merely guides these limits into forms appropriate to the spinning system.

The most important consequence of the compatibility of our two coupled Dirac equations that makes them ideally suited for physical calculations is that they may be rearranged in a consistent fashion into forms convenient for particular applications. These forms often inherit especially simple structure from the original Dirac equations. For example, repeated use of these equations yields an exact reduction from the original two coupled 16-component wave equations to four decoupled four-component Schrödinger-like equations. None of the couplings among the upper-upper, upper-lower, lower-upper, and lower-lower components of a 16-component spinor need to be truncated. Unlike what happens in other approaches that have only one two-body equation, in our approach the existence of two two-body Dirac equations allows one to fold in the effects of all components of the wave function. The resultant quasipotential contains familiar features including the expected Darwin, spin-spin, spin-orbit, and tensor parts embedded in new strong-potential relativistic structure.

II. RELATIVISTIC CONSTRAINT DYNAMICS FOR TWO SPINLESS PARTICLES UNDER MUTUAL VECTOR AND SCALAR INTERACTIONS

We begin by conjecturing that the two-body system governed by vector and scalar interactions resembles that of two relativistic particles interacting via effective potentials introduced through minimal substitution on the constituent four-momenta and masses. In previous work,[2–4] we treated the interactions separately, showing what restrictions are placed on them by the compatibility condition on the constraints. In this section, we show that when both are present, the two parts of the interaction (scalar and vector) lead separately to compatibility. In this demonstration we also include relative-momentum dependence of the interactions, not present in our earlier work. We finally arrive at the system Hamiltonians for scalar and timelike vector interactions, scalar and electromagneticlike vector interactions, and, thirdly, scalar with both timelike and electromagneticlike vector interactions. The compatibility condition restricts the number of independent invariant interaction functions to three in the last case and two in the first and second cases. We show how the constituent mass (M_i) and energy (E_i) potentials depend on these scalars. Surprisingly, we find that the constituent mass potentials depend not only on the invariant function associated with the scalar interaction, but also on that associated with the electromagneticlike interaction. Finally, we display the quantum wave equations corresponding to "stationary states." These are the spinless versions of the "two-body Dirac equations" (1a) and (1b) that are the fundamental results of our paper.

The mass-shell constraint on a free particle's four-momentum is

$$\mathcal{H}=p^2+m^2\approx 0 \ , \tag{2}$$

into which one introduces interaction with external vector and scalar potentials through the modifications

$$p^\mu \to \pi^\mu = p^\mu - A^\mu, \quad m \to M = m+S \ .$$

For a system of two interacting spinless particles we postulate that the corresponding generalized mass-shell constraints are

$$\mathcal{H}_1=\pi_1{}^2+M_1{}^2\approx 0, \quad \mathcal{H}_2=\pi_2{}^2+M_2{}^2\approx 0 \ , \tag{3}$$

where $\pi_i^\mu = p_i^\mu - A_i^\mu$ and $M_i = m_i + S_i$. In this constituent potential approach, each particle appears to be in an external potential with the other particle as source. However, the A_i^μ's and S_i's are not fields but rather effective potentials that are point functions of the particle coordinates and momenta. Eventually, the constituent A_i^μ's and S_i's are identified with appropriate quantum-field-theoretic potentials having the corresponding transformation properties.

The Dirac Hamiltonian for this constrained system is $\mathcal{H}=\lambda_1\mathcal{H}_1+\lambda_2\mathcal{H}_2$, so that a sufficient condition for $\mathcal{H}_1$ and $\mathcal{H}_2$ to be conserved in the single evolution parameter τ (i.e., $\dot{\mathcal{H}}_i=\{\mathcal{H}_i,\mathcal{H}\}\approx 0$) is that the constraints be first class:

$$\{\mathcal{H}_1,\mathcal{H}_2\}\approx 0 \ . \tag{4}$$

The weak equality signs in (3) and (4) mean that the constraints are to be imposed only after working out the Poisson brackets. Condition (4) then confines the motion to the constraint hypersurface defined in (3). The left-hand sides need only vanish on that surface but may vanish identically (strongly). Condition (4) ensures that the action of the constraints is compatible with the evolution generated by the Dirac Hamiltonian governing this system.

The fundamental brackets among the constituent variables are

$$\{x_i^\mu, p_j^\nu\} = g^{\mu\nu}\delta_{ij} \ . \tag{5}$$

The square of the total four-momentum,

$$P=p_1+p_2, \quad P^2=-w^2 \ , \tag{6}$$

defines the total energy w in the c.m. ($\mathbf{P}=0$) frame. As in nonrelativistic mechanics, we introduce canonical relative position and momentum variables defined by

$$x=x_1-x_2, \quad p=\frac{1}{w}(\epsilon_2 p_1-\epsilon_1 p_2) \ . \tag{7}$$

The constituent momenta are related to the total and relative momenta[33] by $p_1=\epsilon_1\hat{P}+p$ and $p_2=\epsilon_2\hat{P}-p$, where $\hat{P}=P/w$ ($\hat{P}^2=-1$), while the requirement $\{x^\mu,p^\nu\}=g^{\mu\nu}$ forces $\epsilon_i=\epsilon_i(P^2)$ and $\epsilon_1+\epsilon_2=w$. Since $\{x^\mu,P^\nu\}=0$ and the $\mathcal{H}_i$ depend only on relative x, the c.m. total energy w is a constant of the motion.

If we assume translation invariance of the system, we can write the constraints in the forms

$$\mathcal{H}_1=p_1{}^2+m_1{}^2+\Phi_1(x,p_1,p_2)\approx 0 \ , \tag{8a}$$

$$\mathcal{H}_2=p_2{}^2+m_2{}^2+\Phi_2(x,p_1,p_2)\approx 0 \ , \tag{8b}$$

where

$$\Phi_1=M_1{}^2-m_1{}^2+\pi_1{}^2-p_1{}^2 \ , \tag{9a}$$

$$\Phi_2=M_2{}^2-m_2{}^2+\pi_2{}^2-p_2{}^2 \ . \tag{9b}$$

In terms of these forms, the compatibility condition (4) then becomes

$$2p_1\cdot\partial\Phi_2+2p_2\cdot\partial\Phi_1+\{\Phi_1,\Phi_2\}\approx 0 \ . \tag{10}$$

If we choose[17]

$$\Phi_1=\Phi_2=\Phi(x_\perp,p_1,p_2) \ , \tag{11}$$

where

$$x_\perp^\mu=(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)x_\nu \ , \tag{12}$$

then the equality in Eq. (10) is strong (the constraints in this case are said to be strongly compatible). The reason for this is that since the only independent nonzero invariants involving $x_\perp$ and $x_\perp{}^2$ and $x_\perp\cdot p$ ($=x_\perp\cdot p_1$), $P\cdot\partial\Phi$ is proportional to $P\cdot x_\perp\equiv 0$ and $P\cdot p_\perp\equiv 0$.

Equation (11) serves as a relativistic counterpart of Newton's third law and leads to the interaction-independent constraint

$$\mathcal{H}_1-\mathcal{H}_2=2P\cdot p+(\epsilon_2-\epsilon_1)w+m_1{}^2-m_2{}^2\approx 0 \ . \tag{13}$$

Using this constraint and $\epsilon_1+\epsilon_2=w$, we can eliminate ϵ_i in $p_i{}^2$ to obtain the system Hamiltonian

$$\begin{aligned}\mathcal{H}_1\approx\mathcal{H}_2&\approx p^2-b^2(w)+(\hat{P}\cdot p)^2+\Phi\\&=p_\perp{}^2-b^2(w)+\Phi\approx 0 \ ,\end{aligned} \tag{14}$$

where $p^2=p_\perp{}^2-(\hat{P}\cdot p)^2$ and

$$\begin{aligned}b^2(w)&\equiv[w^4-2w^2(m_1{}^2+m_2{}^2)+(m_1{}^2-m_2{}^2)^2]/4w^2\\&=\frac{\Delta(w^2,m_1{}^2,m_2{}^2)}{4w^2}\end{aligned} \tag{15}$$

is the invariant square of the relative three-momentum of two free particles in the c.m. system. The appearance of the triangle function is a signal that these equations display the correct two-body relativistic kinematics. In particular, Eq. (14) implies $p_\perp{}^2=b^2$ when both particles are on their respective mass shells ($\Phi=0$). If in addition to being canonically conjugate to x_μ, the relative momentum is determined so that on the constraint hypersurface defined by (3) it is spacelike,

$$P\cdot p\approx 0 \ , \tag{16}$$

then $p\approx p_\perp$ [$=(0,\mathbf{p})$ in the c.m. system] so that p is the covariant extension of the usual relative three-momentum. This statement is supported by other consequences of $\hat{P}\cdot p\approx 0$. First note that this condition follows directly from the constraint difference (13) if we choose

$$\epsilon_1-\epsilon_2=\frac{1}{w}(m_1{}^2-m_2{}^2) \ . \tag{17}$$

Then the variables ϵ_i are equal to $-p_i\cdot\hat{P}$ on the constraint hypersurface and, thus, can be interpreted as the (conserved) c.m. energies of the constituent particles. They are given by

$$\begin{aligned}\epsilon_1&=(w^2+m_1{}^2-m_2{}^2)/2w \ ,\\ \epsilon_2&=(w^2+m_2{}^2-m_1{}^2)/2w \ .\end{aligned} \tag{18}$$

This procedure thus completely determines the canonical variables. Because of Eqs. (11) and (16), two of the variables, the relative energy and relative time in the c.m. system, have effectively disappeared. (Note that in the nonrelativistic limit, $\mathbf{p}$ becomes $(m_2\mathbf{p}_1-m_1\mathbf{p}_2)/m_1m_2$ [$=\mu(d\mathbf{r}/dt)$ for free particles], the conventional nonrelativistic expression for the relative momentum. This is a property not shared by the choice $(\mathbf{p}_1-\mathbf{p}_2)/2$ unless one works in the c.m. frame or with equal masses.)

Given the third-law condition (11), Eq. (17) implies

$$\mathcal{H}_1-\mathcal{H}_2=2P\cdot p \ . \tag{19}$$

[A weak third-law form $\Phi_1(x_\perp,p_1,p_2)-\Phi_2(x_\perp,p_1,p_2)\propto P\cdot p\approx 0$ is still a sufficient condition for the compatibility condition to be satisfied (though weakly), since $P\cdot p$ has zero brackets with Φ_1 and Φ_2. The compatibility condition can also be solved if $\Phi_1(x_\perp,p_1,p_2)-\Phi_2(x_\perp,p_1,p_2)$ is not weakly zero. We show in Appendix A that it is still possible in that case to have (14) or (20) below satisfied.] Since

$$\begin{aligned}\mathcal{H}_1&=p^2-\epsilon_1{}^2+m_1{}^2+\Phi+2\epsilon_1\hat{P}\cdot p \ ,\\ \mathcal{H}_2&=p^2-\epsilon_2{}^2+m_2{}^2+\Phi-2\epsilon_2\hat{P}\cdot p \ ,\end{aligned} \tag{8$'$}$$

the remaining independent combination of the constraints (not involving $\hat{P}\cdot p$) then becomes

$$\mathcal{H}\equiv\frac{\epsilon_2}{w}\mathcal{H}_1+\frac{\epsilon_1}{w}\mathcal{H}_2=p^2-b^2(w)+\Phi\approx\mathcal{H}_1\approx\mathcal{H}_2\approx 0 \ , \tag{20}$$

where the first two weak equalities result from $\hat{P}\cdot p\approx 0$, Eq. (11), and the fact that $\epsilon_1{}^2-m_1{}^2=\epsilon_2{}^2-m_2{}^2=b^2(w)$. Just as does its (weakly) equivalent form (14), the system Hamiltonian Eq. (20) (which is also of the form of Todorov's quasipotential Hamiltonian)[4,24] not only conserves the constraint $\hat{P}\cdot p\approx 0$, but also incorporates the correct relativistic two-body kinematics. In his quasipotential approach,[17,24] Todorov defines the other useful variables

$$m_w=\frac{m_1m_2}{w} \ , \tag{21a}$$

$$\epsilon_w = \frac{w^2 - m_1{}^2 - m_2{}^2}{2w} , \tag{21b}$$

interpreted as the relativistic reduced mass and energy of a fictitious particle of relative motion.[34] In terms of them,

$$b^2 = \epsilon_w{}^2 - m_w{}^2 , \tag{22}$$

reinforcing this interpretation. Using these variables, we can rewrite (20) in an effective one-particle generalized mass-shell or Klein-Gordon form if we define

$$\mathcal{P}^\mu = p^\mu + \epsilon_w \hat{P}^\mu . \tag{23}$$

The resulting form of (20) is weakly equivalent to

$$\mathcal{P}^2 + m_w{}^2 + \Phi \approx 0 . \tag{24}$$

Note that, in the c.m. system, $\mathcal{P}^\mu = (\epsilon_w, \mathbf{p})$.

Beyond the dependence of the interaction on $x_\perp$, the compatibility condition (4) imposes further restrictions on the π_i's and M_i's of Eq. (3) through the "third law," Eq. (11). We shall find that the scalar and vector parts separately lead to compatibility. To see this, first note that the third-law requirement $\Phi_1 = \Phi_2$ can be written as

$$\begin{aligned} \Phi &= M_1{}^2 - m_1{}^2 + \pi_1{}^2 - p_1{}^2 \\ &= M_2{}^2 - m_2{}^2 + \pi_2{}^2 - p_2{}^2 . \end{aligned} \tag{25}$$

When there is no vector interaction, $\pi_i = p_i$ and hence $\Phi = M_1{}^2 - m_1{}^2 = M_2{}^2 - m_2{}^2$, while when there is no scalar interaction, $M_i = m_i$ and $\Phi = \pi_1{}^2 - p_1{}^2 = \pi_2{}^2 - p_2{}^2$. We assume that Φ has a unique "scalar part":

$$\begin{aligned} \Phi_S &\equiv M_1{}^2 - m_1{}^2 \\ &= M_2{}^2 - m_2{}^2 = 2m_1 S_1 + S_1{}^2 = 2m_2 S_2 + S_2{}^2 , \end{aligned} \tag{26}$$

even in the presence of vector interactions. This assumption together with (25) implies that Φ has a unique "vector part":

$$\begin{aligned} \pi_1{}^2 - p_1{}^2 = \pi_2{}^2 - p_2{}^2 \equiv \Phi_A &= -2p_1 \cdot A_1 + A_1{}^2 \\ &= -2p_2 \cdot A_2 + A_2{}^2 . \end{aligned} \tag{27}$$

In the special case in which the S_i are assumed independent of the relative momentum (p), it is necessary that the third-law condition for the scalar part of Φ be split off from the vector part since the vector part is p dependent. However, we will assume that this division persists even when the S_i are p dependent. A consequence (see below) of this assumption is that the parts of $\mathcal{H}_i$ that arise from scalar and vector interactions separately contribute to compatibility. As also shown [see Eqs. (43a) and (43b)], this division does not, however, prevent M_1 and M_2 from developing dependence on the strength of the vector interaction. Our assumption (26) can also be viewed as a statement about the overall independence of the scalar and vector interactions in the general case.

Next we note that the constituent A_i^μ's can be written in the form

$$A_1^\mu = \alpha_1 p_1^\mu + \beta_1 p_2^\mu , \tag{28a}$$

$$A_2^\mu = \alpha_2 p_2^\mu + \beta_2 p_1^\mu . \tag{28b}$$

An alternative collective parametrization, using the independent system variables $\hat{P}^\mu$ and p^μ, yields

$$\pi_1^\mu = E_1 \hat{P}^\mu + G_1 p^\mu , \tag{29a}$$

$$\pi_2^\mu = E_2 \hat{P}^\mu - G_2 p^\mu , \tag{29b}$$

with the scalar functions E_i and G_i used instead of α_i and β_i. This parametrization is an interaction-dependent version of the expressions for the constituent momenta p_i given below (7). In either parametrization, each of the invariant functions is of the form $f(x_\perp, p_1, p_2)$. With no interaction, $E_i = \epsilon_i$ and $G_i = 1$. Notice that we have omitted terms proportional to $x_\perp^\mu$ from (28) and (29) because they would produce unobservable gauge changes. By this we mean that such terms could be removed by canonical transformation on $p_i^\mu \to p_i^\mu + \partial^\mu \chi_i(x_\perp)$. In the quantum case such parts (corresponding to $x_\perp \cdot p$ terms) could be eliminated through a scale transformation. Generalizing our earlier work,[2–4] we assume here that the scalar functions E_i, G_i, and M_i may depend on the relative-momentum variable p as well as on the total momentum P. The assumption that $\Phi_1 = \Phi_{A1} + \Phi_{S1} = \Phi_2 = \Phi_{A2} + \Phi_{S2} \equiv \Phi_A + \Phi_S$, implies that

$$\begin{aligned} \{\mathcal{H}_1, \mathcal{H}_2\} &= \{p_1{}^2 + \Phi_A + \Phi_S, p_2{}^2 + \Phi_A + \Phi_S\} \\ &= \{p_1{}^2 - p_2{}^2, \Phi_A + \Phi_S\} = 0 . \end{aligned} \tag{30}$$

Since Φ_A and Φ_S are each functions of $x_\perp$, the scalar and vector parts of the $\mathcal{H}_i$ separately (and strongly) lead to the compatibility of the constraints.

Equation (26) leaves us with just one independent combination of S_1 and S_2. We choose to call it S. Furthermore, Eq. (27) implies

$$\begin{aligned} &E_1{}^2 - \epsilon_1{}^2 - 2\hat{P} \cdot p(G_1 E_1 - \epsilon_1) - (G_1{}^2 - 1)p^2 \\ &\quad = E_2{}^2 - \epsilon_2{}^2 + 2\hat{P} \cdot p(G_2 E_2 - \epsilon_2) - (G_2{}^2 - 1)p^2 . \end{aligned} \tag{31}$$

We take $G_1 = G_2 \equiv G$, giving us the simplest algebraic solution for G, one that is independent of p^2. This restriction still leaves us enough room to encompass perturbative field-theoretic behavior of the effective potentials. With this assumption, (31) becomes

$$E_1{}^2 - 2GE_1 \hat{P} \cdot p - (E_2{}^2 + 2GE_2 \hat{P} \cdot p) = \epsilon_1{}^2 - \epsilon_2{}^2 - 2w\hat{P} \cdot p , \tag{32}$$

so that E_1 and E_2 are also not independent. The assumption that $G_1 = G_2$ is also equivalent to the statement that the spacelike and timelike parts of Φ_{Ai} obey separate third-law conditions. Equation (32) implies that on the constraint hypersurface,

$$E_1{}^2 - E_2{}^2 = \epsilon_1{}^2 - \epsilon_2{}^2 . \tag{33}$$

If we take this condition as a strong one, it together with the ansatz $G_1 = G_2$ implies there are just two independent scalar combinations of E_1, E_2, and G. We call them $\mathcal{A}$ and $\mathcal{V}$. We choose $G=G(\mathcal{A})$, thus redefining $\mathcal{A}$ instead of G as a basic unit in our potential. [A necessary condition for (29a) and (29b) to possess the correct free particle limit is that $G(0)=1$.] This choice also implies that $E_i=E_i(\mathcal{A},\mathcal{V})$, so that, in general, (29a) and (29b) become

$$\pi_1^\mu=E_1(\mathcal{A},\mathcal{V})\hat{P}^\mu+G(\mathcal{A})p^\mu , \tag{34a}$$

$$\pi_2^\mu=E_2(\mathcal{A},\mathcal{V})\hat{P}^\mu-G(\mathcal{A})p^\mu . \tag{34b}$$

If we define additional scalar variables $\mathcal{A}_i(\mathcal{A},\mathcal{V})$ by

$$G(\epsilon_i-\mathcal{A}_i(\mathcal{A},\mathcal{V}))\equiv E_i ,$$

Eq. (33) then becomes

$$2\epsilon_1\left[\mathcal{A}_1-\frac{w}{2}(1-G^{-2})\right]-\mathcal{A}_1^2 =2\epsilon_2\left[\mathcal{A}_2-\frac{w}{2}(1-G^{-2})\right]-\mathcal{A}_2^2 . \tag{35}$$

Since G is a function only of $\mathcal{A}$, its form can be determined [from (35)] by choosing $\mathcal{V}=0$ and $\mathcal{A}_1=\mathcal{A}_2=\mathcal{F}(\mathcal{A})$, leading to

$$G^2=\frac{1}{1-2\mathcal{F}(\mathcal{A})/w} . \tag{36}$$

Having evaluated G, we note that the choice $\mathcal{F}(\mathcal{A})=0$ ($G=1$) leads to a timelike A_i^μ [$=\mathcal{A}_i(0,\mathcal{V})\hat{P}^\mu$] and, as shown in Ref. 4, the choice $\mathcal{A}_1=\mathcal{A}_2=\mathcal{F}(\mathcal{A})=\mathcal{A}(\mathcal{V}=0)$ leads to an electromagneticlike four-vector. Surprisingly, these two special cases turn out to have the same classical dynamics. However, as shown in Ref. 4, Hermitian ordering of the corresponding quantum operators leads to inequivalent quantum dynamics. For now, $\mathcal{V}$ and $\mathcal{A}$ will be general. In this general case we will find that the constituent scalars (S_i) depend on the strength of the electromagneticlike interaction function $\mathcal{A}$ as well as S.

Our aim now is to relate these scalar invariants ($S,\mathcal{V},\mathcal{A}$) to the energy and mass potentials (M_i,E_i). These relationships merely lead to a parametrization of E_i and M_i and have nothing to do with compatibility.[35] We make one inflexible demand on these relationships: that they lead to the correct static and nonrelativistic limits. We first treat the special case of a combined scalar and timelike vector interaction ($\mathcal{V}\neq 0$, $\mathcal{A}=0$). If we modify the free ($\Phi=0$) form of the effective one-body Klein-Gordon equation (24) through $m_w\to M_w=m_w+S$ and $\mathcal{P}^\mu\to\pi^\mu=\mathcal{P}^\mu-\mathcal{V}\hat{P}^\mu$, the system Hamiltonian (20) becomes

$$\mathcal{H}=\pi^2+M_w^2=p^2-(\epsilon_w-\mathcal{V})^2+(m_w+S)^2 \approx\mathcal{H}_1\approx\mathcal{H}_2\approx 0 . \tag{37}$$

(Note that in the limit that particle two becomes infinitely heavy, $\epsilon_w\to\epsilon_1$, $m_w\to m_1$ and, thus, this equation becomes the one-body Klein-Gordon constraint for a single particle in an external scalar and vector potential.) Then comparison of this form with (3) allows us to relate the energy and mass potentials (E_i and M_i) to the scalars $\mathcal{V}$ and S. In particular, (34a) and (34b) and $\epsilon_i^2-m_i^2=\epsilon_w^2-m_w^2$ imply that E_1 and E_2 are related to the scalar $\mathcal{V}$ by

$$E_1(0,\mathcal{V})^2=(\epsilon_1-\mathcal{A}_1)^2\approx\epsilon_1^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2 , \tag{38a}$$

$$E_2(0,\mathcal{V})^2=(\epsilon_2-\mathcal{A}_2)^2\approx\epsilon_2^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2 , \tag{38b}$$

while M_1 and M_2 are related to S by

$$M_1^2(\mathcal{A}=0,S)=(m_1+S_1)^2=m_1^2+2m_wS+S^2 , \tag{39a}$$

$$M_2^2(\mathcal{A}=0,S)=(m_2+S_2)^2=m_2^2+2m_wS+S^2 . \tag{39b}$$

The equalities in (38) are weak because of (32). For weak potentials and "weak binding"

$$(\,|w-m_1-m_2|\ll m_1,m_2) ,$$

the potential-energy term in (37) becomes $2\mu(\mathcal{V}+S)$, where $\mu=m_1m_2/(m_1+m_2)$ in agreement with the nonrelativistic expression. We can further relate the invariant scalars S and $\mathcal{V}$ to field-theoretic potentials by performing an $O(1/c^2)$ expansion of (37). One finds[4]

$$w=m_1+m_2+\frac{\mathbf{p}^2}{2\mu}+\mathcal{V}+S-\frac{(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2\mathcal{V}}{m_1m_2}+\frac{\mathcal{V}^2}{2(m_1+m_2)}\frac{\mathcal{V}S}{(m_1+m_2)}-\frac{\mathbf{p}^2S}{2}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]+\frac{S^2}{2(m_1+m_2)} . \tag{40}$$

Note that the scalar and vector potentials S and $\mathcal{V}$ give distinctly different $O(1/c^2)$ contributions to the total c.m. energy w. Those potentials can be then determined by comparison with a corresponding field-theoretic expansion of w (see Ref. 4 and example of $\mathcal{A}$ given below).

The other special case, where $\mathcal{V}=0$ and $\mathcal{A}_1=\mathcal{A}_2=\mathcal{F}(\mathcal{A})$, gives A_i^μ that are not timelike, but contain spacelike parts as well. With G^2 as given by (36), we find

$$E_i=E_i(\mathcal{A},0)=G(\epsilon_i-\mathcal{F}(\mathcal{A})) . \tag{41}$$

In this case, $2(E_1+E_2)G=w$ and, thus, (32) and (33) are strongly equivalent. In the absence of the scalar, $\mathcal{H}_i$ is

$$\begin{aligned}\pi_i^2+m_i^2&\approx G^2(p^2-[\epsilon_i-\mathcal{F}(\mathcal{A})]^2)+m_i^2\\&=G^2(p^2-[\epsilon_i-\mathcal{F}(\mathcal{A})]^2+m_i^2/G^2)\\&=G^2(p^2-[\epsilon_w-\mathcal{F}(\mathcal{A})]^2+m_w^2) ,\end{aligned} \tag{42}$$

since $\epsilon_i-m_i^2/w=\epsilon_w$. The exact form of $\mathcal{F}(\mathcal{A})$ is not completely arbitrary. In Ref. 4 we showed that a semirelativistic expansion of (42) [one that includes terms of $O(1/c^2)$] is given by

$$w=m_1+m_2+\frac{\mathbf{p}^2}{2\mu}+\mathcal{F}(\mathcal{A})-\frac{(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2\mathcal{F}(\mathcal{A})}{m_1m_2}+\frac{\mathcal{F}(\mathcal{A})^2}{2(m_1+m_2)}$$

in the c.m. system. In the nonrelativistic limit we must have $\mathcal{F}(\mathcal{A})=\mathcal{A}$ if we identify $\mathcal{A}$ with the potential energy. We also found that if we used this nonrelativistic identification in the two $O(1/c^2)$ parts of the potential at the end of the expression above and let $\mathcal{A}=-\alpha/r$, then we obtained the standard Darwin Hamiltonian (after a canonical transformation) with the last two terms being replaced by

$$-\frac{\alpha}{2m_1m_2}\mathbf{p}\cdot\left[\frac{1}{r}+\frac{\mathbf{rr}}{r^3}\right]\cdot\mathbf{p}\ .$$

This correct semirelativistic limit is preserved by the choice $\mathcal{F}(\mathcal{A})=\mathcal{A}[1+O(\mathcal{A}^2)]$. In the remainder of this paper we assume $\mathcal{F}(\mathcal{A})=\mathcal{A}$ (Ref. 36).

When we put the scalar in as before (so as to give the correct nonrelativistic and heavy-particle limits) using the modification $m_w\rightarrow m_w+S$ then, in place of (39a) and (39b), we obtain

$$M_1^2(\mathcal{A},S)=m_1^2+G^2(2m_wS+S^2)=(m_1+S_1)^2\ , \tag{43a}$$

$$M_2^2(\mathcal{A},S)=m_2^2+G^2(2m_wS+S^2)=(m_2+S_2)^2\ . \tag{43b}$$

Thus, the individual S_i's depend on the strength of the vector potential in addition to that of the scalar. If G^2 did not appear in these equations, then in the semirelativistic limit the effective potential would contain an unacceptable $S\mathcal{A}$ cross term.[37] Since, however, G^2 is just an overall multiplicative factor in (42) (when $m_w\rightarrow m_w+S$), the scalar dynamics is independent of $\mathcal{A}$. It contributes only in the spin-dependent case where scalar spin effects turn out to depend on $\mathcal{A}$.

In the most general case, both $\mathcal{V}$ and $\mathcal{A}$ contribute to the timelike part of the vector interaction, leading to

$$E_1^2(\mathcal{A},\mathcal{V})\approx G^2((\epsilon_1-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2)$$
$$=G^2(\epsilon_1-\mathcal{A}_1)^2\ , \tag{44a}$$

$$E_2^2(\mathcal{A},\mathcal{V})\approx G^2((\epsilon_2-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2)$$
$$=G^2(\epsilon_2-\mathcal{A}_2)^2\ . \tag{44b}$$

[This form has the virtue that it reduces to either the timelike or electromagneticlike result if $\mathcal{A}=0$ or $\mathcal{V}=0$ respectively, and produces no $\mathcal{V}\mathcal{A}$ cross terms in the expression for $\mathcal{H}$ (Ref. 35).] We still obtain $M_i(S,\mathcal{A})$ given in (43a) and (43b), so that

$$\mathcal{H}_i=\pi_i^2+M_i^2\approx 0\approx\mathcal{H}\approx G^2(p^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+2m_wS+S^2+m_w^2-\epsilon_w^2)\approx 0\ . \tag{45}$$

Note carefully that $\mathcal{A}$ and $\mathcal{V}$ do not appear in the form of a sum.

In order to quantize the constraint formalism we must construct the quantum versions of the $\mathcal{H}_i$'s. We wish to maintain the gauge structure in terms of Hermitian π_i^μ and M_i. Classically, the underlying scalars $\mathcal{A}$, $\mathcal{V}$, and S may depend on $x_\perp^2$, $x_\perp\cdot p$, p^2, and w^2 or $x_\perp^2$, l^2, p^2, and w^2, where $l^2=x_\perp^2p^2-(x_\perp\cdot p)^2$ $[=(\mathbf{r}\times\mathbf{p})^2$ in the c.m. system] is the invariant square of the relative angular momentum. (Linear dependence on $x_\perp\cdot p$ can normally be eliminated by a scale transformation.) The system variables p and $\hat{P}$ are well defined if we restrict our space so that P^2 has only timelike eigenvalues. If we assume that the E_i $[=G(\epsilon_i-\mathcal{A}_i)]$ are Hermitian, then the classical expression (34a) and (34b) suggests the Hermitian forms

$$\pi_1^\mu=G\left[\hat{P}^\mu(\epsilon_1-\mathcal{A}_1)+p^\mu+\frac{1}{2i}\nabla^\mu\ln G\right]\ , \tag{46a}$$

$$\pi_2^\mu=G\left[\hat{P}^\mu(\epsilon_2-\mathcal{A}_2)-p^\mu-\frac{1}{2i}\nabla^\mu\ln G\right]\ . \tag{46b}$$

In Ref. 4, we treated the details of such a quantization procedure. Suffice it to say that if we use the quantum brackets

$$[x_\perp^\mu,p^\mu]=i(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)\ ,$$

then we can verify in direct analogy to the classical arguments that the commutator $[\mathcal{H}_1,\mathcal{H}_2]$ vanishes and that the quantum version of the classical constraints $\mathcal{H}_i\approx 0$ become simultaneous compatible conditions on the single wave function,

$$\mathcal{H}_1\psi=(\pi_1^2+M_1^2)\psi=0\ , \tag{47a}$$

$$\mathcal{H}_2\psi=(\pi_2^2+M_2^2)\psi=0\ . \tag{47b}$$

In what follows, we assume that ψ is an eigenfunction of the total momentum, so that P^μ becomes a c number. Then the difference constraint (16) becomes

$$P\cdot p\psi_P=0\ , \tag{48}$$

a differential equation in the relative variable x_μ (since $p^\mu=-i\partial/\partial x_\mu$ in the coordinate representation). Thus, $\psi_P(x)=\psi_P(x_\perp)$, with a possible norm given by[17,38]

$$\int\psi_P^*(x_\perp)\psi_P(x_\perp)d^3x_\perp\ ,$$

where $d^3x_\perp$ is the covariant measure $\delta(\hat{P}\cdot x)d^4x$. The quantum counterpart to the remaining independent constraint (45) is the system wave equation

$$\mathcal{H}\psi=0\ .$$

In the c.m. ($P^0=w$, $\mathbf{P}=0$) system, this is explicitly

$$G^2\left[\mathbf{p}^2-(\epsilon_w-\mathcal{A})^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+(m_w+S)^2+\frac{2}{i}\nabla\ln G\cdot\mathbf{p}-\tfrac{1}{2}\nabla^2\ln G-\tfrac{3}{4}(\nabla\ln G)^2\right]\psi=0\ , \tag{49}$$

where we have used the fact that (48) implies $p^2\psi=\mathbf{p}^2\psi$ in the c.m. rest frame.

We assume that whatever additional Hermitian ordering is necessary may be performed. Notice that (49) will not be local if its potentials have $\mathbf{p}^2$ dependence in addition to $\mathbf{x}^2$ and l^2 dependence. If we let $\phi = G\psi$, then the wave equation takes the simpler form:

$$\left[\mathbf{p}^2-(\epsilon_w-\mathcal{A})^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+(m_w+S)^2 + \tfrac{1}{2}\nabla^2\ln G+\tfrac{1}{4}(\nabla\ln G)^2\right]\phi=0 . \quad (50)$$

Equation (50) describes two spinless particles under mutual scalar, timelike vector, and electromagneticlike vector interaction. The system wave equation has a number of important properties (some new) which are not shared by various truncations of the Bethe-Salpeter equation. First of all, if there is no p dependence in $\mathcal{V}$, $\mathcal{A}$, or S, then this equation has a momentum structure as simple as that appearing in the nonrelativistic Schrödinger equation. (In fact, this equation reduces to the nonrelativistic Schrödinger equation with a potential energy of $S+\mathcal{V}+\mathcal{A}$.) This simplifies calculations considerably compared to the sum of square-root forms $[(\mathbf{p}_1{}^2+m_1{}^2)^{1/2}+(\mathbf{p}_2{}^2+m_2{}^2)^{1/2}]$ in the Salpeter approximation of the Bethe-Salpeter equation or semirelativistic expansions (including p^4 terms) in the Fermi-Breit approximation (the Darwin Hamiltonian) to the Bethe-Salpeter equation. Second, this equation reduces to the Klein-Gordon equation for a single particle in an external scalar and vector potential in the static limit. (The recoil Darwin terms at the end drop out since ∇G vanishes in this limit.) Another important feature with practical value arises from the two terms at the end of this equation. They account for recoil effects in a nonperturbative way and eliminate the need for singularity softening parameters in phenomenological applications.[4] This feature is not shared by various semirelativistic truncations of the Bethe-Salpeter equation or an *ad hoc* relativistic sum of square-root forms. (In our equation, the kinematics and dynamics are tied together; that is, the potential terms are not added on in an *ad hoc* fashion to the relativistic kinetic-energy forms.) Nevertheless, Eq. (50) does have a semirelativistic form that agrees with the Darwin-Hamiltonian form. However, there is no advantage in reducing it to this form since its simple momentum structure is lost if the covariance is unraveled. In later sections, when we include spin, we shall find that wave equations similar to this will appear whose spin-dependent terms are an elaboration of this underlying spinless structure. If one does not carefully build in the correct semirelativistic and heavy-particle limits in the spinless case, generalizations with spin will give misleading information about both heavy- and light-particle limits.

In an earlier paper[22] we applied a special version ($\mathcal{A}=0$, S confining, and $\mathcal{V}$ a QCD modified Coulomb potential) of Eq. (50) to quark model calculation of the mesons with spinless quarks. Lichtenberg, Namgung, and Wills performed a similar calculation[39] (using the same equations, but with a different choice for $\mathcal{V}$) and also applied them to the calculation of glueball masses. Since (50) is a fully covariant equation it has no problem handling zero-mass constituents, such as occurs in semirelativistic approaches. Unfortunately, Lichtenberg and Wills found no positive-energy ground state and attributed this to the fact that the $\mathcal{V}^2$ term is too attractive [it behaves near the origin like

$$(\mathbf{F}_1\cdot\mathbf{F}_2)^2(6\pi/27)^2/r^2\ln^2\lambda r ,$$

where for gluonium, $\mathbf{F}_1\cdot\mathbf{F}_2$, the color-SU(3) operator has the eigenvalue -3 (as opposed to the value $-\frac{4}{3}$ for quarkonium)]. We have found that using Richardson's potential instead of that of Lichtenberg and Wills does not change this result. However, if we assume instead that $\mathcal{V}=0$ and $\mathcal{A}$ is the QCD modified Coulomb potential (since it is the $\mathcal{A}$ potential that is capable of describing a gauge or electromagneticlike structure), then the extra terms in (50) (the two repulsive Darwin recoil terms at the end) give a positive-energy ground state. However, it is much too small (about 300 MeV) for this model to be seriously considered for the glueball candidate around 1440 MeV. We note that the absence of a ground state in the work of Lichtenberg and Wills is of a different nature from that of the Klein paradox that occurs when the charge becomes too large for the point Coulomb potential. In that case, the ground-state energy becomes complex and the wave function becomes rapidly oscillating. In this case, the leading behavior of the wave function near the origin is $\exp(\alpha_0{}^2/\ln\lambda r)$. It is not oscillating and gives a well-defined probability near the origin.[40]

Other possible applications of the spinless equation in two-body QCD phenomenology would be to compute diquark-antidiquark bound states corresponding to exotic mesons and to spinless quark-antiquark bound states that would result from the existence of supersymmetric partners to the quarks.

III. RELATIVISTIC CONSTRAINT DYNAMICS FOR A SINGLE SPIN-$\frac{1}{2}$ PARTICLE

In order to introduce spinning particles into relativistic constraint dynamics, we use a generalized correspondence limit of the Dirac equation in which Dirac matrices "correspond to" elements of a Grassmann algebra. In the resulting "pseudoclassical" mechanics[41,42] the Grassmann variables provide a semiclassical representation of spin. Canonical quantization of relativistic constraints that govern "pseudoclassical" spinning particles then leads to systems of compatible relativistic wave equations suitable for two interacting particles, one or both of which have spin. The details of this procedure were presented in Ref. 2 for scalar interactions. Here we present a brief review of some of the more important results of Ref. 2 and then extend these results to timelike and electromagneticlike vector interactions. To simplify our treatment, we first deal with the structures that appear when only one particle in an interacting pair possess spin, then in the next section take up the more complex case in which both particles have spin.

The Dirac equation for a single free particle is

$$(p_\mu\gamma^\mu+m)\psi=0 ,$$

which can be written in the form

$$\mathscr{S}\psi \equiv (p_\mu \theta^\mu + m\theta_5)\psi = 0 , \quad (51)$$

where

$$\theta^\mu = i \left[\frac{\hbar}{2} \right]^{1/2} \gamma_5 \gamma^\mu, \quad \mu = 0,1,2,3 ,$$

$$\theta_5 = i \left[\frac{\hbar}{2} \right]^{1/2} \gamma_5$$

satisfy

$$[\theta^\mu, \theta^\nu]_+ = -\hbar g^{\mu\nu} , \quad (52a)$$

$$[\theta_5, \theta^\mu]_+ = 0 , \quad (52b)$$

$$[\theta_5, \theta_5]_+ = -\hbar . \quad (52c)$$

These anticommutators, together with

$$[x^\mu, p^\nu]_- = i\hbar g^{\mu\nu} \quad (53)$$

and

$$[x^\mu, \theta^\alpha]_- = 0 = [p^\mu, \theta^\alpha]_-, \quad \alpha = 0,1,2,3,5 , \quad (54)$$

define the algebraic properties of our dynamical variables. Equations (52)–(54) divide the basic variables into two distinct classes: (a) those whose defining quantum brackets are exclusively commutators (called even), and (b) those that participate in fundamental anticommutators (called odd). Clearly, the (bosonic) x and p variables are even, while the (fermionic) θ variables are odd.

For dynamical variables A_α and A_β that have well-defined character (odd or even) we can write the generalized quantum brackets

$$[A_\alpha, A_\beta]_{-\eta_{\alpha\beta}} = A_\alpha A_\beta - \eta_{\alpha\beta} A_\beta A_\alpha , \quad (55)$$

where $\eta_{\alpha\beta} = (-)^{\epsilon_\alpha \epsilon_\beta}$. The variable ϵ_α is 0 if A_α is even and 1 if A_α is odd. Thus, for two even variables, or one odd and one even, $-\eta_{\alpha\beta} = -$ and the brackets represent a commutator. For two odd variables, $-\eta_{\alpha\beta} = +$ and the brackets represent an anticommutator. We define the product quantum brackets such that the bracket of $A_\alpha A_\beta$ with A_γ is

$$[A_\alpha A_\beta, A_\gamma]_{-\eta_{\alpha\gamma}\eta_{\beta\gamma}} .$$

This implies that the product of an odd with an odd is an even, the product of an even with an odd is an odd, and that the product of an even with an even is an even. Using the definition in (55), one finds that

$$[A_\alpha, A_\beta, A_\gamma]_{-\eta_{\alpha\gamma}\eta_{\beta\gamma}} = A_\alpha [A_\beta, A_\gamma]_{-\eta_{\beta\gamma}} + \eta_{\beta\gamma} [A_\alpha, A_\gamma]_{-\eta_{\alpha\gamma}} A_\beta , \quad (56)$$

together with an appropriate Jacobi condition.

Application of the correspondence principle to this generalized bracket leads to the (pseudo)classical bracket

$$\frac{1}{i\hbar} [A_\alpha, A_\beta]_{-\eta_{\alpha\beta}} \rightarrow \{A_\alpha, A_\beta\} . \quad (57)$$

Thus, the nonvanishing brackets among the fundamental dynamical variables become

$$\{\theta^\mu, \theta^\nu\} = i g^{\mu\nu} , \quad (58a)$$

$$\{\theta_5, \theta_5\} = i , \quad (58b)$$

$$\{x^\mu, p^\nu\} = g^{\mu\nu} . \quad (58c)$$

We assume that our classical θ's are real (as are x and p).

In ordinary quantum mechanics the correspondence limit leads not only to the Poisson-bracket algebra, but also to the c-number commutativity of x and p. In a similar way, one finds (see, e.g., Refs. 2, 41, and 42)

$$\theta^\mu \theta^\nu + \theta^\nu \theta^\mu = 0 , \quad (59a)$$

$$\theta_5 \theta^\mu + \theta^\mu \theta_5 = 0 , \quad (59b)$$

$$\theta_5{}^2 = 0 ; \quad (59c)$$

that is, the "classical" θ's are Grassmann variables. In this correspondence limit, the quantum Jacobi condition becomes

$$\sum \eta_{\alpha\gamma} \{A_\alpha, \{A_\beta, A_\gamma\}\} = 0 . \quad (60)$$

A differential realization of (58) in terms of x's, p's, and θ's is provided by[42]

$$\{\ ,\ \} = \frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu} \frac{\overrightarrow{\partial}}{\partial x_\mu} + i \frac{\overleftarrow{\partial}}{\partial \theta^\mu} \frac{\overrightarrow{\partial}}{\partial \theta_\mu} + i \frac{\overleftarrow{\partial}}{\partial \theta_5} \frac{\overrightarrow{\partial}}{\partial \theta_5} .$$

In terms of these pseudoclassical variables,[41] the Dirac equation (51) "corresponds to" a constraint on the dynamical variables:

$$\mathscr{S} = p \cdot \theta + m\theta_5 \approx 0 . \quad (61)$$

This is not the only constraint. Since $\mathscr{S}$ is odd, use of (58) allows us to find another constraint: the mass-shell condition

$$\frac{1}{i} \{\mathscr{S}, \mathscr{S}\} = \mathscr{H} = p^2 + m^2 \approx 0 , \quad (62)$$

which must be imposed on the pseudoclassical dynamical scheme if it is to define a compatible system. The Jacobi identity (60) gives us a very simple way to see that these two constraints are themselves compatible in the sense that[43]

$$\{\mathscr{S}, \mathscr{H}\} = -i \{\mathscr{S}, \{\mathscr{S}, \mathscr{S}\}\} = 0 .$$

Thus, we obtain a closed canonical algebra of only two constraints.

For a single spinning particle one would introduce interactions with external vector and scalar potentials by replacing (61) by

$$\mathscr{S} = \pi \cdot \theta + M\theta_5 \quad (63)$$

with $\pi^\mu = p^\mu - A^\mu$ and $M = m + S$. The $\mathscr{H}$ constraint then becomes

$$\mathscr{H} = \pi^2 - i\theta^\mu \theta^\nu F_{\mu\nu} + M^2 + 2i\partial M \cdot \theta\theta_5 , \quad (64)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

This capsule summary would complete our treatment of the single spinning particle were it not for the fact that important structures hidden in (63) and (64) play a crucial role in the construction of compatible constraints for the analogous two-body system. We fail to find compatible constraints for spinning particles if we naively parallel the procedure used in the spinless two-body problem. That is, if we simply assume that π_i and M_i are the same functions of $x_\perp$ as appears in the spinless case, and that these interaction functions appear in the constraints $\mathcal{S}_1=\pi_1\cdot\theta_1+M_1\theta_5$ and $\mathcal{H}_2=\pi_2{}^2+M_2{}^2$ (for a spin-$\frac{1}{2}$ particle interacting with a spin-zero particle), we fail to obtain a system of compatible constraints. We must resort to a different procedure. We found a way out of this difficulty in our earlier work on scalar interactions through a study of the way in which a supersymmetry of the free-particle dynamics generated by (61) and (62) survives intact in the presence of external interaction in the dynamical system governed by (63) and (64). We found that the naive extension of spin-independent interactions into spin-dependent free-particle constraints destroyed this supersymmetry and, with it, compatibility of constraints. If one correctly built its interaction-dependent version into the dynamics, however, the constraints $\mathcal{S}_1$ and $\mathcal{H}_2$ became compatible. As we shall see, the extra terms induced by this procedure are physically important recoil terms. The structures in which they participated became even more elaborate when we took up the interacting system of two spin-$\frac{1}{2}$ particles. In that case, careful maintenance of supersymmetries for each spinning particle led to compatible classical constraints $\mathcal{S}_1$ and $\mathcal{S}_2$. We review here some of the highlights of this approach to solving the compatibility problem. We then show that we can easily extend our treatment of the scalar interaction to systems governed by timelike vector interactions. We combine these interactions and introduce electromagneticlike interactions as well, eventually finding compatible constraints that are the pseudoclassical extensions of the constraints $\mathcal{H}_1=\pi_1{}^2+M_1{}^2$, $\mathcal{H}_2=\pi_2{}^2+M_2{}^2$ appearing in the spinless case. Canonical quantization leads ultimately to quantum wave equations that are the spin-dependent counterparts of (47) and (50). The source of the extra structure in these equations lies in the supersymmetries of a free spinning particle.

The pseudoclassical and quantum descriptions of the free spin-$\frac{1}{2}$ particle provided by the constraint (61) and the Dirac equation (51) are invariant on the solution surfaces generated by (62) and the Klein-Gordon equation, respectively, under the transformation

$$\begin{aligned}\delta\theta^\mu&=-i\varepsilon p^\mu ,\\ \delta\theta_5&=-i\varepsilon\sqrt{-p^2}\approx -i\varepsilon m ,\\ \delta p^\mu&=0 ,\end{aligned} \tag{65}$$

where ε, like θ^α, is an odd variable. Under this supersymmetry transformation[43] the Dirac constraint (61) is transformed into ε times the Klein-Gordon constraint (62) which already vanishes on the surface of solution. The Klein-Gordon constraint itself is trivially supersymmetric. The generator of the transformation (65) is

$$\mathcal{G}=p\cdot\theta+\sqrt{-p^2}\,\theta_5 \tag{66}$$

and is self-Abelian ($[\mathcal{G},\mathcal{G}]=0$). Invariance of the dynamics under (65) then becomes the statement that

$$\{\varepsilon\mathcal{G},\mathcal{S}\}\approx i\varepsilon\mathcal{H}\approx 0 . \tag{67}$$

Since we have determined the generator of (65), we can complete our set of supersymmetry transformations by computing the action of $\mathcal{G}$ on the remaining canonical variable x^μ:

$$\delta x^\mu=-\{\varepsilon\mathcal{G},x^\mu\}=\varepsilon(\theta^\mu-\hat{p}^\mu\theta_5) , \tag{68}$$

where $\hat{p}^\mu=p^\mu/\sqrt{-p^2}$.

In Ref. 2 we determined the spin dependence for scalar interactions by requiring that our interaction maintain the invariance of $\mathcal{S}$ under the supersymmetry transformation generated by $\mathcal{G}$. This requirement was satisfied through the use of a special supersymmetric-invariant position variable. This variable for a free particle,[44]

$$\tilde{x}^\mu=x^\mu+\frac{i\theta^\mu\theta_5}{m} , \tag{69}$$

satisfies

$$\{\mathcal{G},\tilde{x}^\mu\}\approx 0 ,$$

with the equality a weak one because the result vanishes with $\mathcal{H}$. This suggests that supersymmetry may be maintained in the presence of an external interaction if x dependence of the interaction appears only through $\tilde{x}$. We introduce scalar interactions through

$$m\to\tilde{M}\equiv m+S(\tilde{x}) , \tag{70}$$

where

$$\tilde{x}^\mu=x^\mu+\frac{i\theta^\mu\theta_5}{M(\tilde{x})} . \tag{71}$$

Then using the Grassmann Taylor expansion (and the fact that $\theta_5{}^2=0$), we find that

$$\mathcal{S}=p\cdot\theta+\tilde{M}\theta_5=p\cdot\theta+M\theta_5\approx 0 \tag{72}$$

and

$$\mathcal{H}\equiv\frac{1}{i}\{\mathcal{S},\mathcal{S}\}=p^2+M^2+2i\partial M\cdot\theta\theta_5=p^2+\tilde{M}^2\approx 0 . \tag{73}$$

Then using this constraint, we find that

$$\{\mathcal{G},\tilde{x}^\mu\}=\frac{\theta^\mu}{M}\left[\sqrt{-p^2}-M-i\frac{\partial M\cdot\theta}{M}\theta_5\right]+p^\mu\theta_5\left[\frac{1}{\sqrt{-p^2}}-\frac{1}{M}\right]$$

vanishes weakly. Thus, we have constructed in (71) an interaction-dependent version of the free-particle variable (69) which is invariant under the transformation (65), but on the interaction-dependent solution surface defined by the constraint (73). As an immediate consequence,

$$\{\mathcal{G},\mathcal{S}\}\approx 0 \tag{74a}$$

and

$$\{\mathcal{G},\mathcal{H}\}\approx 0 , \tag{74b}$$

so that our constraint system is completely supersymmetric. (That these constraints are compatible follows directly from the Jacobi identity.) Note that, in the case of a single spinning particle in an external scalar potential, although we have introduced interaction through the use of $\tilde{x}$ in order to preserve a supersymmetry, the resulting spin-dependent interaction in (73) is nothing but the standard one that we could have produced directly by taking the correspondence limit of the square of the ordinary Dirac equation with external scalar potential. One of the benefits of our procedure is that its new consequences appear only when we apply it to a multiparticle system. It generates new two-body equations that really are consistent two-body extensions of the Dirac equation.

IV. RELATIVISTIC CONSTRAINT DYNAMICS FOR A SPIN-$\frac{1}{2}$, SPIN-ZERO, TWO-PARTICLE SYSTEM UNDER MUTUAL SCALAR AND VECTOR INTERACTION

In order to extend our method to two-body pseudoclassical systems that include one spinning particle, we must simultaneously preserve the Dirac spin structure connected with supersymmetries and satisfy the requirements of "third-law" restrictions that we encountered in the case of two interacting spinless particles. Thus, we expect that the conditions for compatibility of the resulting constraints will include those already encountered in the bosonic system. (At the very least we must recover the bosonic conditions in the "zero-spin" limit in which the Grassmann variables vanish.) We anticipate that the potentials will be forced to satisfy

$$(M_1^{\,2}-M_2^{\,2})=m_1^{\,2}-m_2^{\,2} \tag{75}$$

and

$$M_i^{\,2}=M_i^{\,2}(x_\perp,p_1,p_2)=m_i^{\,2}+2m_w S+S^2 , \tag{76}$$

where

$$x_\perp^\mu=(g^{\mu\nu}+\hat{P}^{\mu}\hat{P}^{\nu})(x_1-x_2)_\nu . \tag{77}$$

Equation (75) is the third-law condition that together with the first equality in (76) ensured compatibility of the spinless $\mathcal{H}_i$ constraints. The second equality in (76) is a parametrization that not only obeys the third-law condition (75), but also leads to the correct nonrelativistic and heavy-particle limits. In Ref. 2 we showed how for the spin-$\frac{1}{2}$ spin-zero system, once supersymmetry is satisfied, (75) and (76) are forced on us by the requirement of compatibility.

We introduce pseudoclassical spin in a way that preserves supersymmetry for the spinning particle in the interacting two-particle system. Thus, following the pattern set by the ordinary one-particle Dirac equation, we replace (77) by the supersymmetric variable

$$\tilde{x}_\perp^\mu=(g^{\mu\nu}+\hat{P}^{\mu}\hat{P}^{\nu})(\tilde{x}_1-x_2)_\nu , \tag{78}$$

where, with $\tilde{M}_1\equiv M_1(\tilde{x}_\perp)$,

$$\tilde{x}_1^\mu=x_1^\mu+\frac{i\theta_1^\mu\theta_{51}}{\tilde{M}_1}=x_1^\mu+\frac{i\theta_1^\mu\theta_{51}}{M_1} . \tag{79}$$

For the fermionic particle, the odd constraint becomes

$$\mathcal{S}_1=p_1\cdot\theta_1+\tilde{M}_1\theta_{51}=p_1\cdot\theta_1+M_1\theta_{51}\approx 0 . \tag{80}$$

The "squared" version of this constraint is just

$$\mathcal{H}_1=\frac{1}{i}\{\mathcal{S}_1,\mathcal{S}_1\}=p_1^{\,2}+M_1^{\,2}+2i\partial M_1\cdot\theta_1\theta_{51}$$
$$=p_1^{\,2}+\tilde{M}_1^{\,2}\approx 0 , \tag{81}$$

where the last form follows from the Grassmann Taylor expansion [with use of (59)]

$$\tilde{M}_1=M_1+i\frac{\partial M_1}{M_1}\cdot\theta_1\theta_{51} .$$

Since particle two is spinless, it has no Dirac-type $\mathcal{S}$ constraint, but obeys only the supersymmetric Klein-Gordon-type constraint

$$\mathcal{H}_2=p_2^{\,2}+\tilde{M}_2^{\,2}=p_2^{\,2}+M_2^{\,2}+2iM_2\frac{\partial M_2}{M_1}\cdot\theta_1\theta_{51} , \tag{82}$$

where

$$\tilde{M}_2=M_2+i\frac{\partial M_2}{M_1}\cdot\theta_1\theta_{51} .$$

(Note that both $\tilde{M}_1$ and $\tilde{M}_2$ are supersymmetric.) Both (81) and (82) can be written in the form $\mathcal{H}_i=p_i^{\,2}+m_i^{\,2}+\Phi_i$, where $\Phi_i=\tilde{M}_i^{\,2}-m_i^{\,2}$. The third-law condition, $\Phi_1=\Phi_2$ in the spinless case led to (75). In Ref. 2, we showed how the more complicated spin-dependent compatibility calculation also leads to (75). What is remarkable about the supersymmetry requirement is that the spin-dependent interaction that it generates also makes (75) equivalent to the statement that $\Phi_1=\Phi_2$ in the spinning case. Thus, the supersymmetric structure introduced by $\tilde{x}$ has the virtue that it not only requires the same third law that appeared in the spinless case, but also reproduces a single effective relativistic interaction (Φ) that appears in each mass-shell constraint just as in the spinless case. Thus, (75) simplifies the spin-dependent terms so that $\tilde{M}_1^{\,2}-\tilde{M}_2^{\,2}=m_1^{\,2}-m_2^{\,2}$. (75) also simplifies the form of $\mathcal{H}_2$ to

$$\mathcal{H}_2=p_2^{\,2}+\tilde{M}_2^{\,2}=p_2^{\,2}+M_2^{\,2}+2i\partial M_1\cdot\theta_1\theta_{51} . \tag{83}$$

The dynamical system defined by (80)–(83), though an interacting one, retains the supersymmetry of the free case. Since only one of the particles has spin, we have just a single supersymmetry generator of the type we found for the single particle in (66):

$$\mathcal{G}_1=p_1\cdot\theta_1+\sqrt{-p_1^{\,2}}\,\theta_{51} . \tag{84}$$

Its brackets with $\tilde{x}_1$, $\tilde{x}_\perp$, $\mathcal{S}_1$, $\mathcal{H}_1$, and $\mathcal{H}_2$ all vanish (weakly). This makes it plausible that the three constraints $\mathcal{S}_1$, $\mathcal{H}_1$, and $\mathcal{H}_2$ will turn out to be compatible. We already know $\{\mathcal{S}_1,\mathcal{H}_1\}=0$ (as a consequence of the pseudoclassical bracket Jacobi condition just as in the single-particle case). Next, we find by direct calculation [with repeated use of the pseudoclassical analog of the quantum product rule (56)] that

$$\{\mathcal{S}_1,\mathcal{H}_2\}=\partial(M_1{}^2-M_2{}^2)\cdot\theta_1-2\partial M_1\cdot P\theta_{51}=0 \ . \quad (85)$$

The first term vanishes if we use (75) as does the second term, since M_1 depends on x only through $x_\perp$ (this includes dependence on $x_\perp{}^2$ and $x_\perp\cdot p=x_\perp\cdot p_\perp$).

Thus, once we build in supersymmetry, the compatibility restrictions drive us not only to the third law but also to the $x_\perp$ dependence given in (76) and (77). If we use the same parametrization of $M_i(S)$ as we did in the spinless case, we will ensure that our quantum wave equations (arising from the squared constraints) will have the same spin-independent terms as already appeared in the spinless case.[45] The one remaining compatibility condition $\{\mathcal{H}_1,\mathcal{H}_2\}=0$ can be evaluated using the Jacobi condition. We find that

$$\begin{aligned}\{\mathcal{H}_2,\mathcal{H}_1\}&=-i\{\mathcal{H}_2,\{\mathcal{S}_1,\mathcal{S}_1\}\}\\&=2i\{\mathcal{S}_1,\{\mathcal{S}_1,\mathcal{H}_2\}\}=0 \ . \end{aligned} \quad (86)$$

This vanishes strongly since $\{\mathcal{S}_1,\mathcal{H}_2\}$ does. Moreover, if we observe that (just as in the spinless case) the difference of $\mathcal{H}_1$ and $\mathcal{H}_2$ is independent of interaction (and spin),

$$\mathcal{H}_1-\mathcal{H}_2=p_1{}^2-p_2{}^2+m_1{}^2-m_2{}^2=2P\cdot p\approx 0 \ , \quad (87)$$

we can easily verify compatibility by a route other than the direct calculation of (85):

$$\begin{aligned}\{\mathcal{S}_1,\mathcal{H}_2\}&=-\{\mathcal{S}_1,\mathcal{H}_1-\mathcal{H}_2\}\\&=-2\{M_1\theta_{51},P\cdot p\}\\&=-2\partial M_1\cdot P\theta_{51}=0 \ . \end{aligned} \quad (88)$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ are weakly equivalent to each other and to

$$\mathcal{H}=p^2-b^2+\Phi=\frac{\epsilon_2}{w}\mathcal{H}_1+\frac{\epsilon_1}{w}\mathcal{H}_2\approx 0 \ , \quad (89)$$

where (with $M_1=m_1+S_1$)

$$\begin{aligned}\Phi&=2m_1S_1+S_1{}^2+2i\partial S_1\cdot\theta_1\theta_{51}\\&=2m_2S_2+S_2{}^2+2i\partial S_1\cdot\theta_1\theta_{51} \ . \end{aligned} \quad (90)$$

Use of the effective particle variables m_w, ϵ_w, and $\mathcal{P}$ defined in (21) and (23) turns (89) into

$$\mathcal{P}^2+(m_w+S)^2+2i\partial M_1\cdot\theta_1\theta_{51}\approx 0 \ . \quad (91)$$

We now wish to extend our results for the scalar interaction to a system of a spinless and a spin-$\frac{1}{2}$ particle interacting through timelike and electromagneticlike vector interactions in addition to scalar interactions. To see the role played by supersymmetry in the presence of these additional interactions we first consider the timelike vector interaction by itself. Our procedure will imitate the steps from (75)–(91) in the case of scalar interactions. From our work in the spinless case, we expect that the potential will be forced to satisfy [see (32) with $G=1$]

$$(E_1{}^2-E_2{}^2)=\epsilon_1{}^2-\epsilon_2{}^2+2(E_1+E_2-w)P\cdot p \ , \quad (92)$$

where

$$E_i{}^2=E_i{}^2(x_\perp,p_1,p_2)\approx\epsilon_i{}^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2 \ . \quad (93)$$

We choose the timelike vector interaction to point in the direction of $\hat{P}^\mu$, the only constant timelike vector intrinsic to the two-body system. In terms of it, we rewrite the free Dirac constraint as

$$p_1\cdot\theta_1+m_1\theta_{51}=\epsilon_1\hat{P}\cdot\theta_1+p\cdot\theta_1+m_1\theta_{51}\approx 0 \quad (94)$$

with p the relative momentum variable (7). Hence, in analogy to scalar interactions where this free Dirac constraint is replaced by the interacting form

$$\mathcal{S}_1=\epsilon_1\hat{P}\cdot\theta_1+p\cdot\theta_1+M_1\theta_{51}\approx 0 \ , \quad (80')$$

we write the Dirac constraint with timelike four-vector interaction as

$$\mathcal{S}_1=E_1\hat{P}\cdot\theta_1+p\cdot\theta_1+m_1\theta_{51}\approx 0 \ . \quad (95)$$

Whereas the scalar supersymmetry generator left invariant the constituent $\tilde{x}_i^\mu$ as well as the collective variable $\tilde{x}_\perp^\mu$, for the case of timelike vector interactions our supersymmetry generator $\mathcal{G}_E$ will be one that generates a supersymmetry transformation that only leaves invariant a collective-coordinate variable. For our generator we choose the self-Abelian form

$$\mathcal{G}_{1E}=(p_\perp{}^2+m_1{}^2)^{1/2}\hat{P}\cdot\theta_1+p_\perp\cdot\theta_1+m_1\theta_{51} \ . \quad (96)$$

The transformation it generates when the interaction is turned off is

$$\begin{aligned}\delta x^\mu&=\varepsilon\left[\theta_1^\mu-\frac{p^\mu}{(p_\perp{}^2+m_1{}^2)^{1/2}}\hat{P}\cdot\theta_1\right]_\perp ,\\ \delta\theta_1^\mu&=-i\varepsilon[p_\perp^\mu+(p^2+m^2)^{1/2}\hat{P}^\mu] \ ,\\ \delta\theta_{51}&=-i\varepsilon m_1, \quad \delta p^\mu=0 \ .\end{aligned}$$

Under this set of transformations, the variable

$$(\tilde{x}_\perp)_E=[(\tilde{x}_1)_E-x_2]_\nu(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu) \quad (97)$$

with

$$(\tilde{x}_1)_E^\mu=x_1^\mu-\frac{i\theta_1^\mu\hat{P}\cdot\theta_1}{\epsilon_1} \quad (98)$$

is a supersymmetric invariant. When the interaction is turned on, the corresponding supersymmetric position variable is still as it appears in (97), but with [analogous to (79)]

$$(\tilde{x}_1)_E^\mu=x_1^\mu-\frac{i\theta_1^\mu\theta_1\cdot\hat{P}}{\tilde{E}_1}=x_1^\mu-i\frac{\theta_1^\mu\theta_1\cdot\hat{P}}{E_1} \ , \quad (99)$$

where

$$\tilde{E}_1=E_1((\tilde{x}_\perp)_E,p_1,p_2)$$

is a supersymmetric potential. Note that an alternative form of the scalar supersymmetry generator that is also self-Abelian is

$$\mathcal{G}_{1M}=\epsilon_1\hat{P}\cdot\theta_1+p_\perp\cdot\theta_1+(\epsilon_1{}^2-p_\perp{}^2)^{1/2}\theta_{51}\ . \tag{84'}$$

Unlike (84), this generator resembles that of (96) in that, although leaving invariant the supersymmetric collective variable $(\tilde{x}_\perp)$, it does not leave invariant the constituent variables $\tilde{x}_i^\mu$. The squared version of the Dirac constraint (95) is

$$\begin{aligned}\mathcal{H}_1&=\frac{1}{i}\{\mathcal{S}_1,\mathcal{S}_1\}\\&=p^2+m_1{}^2-E_1{}^2+2i\partial E_1\cdot\theta_1\theta_1\cdot\hat{P}+2E_1\hat{P}\cdot p\\&=p^2+m_1{}^2-\tilde{E}_1{}^2+2E_1\hat{P}\cdot p\ ,\end{aligned} \tag{100}$$

where the last form follows from the Grassmann Taylor expansion

$$\tilde{E}_1=E_1-i\frac{\partial E_1}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1\ .$$

If we use the definition

$$\pi_1^\mu=E_1\hat{P}^\mu+p^\mu=p_1^\mu-A_1^\mu\ , \tag{101}$$

we may rewrite $\mathcal{S}_1$ and $\mathcal{H}_1$ in (95) and (100) in the alternate forms

$$\mathcal{S}_1=\pi_1\cdot\theta_1+m_1\theta_{51}\ , \tag{102a}$$

$$\mathcal{H}_1=\pi_1{}^2-i\theta_1^\mu\theta_1^\nu F_{1\mu\nu}+m_1{}^2\ , \tag{102b}$$

where

$$F_{1\mu\nu}=\{\pi_{1\mu},\pi_{1\nu}\}=(\hat{P}_\mu\partial_\nu E_1-\hat{P}_\nu\partial_\mu E_1)\ . \tag{103}$$

The spinless counterpart to (100) is $\mathcal{H}_1=p^2+m_1{}^2-E_1{}^2+2E_1\hat{P}\cdot p$. One may obtain (100) directly from it (weakly) by replacing $E_1{}^2$ by the supersymmetric invariant $\tilde{E}_1{}^2$. The single constraint for the spinless particle (two) would take the form $\mathcal{H}_2=p^2+m_2{}^2-E_2{}^2-2E_2\hat{P}\cdot p$ if particle one were also spinless. To introduce interaction with a spinning particle, we replace $E_2{}^2$ by the supersymmetric invariant form $\tilde{E}_2{}^2$, where

$$\tilde{E}_2=E_2-i\frac{\partial E_2}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1\ .$$

Then $\mathcal{H}_2$ is given by

$$\begin{aligned}\mathcal{H}_2&=p^2+m_2{}^2-\tilde{E}_2{}^2-2E_2\hat{P}\cdot p\\&=p^2+m_2{}^2-E_2{}^2+2iE_2\frac{\partial E_2}{E_1}\cdot\theta_1\theta_1\cdot\hat{P}-2E_2\hat{P}\cdot p\ .\end{aligned} \tag{104}$$

If we define Φ_1 and Φ_2 by $\mathcal{H}_i=p_i{}^2+m_i{}^2+\Phi_i$, then

$$\Phi_1-\Phi_2=E_1{}^2-E_2{}^2-(\epsilon_1{}^2-\epsilon_2{}^2)-2(E_1+E_2-w)\hat{P}\cdot p+2i\left[\frac{E_2\partial E_2}{E_1}-\partial E_1\right]\cdot\theta_1\theta_1\cdot\hat{P}\ .$$

In the scalar case, the third-law condition $\Phi_1=\Phi_2$ for the spin-dependent $\mathcal{H}_1$ and $\mathcal{H}_2$ led to the same condition (75) as in the spinless case. However, for the timelike four-vector interaction, the spinless third-law condition (92) does not lead to $\Phi_1-\Phi_2=0$, but instead to

$$\Phi_1-\Phi_2=\frac{i}{E_1}\partial(E_1+E_2)\cdot\theta_1\theta_1\cdot\hat{P}\hat{P}\cdot p\approx 0\ .$$

The supersymmetric system, thus, requires the third law but in a weak form. As pointed out below (19), a weak form of the third law is sufficient (along with $x_\perp$ dependence of interaction) to ensure compatibility. Note that the third-law expression (92) leads to

$$\mathcal{H}_1-\mathcal{H}_2=\left[2w+2i\frac{\partial(E_1+E_2)}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1\right]\hat{P}\cdot p\approx 0\ , \tag{105}$$

which, as in the scalar and spinless cases, is proportional to the constraint $\hat{P}\cdot p\approx 0$. Using this constraint, $\mathcal{H}_1$ and $\mathcal{H}_2$ in (100) and (104) can be compactly written as[46]

$$\begin{aligned}\mathcal{H}_1&=\tilde{\pi}_1{}^2+2(E_1-\tilde{E}_1)\hat{P}\cdot p+m_1{}^2\\&\approx p^2+m_1{}^2-\tilde{E}_1{}^2\approx 0\ ,\\\mathcal{H}_2&=\tilde{\pi}_2{}^2-2(E_2-\tilde{E}_2)\hat{P}\cdot p+m_2{}^2\\&\approx p^2+m_2{}^2-\tilde{E}_2{}^2\approx 0\ ,\end{aligned} \tag{106}$$

where $\tilde{\pi}_1^\mu=\tilde{E}_1\hat{P}^\mu+p$ and $\tilde{\pi}_2=\tilde{E}_2\hat{P}^\mu-p$.

The construction of the $\mathcal{H}$'s from $\mathcal{S}$, as well as the verification of their invariance under the supersymmetry transformations generated by $\mathcal{G}_{1E}$ in (96), parallel their counterparts given in Eqs. (72)–(74) for a single particle in an external scalar potential and will not be repeated here. The check of compatibility is slightly more complicated than it was in the scalar case. Just as before, the Jacobi identity leads to $\{\mathcal{S}_1,\mathcal{H}_1\}=0$. Using this result along with the difference (105), we see that the compatibility condition $\{\mathcal{S}_1,\mathcal{H}_2\}\approx 0$ is also satisfied since

$$\{\mathcal{S}_1,\mathcal{H}_2\}=-2\left\{\mathcal{S}_1,\left[w+i\frac{\partial(E_1+E_2)}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1\right]\hat{P}\cdot p\right\}\ , \tag{107}$$

which vanishes (at least weakly) because of the $x_\perp$ dependence of $\mathcal{S}_1$. Likewise,

$$\{\mathcal{H}_1,\mathcal{H}_2\}=2\left\{\mathcal{H}_1,\left[w+i\frac{\partial(E_1+E_2)}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1\right]\hat{P}\cdot p\right\} \tag{108}$$

vanishes, since all the forms that appear in $\mathcal{H}_1$ (including $\partial E_1\cdot\theta_1\hat{P}\cdot\theta_1\equiv\partial E_1\cdot\theta_{1\perp}\hat{P}\cdot\theta_1$) have vanishing brackets with $\hat{P}\cdot p$. Again, as in the scalar case, the requirements of supersymmetry and compatibility drive us to the third law and $x_\perp$ dependence given in (92) and (93). The weak equality in (93) is consistent with (92) and also gives the correct nonrelativistic and heavy-particle limits. If we use the same parametrization that we did in the spinless case we will ensure that our quantum wave equation will have the same spin-independent terms that appear in the corresponding spinless case.[45]

As our next step we combine scalar and timelike vector interactions for the spin-zero spin-$\frac{1}{2}$ system. When both interactions are turned off, the two supersymmetry generators

$$\mathcal{G}_{1M}=\epsilon_1\hat{P}\cdot\theta_1+p_\perp\cdot\theta_1+(\epsilon_1{}^2-p_\perp{}^2)^{1/2}\theta_{51} \;, \tag{109a}$$

$$\mathcal{G}_{1E}=(p_\perp{}^2+m_1{}^2)^{1/2}\hat{P}\cdot\theta_1+p_\perp\cdot\theta_1+m_1\theta_{51} \tag{109b}$$

generate dynamical symmetries. In the presence of both scalar and timelike four-vector interactions, these become

$$\mathcal{G}_{1M}=E_1\hat{P}\cdot\theta_1+p_\perp\cdot\theta_1+(\tilde{E}_1{}^2-p_\perp{}^2)^{1/2}\theta_{51} \;, \tag{110a}$$

$$\mathcal{G}_{1E}=(p_\perp{}^2+\tilde{M}_1{}^2)^{1/2}\hat{P}\cdot\theta_1+p_\perp\cdot\theta_1+M_1\theta_{51} \;, \tag{110b}$$

where

$$\tilde{E}_1\equiv E_1((\tilde{x}_\perp)_E,p_1,p_2)$$

and

$$\tilde{M}_1\equiv M_1((\tilde{x}_\perp)_M,p_1,p_2)$$

and

$$(\tilde{x}_\perp)_E=[(\tilde{x}_1)_E-x_2]^\mu_\perp \;, \tag{111a}$$

$$(\tilde{x}_\perp)_M=[(\tilde{x}_1)_M-x_2]^\mu_\perp \;, \tag{111b}$$

with

$$(\tilde{x}_1)^\mu_E=x_1^\mu-i\frac{\theta_1^\mu\theta_1\cdot\hat{P}}{\tilde{E}_1}=x_1^\mu-i\frac{\theta_1^\mu\theta_1\cdot\hat{P}}{E_1} \;, \tag{112a}$$

$$(\tilde{x}_1)^\mu_M=x_1^\mu+\frac{i\theta_1^\mu\theta_{51}}{\tilde{M}_1}=x_1^\mu+i\frac{\theta_1^\mu\theta_{51}}{M_1} \;. \tag{112b}$$

The Dirac constraint and its square are

$$\begin{aligned}\mathcal{S}_1&=\tilde{E}_1\hat{P}\cdot\theta_1+p\cdot\theta_1+\tilde{M}_1\theta_{51}\\&=E_1\hat{P}\cdot\theta_1+p\cdot\theta_1+M_1\theta_{51} \;,\end{aligned} \tag{113a}$$

$$\begin{aligned}\mathcal{H}_1&=\frac{1}{i}\{\mathcal{S}_1,\mathcal{S}_1\}\\&=p^2+M_1{}^2-E_1{}^2+2E_1\hat{P}\cdot p+2i\partial M_1\cdot\theta_1\theta_{51}\\&\quad+2i\partial E_1\cdot\theta_1\theta_1\cdot\hat{P}-2i\{E_1,M_1\}\theta_1\cdot\hat{P}\theta_{51} \;.\end{aligned} \tag{113b}$$

The last term in (113b) would be absent if the scalar and vector interactions were both independent of the relative momentum or dependent on it only through the angular momentum l^2. The two preceding terms can be viewed, just as when only one interaction is turned on, as coming from the replacements of $x_\perp$ in E_1 and M_1 by the $\tilde{x}_\perp$ given in (111) and (112). In fact, the last brackets in (113b) can be absorbed into a new $\tilde{x}$ definition. That is, we can write

$$\mathcal{H}_1=p^2-\tilde{E}_1{}^2+\tilde{M}_1{}^2+2E_1\hat{P}\cdot p \;, \tag{114}$$

where $\tilde{E}_1$ and $\tilde{M}_1$ are defined below (110), and in (111) if the expressions in (112) are replaced by

$$(\tilde{x}_1^\mu)_E=x_1^\mu-\frac{i\theta_1^\mu\theta_1\cdot\hat{P}}{E_1}+\frac{i\partial_p^\mu M_1\hat{P}\cdot\theta_1\theta_{51}}{E_1} \;, \tag{115a}$$

$$(\tilde{x}_1^\mu)_M=x_1^\mu+\frac{i\theta_1^\mu\theta_{51}}{M_1}+\frac{i\partial_p^\mu E_1\hat{P}\cdot\theta_1\theta_{51}}{M_1} \;. \tag{115b}$$

This has no effect on the definitions of the generators given in (110).

$\mathcal{G}_{1M}$ and $\mathcal{G}_{1E}$ no longer generate supersymmetries of the dynamical system when both interactions are present. One can in fact show that $\mathcal{S}_1$ is supersymmetric, that is,

$$\{\mathcal{G}_{1M},\mathcal{S}_1\}\approx i\mathcal{H}_1\approx 0 \tag{116}$$

and

$$\{\mathcal{G}_{1E},\mathcal{S}_1\}\approx i\mathcal{H}_1\approx 0 \;. \tag{117}$$

We will omit the details of the proof. The interested reader may need the Grassmann Taylor expansions given below in order to expand $\mathcal{G}_{1M}$ and $\mathcal{G}_{1E}$:

$$\tilde{M}_1=M_1+\frac{i\partial M_1\cdot\theta_1}{M_1}\theta_{51}+\frac{i\partial M_1\cdot\partial_p E_1}{M_1}\theta_1\cdot\hat{P}\theta_{51} \tag{118a}$$

and

$$\tilde{E}_1=E_1-\frac{i\partial E_1\cdot\theta_1}{E_1}\hat{P}\cdot\theta_1+\frac{i\partial E_1\cdot\partial_p M_1}{E_1}\theta_1\cdot\hat{P}\theta_{51} \;. \tag{118b}$$

One can show that each $\tilde{x}$ variable is a supersymmetric invariant under the transformations generated by the corresponding $\mathcal{G}$: $\{\mathcal{G}_{1M},(\tilde{x}_\perp)_M\}\approx 0$ and $\{\mathcal{G}_{1E},(\tilde{x}_\perp)_E\}\approx 0$. However, the facts that $\{\mathcal{G}_{1M},(\tilde{x}_\perp)_E\}\not\approx 0$ and $\{\mathcal{G}_{1E},(\tilde{x}_\perp)_M\}\not\approx 0$ prevent the system from being completely supersymmetric. Thus, unlike $\mathcal{S}_1$, $\mathcal{H}_1$ is not supersymmetric invariant under the transformations generated by the $\mathcal{G}$'s (Ref. 47). (One can also show[48] that $\{\mathcal{G}_{1M},\mathcal{G}_{1E}\}\not\approx 0$.) Hence, when both interactions are present, there is no overall supersymmetry (of the $\mathcal{G}$ type) in the sense that there was for scalar or timelike vector interactions alone.[49] Nonetheless, the spin-dependent structures introduced by the (formerly) supersymmetric position variables $(\tilde{x}_\perp)_M$ and $(\tilde{x}_\perp)_E$ are still sufficient to lead to compatible constraints. At the least, they lead to a system of compatible constraints in either limit in which only one of the interactions is left on. Thus, when we combine the interactions, we obtain the spin structure that we must obtain in either limit. Beyond this, the $\tilde{x}$'s lead to compatible constraints even when both interactions are present because they are sufficient along with the usual third-law conditions to produce $\mathcal{H}_i$ constraints whose difference is proportional to $P\cdot p$.

With combined scalar and timelike vector interactions, we have shown (see Ref. 46).

$$\mathcal{S}_1=\tilde{\pi}_1\cdot\theta_1+\tilde{M}_1\theta_{51}=\pi_1\cdot\theta_1+\tilde{M}_1\theta_{51} , \quad (119)$$

$$\mathcal{H}_1=p^2+2E_1\hat{P}\cdot p-\tilde{E}_1^{\,2}+\tilde{M}_1^{\,2}\approx\tilde{\pi}_1^{\,2}+\tilde{M}_1^{\,2} , \quad (120)$$

where $\tilde{\pi}_1=\tilde{E}_1\hat{P}+p$ with $\tilde{M}_1$ and $\tilde{E}_1$ defined in (118a) and (118b). For $\mathcal{H}_2$ we take

$$\mathcal{H}_2=p^2-2E_2\hat{P}\cdot p-\tilde{E}_2^{\,2}+\tilde{M}_2^{\,2}\approx\tilde{\pi}_2^{\,2}+\tilde{M}_2^{\,2} \quad (121)$$

with $\tilde{\pi}_2=\tilde{E}_2\hat{P}-p$, where, in analogy to (83), (118a), and (118b),

$$\tilde{M}_2=M_2+\frac{i\partial M_2\cdot\theta_1}{M_1}\theta_{51}+\frac{i\partial M_2\cdot\partial_pE_1}{M_1}\theta_1\cdot\hat{P}\theta_{51} , \quad (122a)$$

$$\tilde{E}_2=E_2-\frac{i\partial E_2\cdot\theta_1}{E_1}\hat{P}\cdot\theta_1+\frac{i\partial E_2\cdot\partial_pM_1}{E_1}\theta_1\cdot\hat{P}\theta_{51} . \quad (122b)$$

Defining Φ_i as before, we find that

$$\Phi_1-\Phi_2=\tilde{M}_1^{\,2}-\tilde{M}_2^{\,2}-m_1^{\,2}+m_2^{\,2}$$
$$-\tilde{E}_1^{\,2}+\tilde{E}_2^{\,2}+\epsilon_1^{\,2}-\epsilon_2^{\,2}+2(E_1+E_2-w)\hat{P}\cdot p .$$

The weak third-law requirement, $\Phi_1\approx\Phi_2$ plus the use of the supersymmetric forms given in (118a) (118b), (122a), and (122b), again lead us to the same spinless potential conditions (75) and (93). The resulting form of $\mathcal{H}_2$ implies $\mathcal{H}_1-\mathcal{H}_2\approx\hat{P}\cdot p$, so that proof of compatibility, $\{\mathcal{S}_1,\mathcal{H}_1\}\approx 0$, $\{\mathcal{S}_1,\mathcal{H}_2\}\approx 0$, $\{\mathcal{H}_1,\mathcal{H}_2\}\approx 0$, is straightforward and will not be given here.

In order to complete our discussion of this section, we must include electromagneticlike interactions. From our discussions above, we do not expect there to be an overall supersymmetry generated by a $\mathcal{G}$-type Grassmann form. Rather our motivation arises from the tilde forms $\tilde{\pi}_i$ and $\tilde{M}_i$ that, combined with the third law, have consistently led to conditions on the potential forms M_i and E_i that are the same as those appearing in the spinless case. These tilde forms reflect the partial supersymmetries that are sufficient to lead to compatible constraints. In the case of scalar and timelike four-vector interactions, the parametrization we used for both parts of the potential were independent [$M_i=M_i(S)$, $E_i=E_i(\mathcal{V})$]. This allowed us to treat their supersymmetric extensions separately. When the electromagneticlike vector interactions are included, the potentials are correlated [$M_i=M_i(S,\mathcal{A})$, $E_i=E_i(\mathcal{V},\mathcal{A})$, $G=G(\mathcal{A})$]. (This correlation has nothing whatsoever to do with compatibility. It is a feature of our parametrization of M_i and E_i.) Because of this correlation, we do not expect separate supersymmetric extensions. Consequently, we will not be using supersymmetries (of the $\mathcal{G}$ type) to construct the spin-dependent extentions $\tilde{\pi}_i$ and $\mathcal{M}_i$ of the spinless potential forms. However, as we shall see below, the spin-dependent extensions given in (118a), (118b), (122a), and (122b) can be readily generalized, so that the resulting third-law requirement on the spin-dependent Φ_i still implies the third-law requirement found in the spinless case. These tilde forms are inferred directly from the constraints themselves. The $\mathcal{S}_1$ constraint for the spinning particle, when electromagneticlike interactions are present, is

$$\mathcal{S}_1=\tilde{\pi}_1\cdot\theta_1+\tilde{M}_1\theta_{51}=\pi_1\cdot\theta_1+M_1\theta_{51} , \quad (123)$$

where $\pi_1^\mu=E_1\hat{P}^\mu+Gp_\mu$ and $M_1=m_1+S_1$. We will restrict our attention to the case when the p dependence of the potentials is restricted to its appearance through l^2 with $\{x_\perp^{\,2},l^2\}=0$. Those potential forms E_1, G, and S_1 are the same as those defined in the spinless section. The tilde variable will be left undefined for now other than the required equivalence of the middle and right-hand sides of (123). The quadratic constraint is

$$\mathcal{H}_1=\frac{1}{i}\{\mathcal{S}_1,\mathcal{S}_1\}=\pi_1^{\,2}-i\theta_1^\mu\theta_1^\nu F_{1\mu\nu}$$
$$+M_1^{\,2}-2i\theta_1^\mu\theta_{51}\{\pi_{1\mu},M_1\} . \quad (124)$$

Note that the interaction term $\theta_1^\mu\theta_1^\nu F_{1\mu\nu}$ is the pseudoclassical analog of the $\sigma_1^{\mu\nu}F_{1\mu\nu}$ interaction that appears in the quantum case. Unlike the case of (103), there are **B**-like components of the field in the c.m. system arising from recoil effects (**p** and ∇G dependent terms). Explicitly,

$$F_{1\mu\nu}=\{\pi_{1\mu},\pi_{1\nu}\}$$
$$=G(\hat{P}_\mu\partial_\nu E_1-\hat{P}_\nu\partial_\mu E_1)+G(p_\mu\partial_\nu G-p_\nu\partial_\mu G) , \quad (125a)$$

and together with

$$\{\pi_{1\mu},M_1\}=-G\partial_\mu M_1 \quad (125b)$$

leads to

$$\mathcal{H}_1=G^2p^2-E_1^{\,2}+M_1^{\,2}+2iG\partial G\cdot\theta_1p\cdot\theta_1$$
$$+2iG\partial E_1\cdot\theta_1\hat{P}\cdot\theta_1+2iG\partial M_1\cdot\theta_1\theta_{51}+2GE_1\hat{P}\cdot p . \quad (126)$$

This constraint takes the by now familiar form[46]

$$\mathcal{H}_1\approx\tilde{\pi}_1^{\,2}+\tilde{M}_1^{\,2}\approx 0 ,$$

if we define

$$\tilde{\pi}_1^\mu=\tilde{E}_1\hat{P}^\mu+Gp^\mu+i\theta_1\cdot\partial G\theta_{1\perp}^\mu ,$$

$$\tilde{E}_1=E_1-iG\frac{\partial E_1}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1 , \quad (127a)$$

$$\tilde{M}_1=M_1+G\frac{i\partial M_1\cdot\theta_1}{M_1}\theta_{51} . \quad (127b)$$

This $\tilde{M}_1$ differs from that given by our earlier supersymmetric form by the factor of G (which becomes 1 when $\mathcal{A}=0$, i.e., no electromagneticlike interactions). Notice that if we define

$$(\tilde{x}_\perp)_M = x_\perp + iG\theta_1\theta_{51}/M_1$$

then

$$\tilde{M}_1 = M_1((\tilde{x}_\perp)_M, p_1, p_2) \ .$$

Likewise defining

$$(\tilde{x}_\perp)_E = x_\perp - iG\theta_1\hat{P}\cdot\theta_1/E_1 \ ,$$

implies that

$$\tilde{E}_1 = E_1((\tilde{x}_\perp)_E, p_1, p_2) \ .$$

These tilde variables, unlike earlier ones defined in the case $G=1$, are not supersymmetric. That is,

$$\{\mathcal{G}_{1M}, (\tilde{x}_\perp)_M\} \neq 0 \neq \{\mathcal{G}_{1E}, (\tilde{x}_\perp)_E\} \ .$$

However, we retain the tilde notation to remind ourselves of their supersymmetric forms as $G \to 1$. Finally, defining $\tilde{p} = p + i\theta_1\cdot\nabla G\theta_{1\perp}$ we have $\tilde{\pi}_1 = \tilde{E}_1\hat{P} + G\tilde{p}$. For the spinless particle there is no $\mathcal{S}_2\ \mathcal{S}_2$; for $\mathcal{H}_2$ we use the form[45]

$$\mathcal{H}_2 = G^2p^2 - E_2{}^2 + M_2{}^2 + 2iG\partial G\cdot\theta_1 p\cdot\theta_1 + 2iG\frac{\partial E_2}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1 + 2iG\partial M_1\cdot\theta_1\theta_{51} - 2GE_1\hat{P}\cdot p \approx \tilde{\pi}_2{}^2 + \tilde{M}_2{}^2 \ ,$$

where

$$\tilde{\pi}_2^\mu = \tilde{E}_2\hat{P} - G\tilde{p} = \left[E_2 - iG\frac{\partial E_2}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1\right]\hat{P} - Gp^\mu - i\theta_1\cdot\partial G^\mu_{1\perp} \ , \tag{128a}$$

$$\tilde{M}_2 = M_2 + \frac{iG\partial M_2\cdot\theta_1}{M_1}\theta_{51} \ . \tag{128b}$$

It is straightforward to show that this $\mathcal{H}_2$ is compatible with $\mathcal{H}_1$ and $\mathcal{S}_1$ and that $\mathcal{H}_1 - \mathcal{H}_2 \sim P\cdot p$. Even though there are no longer $\mathcal{G}$-like types of supersymmetry generators to go along with $\tilde{E}_1$ and $\tilde{M}_1$, these generalized tilde forms plus the spacelike spin-dependent modification of p are sufficient to give compatibility and covariant control of the relative energy. We can see the physical significance of the extra terms more easily by leaving out the timelike vector interaction. In that case, this form for $\tilde{\pi}_2$ simplifies to $[E_2 = G(\epsilon_2 - \mathcal{A})$ and $-G(\partial E_2/E_1) = \partial G]$

$$\tilde{\pi}_2 = E_2\hat{P} - Gp - i\theta_1\cdot\partial G\theta_1 = p_2 - \tilde{A}_2 = \epsilon_2\hat{P} - p - \tilde{A}_2 \ . \tag{129}$$

Hence, the vector potential has the Gordon decomposition of the electromagnetic current built in. In particular,

$$\tilde{A}_2^\mu = (\epsilon_2 - E_2)\hat{P} - (1-G)p + i\theta_1\cdot\partial G\theta_1 \ . \tag{130}$$

Our quantum equation for the spin-$\frac{1}{2}$, spin-zero system with simultaneous electromagneticlike, timelike, and scalar interactions are the following simultaneous Dirac and Klein-Gordon equations:

$$\mathcal{S}_1\psi = (\pi_1\cdot\theta_1 + M_1\theta_{51})\psi = 0 \ , \tag{131}$$

$$\mathcal{H}_2\psi = (\tilde{\pi}_2{}^2 + \tilde{M}_2{}^2)\psi = 0 \tag{132}$$

(with appropriately Hermitized π operators [see Eqs. (46a) and (46b)]). Problems with Hermitian orderings are greatly simplified if the relative momentum dependence in the interaction functions is restricted to l^2. These are compatible wave equations since $[\mathcal{S}_1, \mathcal{H}_2]_- = 0$. The proof can be made isomorphic to the proof of classical compatibility by using the product rule (56) when necessary in place of its pseudoclassical counterpart.[50] Explicitly, then

$$\mathcal{S}_1\psi = \left[E_1\hat{P}\cdot\theta_1 + Gp\cdot\theta_1 - i\frac{\partial G}{2}\cdot\theta_1 + M_1\theta_{51}\right]\psi = 0 \ . \tag{133}$$

(This equation reduces to the Dirac equation for a single spin-$\frac{1}{2}$ particle in an external scalar and vector potential in the limit that m_2 becomes very heavy.) Its squared form is

$$\mathcal{H}_1\psi = [\mathcal{S}_1, \mathcal{S}_1]_+\psi = G^2\left[p^2 - (E_1{}^2 - M_1{}^2)/G^2 + 2E_1P\cdot p/G - 2i\partial\ln G\cdot p - \tfrac{1}{2}\partial^2\ln G - \tfrac{3}{4}(\partial\ln G)^2 + 2i\partial\ln G\cdot\theta_1 p\cdot\theta_1 + 2i\frac{\partial E_1}{G}\cdot\theta_1\hat{P}\cdot\theta_1 + 2i\frac{\partial M_1}{G}\cdot\theta_1\theta_{51}\right]\psi = 0 \ . \tag{134}$$

(This equation reduces to the squared Dirac equation for a single spin-$\frac{1}{2}$ particle in an external scalar and vector potential in the limit that m_2 becomes very heavy.) The quantum form of the spinless particle's constraint is

$$\mathcal{H}_2\psi = G^2\left[p^2 - (E_2{}^2 - M_2{}^2)/G^2 - 2E_2P\cdot p/G - 2i\partial\ln G\cdot p - \tfrac{1}{2}\partial^2\ln G - \tfrac{3}{4}(\partial\ln G)^2 + 2i\partial\ln G\cdot\theta_1 p\cdot\theta_1 + 2i\frac{E_2\partial E_2}{E_1G}\cdot\theta_1\hat{P}\cdot\theta_1 + 2i\frac{M_2\partial M_2}{M_1G}\cdot\theta_1\theta_{51}\right]\psi = 0 \ . \tag{135}$$

(This equation reduces to the Klein-Gordon equation for a single spinless particle in an external scalar and vector potential in the limit that m_1 becomes very heavy.) The constraint $P\cdot p\psi = 0$ together with $M_2\partial M_2 = M_1\partial M_1$ and

$E_1 \partial E_1 = E_2 \partial E_2$ on ψ implies that $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent to each other on ψ and yield in the c.m. system the Schrödinger-like equation

$$G^2 \left[\mathbf{p}^2 - (\epsilon_w - \mathcal{A})^2 + 2\epsilon_w \mathcal{V} - \mathcal{V}^2 + (m_w + S)^2 + \frac{2}{i} \nabla \ln G \cdot \mathbf{p} - \tfrac{1}{2} \nabla^2 \ln G - \tfrac{3}{4}(\nabla^2 \ln G)^2 + i \nabla \ln G \cdot \gamma \mathbf{p} \cdot \gamma - i \frac{\nabla E_1 \cdot \gamma \gamma^0}{G} + \frac{i \nabla M_1 \cdot \gamma}{G} \right] \psi = 0 \ . \quad (136)$$

The two off-diagonal terms at the end of this equation are brought to diagonal form by using the Dirac equation (133) to rewrite ψ as

$$\psi = G (M_1 - E_1 \gamma^0)^{-1} (\mathbf{p} - i \nabla \ln G) \cdot \gamma \psi \ . \quad (137)$$

This procedure is the same as that used to reduce the one-body Dirac equation to quadratic two-component form. One then performs the scale change $\psi = G \sqrt{\mathcal{X}_1}\, \Psi$. This leads to the Pauli form [see Eq. (50) for comparison]

$$\left[\mathbf{p}^2 - (\epsilon_w - \mathcal{A})^2 + 2\epsilon_w \mathcal{V} - \mathcal{V}^2 + (m_w + S)^2 - \nabla^2 \ln \mathcal{X}_1 / 2 + (\nabla \ln \mathcal{X}_1)^2 / 4 - \frac{1}{r} \partial \ln \mathcal{X}_1 / \partial r \mathbf{L} \cdot \sigma_1 \right] \Psi = 0 \ , \quad (138)$$

where $\mathcal{X}_1 = (E_1 \gamma_1^0 + M_1)/G$. This wave equation is a spin-dependent elaboration of (50). Its spinless part is the same as given in (50), including the $\nabla \ln G$ terms at the end of (50) buried in the $\nabla \ln \mathcal{X}_1$ term above. Just as with that equation, the momentum structure of (138) is as simple as that appearing in the nonrelativistic Schrödinger equation. Notice that the Darwin and spin-orbit terms contain denominator forms that temper their singular nature when the underlying potential is Coulomb-like. This is a feature that Eq. (138) inherits partly from our two-body spinless formalism and partly from Dirac's equation itself. Equation (138) also has the correct semirelativistic limit, although we shall not display it here.

V. CONSTRAINT DYNAMICS FOR TWO INTERACTING SPIN-$\frac{1}{2}$ PARTICLES UNDER MUTUAL VECTOR AND SCALAR INTERACTIONS: THE TWO-BODY DIRAC EQUATION

We now come to the case of greatest physical interest and greatest complexity—that of two spinning particles in mutual relativistic interaction. We shall treat this case using the same procedures that we employed in the simpler case treated in the previous section. Just as happened in that case, we will find that the various supersymmetries we introduce for each spinning particle (including broken supersymmetries) drive us again to the third-law condition that in turn simplifies the conditions for compatibility to those that appear in the spinless case. We will begin by introducing the scalar interaction alone, then examine the timelike four-vector interaction alone and in combination with the scalar. Finally, we add in the electromagneticlike interaction and quantize the resulting system. This leads to the two simultaneous Dirac equations (1a) and (1b) that are the most important result of our paper. Before investigating the detailed description of the dynamics of two spin-$\frac{1}{2}$ particles, however, we wish to remind the reader of some important results found in the previous section. We discovered there that three independent invariant functions describe the mutual scalar and vector interactions between two spinless particles or between one spinless and one spin-$\frac{1}{2}$ particle. The interactions are local functions dependent on $x_\perp{}^2$, l^2 (becoming nonlocal if additional p^2 dependence is present). They are the consequences of the compatibility dictated by constraint mechanics. When only scalar interactions or timelike four-vector interactions were present, we were guided in finding compatible constraints (and as a consequence compatible Dirac and Klein-Gordon operators) by demanding, in addition to the requirements of spinless compatibility, that the interacting systems have the same supersymmetries displayed by systems of free particles. Associated with each type of interaction was an $\tilde{x}_\perp$ variable that was supersymmetric and that replaced the spinless variable $x_\perp$ in the potential-dependent mass (M) or energy (E) functions. The $\tilde{x}_\perp$ dependence generated spin-dependent corrections of M_i or E_i that in turn led to compatible constraints.

When both types of interaction were present at the same time, there was no overall supersymmetry although the $\tilde{E}_i$ and $\tilde{M}_i$ parts of the interactions were each invariant under its associated supersymmetry. These partial supersymmetries were nevertheless sufficient to generate compatible constraints. As we saw in the case of two spinless particles, the various parts of the interaction can be made separately compatible. In the case of one spinning particle, the mechanism that guaranteed compatibility was the presence of supersymmetries in each limit when only one of the interactions was turned on. With the addition of electromagneticlike vector interactions, we encountered the added complexity of constituent scalar interactions dependent on the vector interaction in addition to the underlying scalar. Thus, the structure induced by various $\tilde{x}_\perp$'s, associated with different parts of the interaction, became mixed. Nevertheless, the tilde forms $\tilde{\pi}_i$ and $\tilde{M}_i$ derived from $\tilde{x}_\perp$ expansions with just scalar and timelike vector interactions readily generalized in the presence of electromagneticlike vector interactions. The generalizations were determined by the requirement that the $\mathcal{H}_1$ constraint (derived from $\mathcal{S}_1$) be expressible entirely in terms of $\tilde{\pi}_1$ and $\tilde{M}_1$. The most important new feature introduced by electromagneticlike interactions was that the transverse part of π_1 must include Grassmann corrections. This in turn induced appropriate changes in both the mass potential $\tilde{M}_1$ and the longitudinal (or timelike) part of $\tilde{\pi}_1$. The importance of the tilde variables was that they allowed a construction of $\tilde{\pi}_2$ and $\tilde{M}_2$ that led to an $\mathcal{H}_2$ constraint compatible with $\mathcal{S}_1$. Without them, construction of this constraint (the only one for the spinless particle) would have been sheer guess work.

A knowledge of the tilde structures was really not essential for the construction of the $\mathcal{S}_1$ and $\mathcal{H}_1$ constraints when there was only one spinning particle. The form $\mathcal{S}_1=\pi_1\cdot\theta_1+M_1\theta_{51}$ was already correct when π_1 and M_1 were given by their spinless forms. The reason for this is that the Grassmann corrections to π_1 and M_1 generated by $\tilde{x}$'s disappeared automatically from $\mathcal{S}_1$ through internal multiplications by the Grassmann variables θ_1 and θ_{51}, respectively. When both particles have spin, however, some of the extra structure will survive since there will be two (mutually commuting) sets of Grassmann variables present. This structure is discussed in detail in Ref. 2 for the case when just scalar interactions are present. Here we review the highlights.

For two spin-$\frac{1}{2}$ particles, our pseudoclassical description employs two sets of Grassmann variables $\theta_{1\alpha}$, $\theta_{2\alpha}$, $\alpha=0,1,2,3,5$ (one for each particle) that for a given particle anticommute among themselves while commuting with the θ's belonging to the other particle ($\theta_{1\alpha}\theta_{2\beta}=\theta_{2\beta}\theta_{1h\alpha}$). These correspond to commuting sets of Dirac γ matrices ($[\gamma_\alpha^{(1)},\gamma_\beta^{(2)}]=0$). All the fundamental dynamical variables we deal with have definite even or odd character with respect to each space. Oddness or evenness is expressed by the relation

$$A_\alpha A_\beta=\eta_{\alpha\beta}A_\beta A_\alpha \ .$$

As we have seen, for a system employing one set of Grassmann variables, $\eta_{\alpha\beta}=(-)^{\epsilon_\alpha\epsilon_\beta}$. However, for a system employing two commuting sets of Grassmann variables,

$$\eta_{\alpha\beta}=(-)^{\epsilon_{\alpha1}\epsilon_{\beta2}} \ . \tag{139}$$

Thus, for example, $\theta_{1\alpha}$ is odd in space 1 and even in space 2, so that $\epsilon_{\alpha1}=1$, $\epsilon_{\alpha2}=0$. Likewise, $\theta_{2\alpha}$ is odd in its own space and even in the other, so that $\epsilon_{\beta1}=0$, $\epsilon_{\beta2}=1$. As a consequence

$$\theta_{1\alpha}\theta_{2\beta}=(-)^0\theta_{2\beta}\theta_{1\alpha} \ .$$

For this system, both the product rule (56) and the pseudoclassical Jacobi identity (60) retain their form with the appropriate η. The only new fundamental pseudoclassical Poisson bracket is $\{\theta_{1\alpha},\theta_{2\beta}\}=0$.

We begin our treatment of the interacting pseudoclassical system of two spinning particles by first introducing scalar interactions. Just as happened in the case of systems that consist of one spinning particle and one spinless particle, we expect that the conditions for compatibility will include the spinless ones (75) and (76). We introduce pseudoclassical spin for each particle in such a way that supersymmetry is preserved for each spinning particle during interaction. This means that $x_\perp$ is replaced by

$$\tilde{x}_\perp=(g_{\mu\nu}+\hat{P}_\mu\hat{P}_\nu)(\tilde{x}_1-\tilde{x}_2)_\nu \ , \tag{140}$$

where [with $\tilde{M}_i\equiv M_i(\tilde{x}_\perp)$]

$$\tilde{x}_1=x_1^\mu+\frac{i\theta_1^\mu\theta_{51}}{\tilde{M}_1} \ , \tag{141a}$$

$$\tilde{x}_2=x_2^\mu+\frac{i\theta_2^\mu\theta_{52}}{\tilde{M}_2} \ . \tag{141b}$$

Equations (141a) and (141b) are self-referent definitions. However, since the Grassmann Taylor expansions for $\tilde{M}_1$ and $\tilde{M}_2$ terminate, they completely determine the $\tilde{x}_i$ in terms of the x_i. Substitution of the resulting $\tilde{M}_i$ into the Dirac-like constraints then yields

$$\mathcal{S}_1=p_1\cdot\theta_1+M_1\theta_{51}-i\frac{\partial M_1}{M_2}\cdot\theta_2\theta_{52}\theta_{51}\approx 0 \tag{142}$$

and

$$\mathcal{S}_2=p_2\cdot\theta_2+M_2\theta_{52}+i\frac{\partial M_2}{M_1}\cdot\theta_1\theta_{51}\theta_{52}\approx 0 \ . \tag{143}$$

The mass potential for particle one ($\tilde{M}_1$) has modifications due to the presence of particle two (and vice versa) that are not removed by multiplication by θ_{51} in $\mathcal{S}_1$. The canonical square of each $\mathcal{S}_i$ yields its mass-shell companion

$$\mathcal{H}_i=\frac{1}{i}\{\mathcal{S}_i,\mathcal{S}_i\}=p_i^{\,2}+\tilde{M}_i^{\,2} \ . \tag{144}$$

Note that just as in the spin-zero spin-$\frac{1}{2}$ case $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same spin dependence, so that $\mathcal{H}_1-\mathcal{H}_2=2P\cdot p\approx 0$. Thus, for an interacting system of two spin-$\frac{1}{2}$ particles, once supersymmetry is imposed, compatibility restrictions drive us once again to the third law and the $x_\perp$ dependence of the underlying spinless potential given in (75) and (76). As shown in Refs. 2 and 3, $\tilde{x}_1$, $\tilde{x}_2$, $\mathcal{S}_1$, and $\mathcal{S}_2$ are supersymmetric under transformations generated by

$$\mathcal{G}_i=p_i\cdot\theta_i+\sqrt{-p_i^{\,2}}\theta_{5i} \ .$$

The $\mathcal{S}_i$'s would not be supersymmetric without the extra spin-dependent terms at the end, nor would they be compatible. In Ref. 2 we found that use of (75) along with repeated use of the pseudoclassical counterpart of the product rule (56) leads to

$$\begin{aligned}\{\mathcal{S}_1,\mathcal{S}_2\}&=-(p_1\cdot\partial M_2/M_1-p_2\cdot\partial M_1/M_2)\theta_{51}\theta_{52}\\&=(P\cdot\partial M_2/M_1)\theta_{51}\theta_{52} \ .\end{aligned} \tag{145}$$

This vanishes strongly even if M_i depends on the relative momentum (since $x_\perp\cdot p=x_\perp\cdot p_\perp$), so that $\mathcal{S}_1$ and $\mathcal{S}_2$ are compatible. In fact, as we showed in Ref. 2 using the Jacobi identity, all four constraints are compatible if $\mathcal{S}_1$ and $\mathcal{S}_2$ are strongly compatible.

The primary utility of supersymmetry in the construction of $\mathcal{S}_1$ and $\mathcal{S}_2$ is that it eliminates all spin complications and reduces pseudoclassical compatibility problems to those of the purely spin-zero system. When only timelike vector interactions are present, similar supersymmetry arguments succeed in generating compatible constraints. As in the case of one spinless and one spinning particle, we expect that the conditions for compatibility will include the spinless ones (92) and (93). In this case we introduce supersymmetric interactions by replacing $x_\perp$ by

$$\tilde{x}_\perp=(g_{\mu\nu}+\hat{P}_\mu\hat{P}_\nu)(\tilde{x}_1-\tilde{x}_2)_\nu \ , \tag{146}$$

where now, with $\tilde{E}_i=E_i(\tilde{x}_\perp)$,

$$(\tilde{x}_1)^\mu_E=x^\mu_1-\frac{i\theta^\mu_1\theta_1\cdot\hat{P}}{\tilde{E}_1} \ , \tag{147a}$$

$$(\tilde{x}_2)^\mu_E=x^\mu_2-\frac{i\theta^\mu_2\theta_2\cdot\hat{P}}{\tilde{E}_2} \ . \tag{147b}$$

Again self-referent definitions may be evaluated using terminating Taylor expansions for the $\tilde{E}_i$. Substitution of these into the Dirac-like constraints leads to

$$\begin{aligned}\mathcal{S}_1&=\tilde{\pi}_1\cdot\theta_1+m_1\theta_{51}\\&=\tilde{E}_1\hat{P}\cdot\theta_1+p\cdot\theta_1+m_1\theta_{51}\\&=E_1\hat{P}\cdot\theta_1+p\cdot\theta_1+m_1\theta_{51}+i\frac{\partial E_1}{E_2}\cdot\theta_1\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2 \ ,\end{aligned} \tag{148a}$$

$$\begin{aligned}\mathcal{S}_2&=\tilde{\pi}_2\cdot\theta_2+m_2\theta_{52}\\&=\tilde{E}_2\hat{P}\cdot\theta_2-p\cdot\theta_2+m_2\theta_{52}\\&=E_2\hat{P}\cdot\theta_2-p\cdot\theta_2+m_2\theta_{52}-i\frac{\partial E_2}{E_1}\cdot\theta_2\hat{P}\cdot\theta_2\hat{P}\cdot\theta_1 \ .\end{aligned} \tag{148b}$$

As usual, the signature of a vector interaction is its alteration of the effective mechanical momentum of each particle to some π_i (here $\tilde{\pi}_i$) that depends on the mutual interaction. However, a new feature arises when both particles have spin. In that case, the $\tilde{\pi}_i$ for each particle contains terms depending on the other particle's spin that survive the Grassmann multiplications in $\mathcal{S}_i$ and product recoil corrections to each $\mathcal{S}_i$ that are electromagnetic-moment-like interactions. The canonical squares of the $\mathcal{S}_i$'s,

$$\mathcal{H}_i=\frac{1}{i}\{\mathcal{S}_i,\mathcal{S}_i\}=\tilde{\pi}_i{}^2+m_i{}^2, \quad i=1,2 \ ,$$

produces $\mathcal{H}_i$'s with the same spin dependence, so that again $\mathcal{H}_1-\mathcal{H}_2\sim P\cdot p$. As in the scalar case, compatibility of $\mathcal{S}_1$ and $\mathcal{S}_2$ depends on the inclusion of the extra spin-dependent terms at the end of (148a) and (148b). These terms help to guarantee the supersymmetry of $\mathcal{S}_1$ and $\mathcal{S}_2$ under transformations generated by

$$\mathcal{G}_i=\hat{P}\cdot\theta_i(p_\perp{}^2+m_i{}^2)^{1/2}+p_\perp\cdot\theta_i+m_i\theta_{5i} \ .$$

Their supersymmetry ultimately depends on the fact that $\{\mathcal{G}_j,\tilde{x}_i\}=0$, i.e., that the constituent variables are supersymmetric. As in the scalar case this guarantees the third-law condition. The details of this demonstration are straightforward as are the proofs that $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are compatible with one another.

When both scalar and timelike vector interactions are present, there is no overall supersymmetry of the like described above. However, just as in the spin-$\frac{1}{2}$, spin-zero case, the structures necessary to guarantee invariance in single-interaction limits are sufficient to guarantee the compatibility of the constraints. The basic Dirac-like constraints are again of the form $\mathcal{S}_i=\tilde{\pi}_i\cdot\theta_i+\tilde{M}_i\theta_{5i}$, depending on interaction-dependent (and spin-dependent) momenta and masses dictated by our previous analyses of the separate interactions. We restrict our attention here to the case in which E_i and M_i depend on the relative momentum at most through the angular momentum l^2. Then we obtain the spin-dependent corrections to the potential forms that lead to compatible constraints by replacing $E_i(x_\perp,p_1,p_2)$, $M_i(x_\perp,p_1,p_2)$ by $E_i((\tilde{x}_\perp)_E,p_1,p_2)$, $M_i((\tilde{x}_\perp)_M,p_1,p_2)$, where

$$(\tilde{x}_\perp)^\mu_M=[(\tilde{x}_1)_M-(\tilde{x}_2)_M]^\mu_\perp \ , \tag{149a}$$

$$(\tilde{x}_\perp)^\mu_E=[(\tilde{x}_1)_E-(\tilde{x}_2)_E]^\mu_\perp \ , \tag{149b}$$

with $\tilde{x}_{iM}$ and $\tilde{x}_{iE}$ given in (141a), (141b), (147a), and (147b). This leads to

$$\begin{aligned}\tilde{M}_1=M_1&+i\frac{\partial M_1}{M_1}\cdot\theta_1\theta_{51}-i\frac{\partial M_1}{M_2}\cdot\theta_2\theta_{52}\\&+\theta_1\cdot\partial\frac{\theta_2\cdot\partial M_1}{M_1M_2}\theta_{51}\theta_{52} \ ,\end{aligned} \tag{150a}$$

$$\begin{aligned}\tilde{M}_2=M_2&+i\frac{\partial M_2}{M_1}\cdot\theta_1\theta_{51}-i\frac{\partial M_2}{M_2}\cdot\theta_2\theta_{52}\\&+\theta_2\cdot\partial\frac{\theta_1\cdot\partial M_2}{M_1M_2}\theta_{51}\theta_{52} \ ,\end{aligned} \tag{150b}$$

$$\begin{aligned}\tilde{E}_1=E_1&-i\frac{\partial E_1}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1+i\frac{\partial E_1}{E_2}\cdot\theta_2\hat{P}\cdot\theta_2\\&+\theta_1\cdot\partial\frac{\theta_2\cdot\partial E_1}{E_1E_2}\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2 \ ,\end{aligned} \tag{150c}$$

$$\begin{aligned}\tilde{E}_2=E_2&-i\frac{\partial E_2}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1+i\frac{\partial E_2}{E_2}\cdot\theta_2\hat{P}\cdot\theta_2\\&+\theta_2\cdot\partial\frac{\theta_1\cdot\partial E_2}{E_1E_2}\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2 \ .\end{aligned} \tag{150d}$$

The basic constraints then become

$$\begin{aligned}\mathcal{S}_1&=\tilde{\pi}_1\cdot\theta_1+\tilde{M}_1\theta_{51}\\&=p\cdot\theta_1+E_1\hat{P}\cdot\theta_1+i\frac{\partial E_1}{E_2}\cdot\theta_2\hat{P}\cdot\theta_2\hat{P}\cdot\theta_1\\&\quad+M_1\theta_{51}-i\frac{\partial M_1}{M_2}\cdot\theta_2\theta_{51}\theta_{52} \ ,\end{aligned} \tag{151}$$

$$\mathcal{S}_2=\tilde{\pi}_2\cdot\theta_2+\tilde{M}_2\theta_{52}$$
$$=-p\cdot\theta_2+E_2\hat{P}\cdot\theta_2-i\frac{\partial E_2}{E_1}\cdot\theta_1\hat{P}\cdot\theta_2\hat{P}\cdot\theta_1$$
$$+M_2\theta_{52}-i\frac{\partial M_2}{M_1}\cdot\theta_1\theta_{51}\theta_{52}\ . \quad (152)$$

One then finds that[46]

$$\mathcal{H}_1\equiv\frac{1}{i}\{\mathcal{S}_1,\mathcal{S}_1\}\approx\tilde{\pi}_1{}^2+\tilde{M}_1{}^2\approx 0\ , \quad (153)$$

$$\mathcal{H}_2\equiv\frac{1}{i}\{\mathcal{S}_2,\mathcal{S}_2\}\approx\tilde{\pi}_2{}^2+\tilde{M}_2{}^2\approx 0\ . \quad (154)$$

Each $\tilde{x}$ turns out to be invariant under its own associated supersymmetry (but not under the other type):

$$\{\mathcal{G}_{iM},(\tilde{x}_j)_M\}\approx 0,\quad \{\mathcal{G}_{iM},(\tilde{x}_j)_E\}\not\approx 0\ ,$$
$$\{\mathcal{G}_{iE},(\tilde{x}_j)_E\}\approx 0,\quad \{\mathcal{G}_{iEM},(\tilde{x}_j)_M\}\not\approx 0\ ,$$

with the $\mathcal{G}_i$'s defined just as in (110) (but with $E_i\to\tilde{E}_i$, $M_i\to\tilde{M}_i$). The construction of the $\mathcal{S}_i$'s from these would-be invariants turns out to be sufficient to guarantee that all four constraints ($\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{H}_1$, $\mathcal{H}_2$) are compatible with one another.

Finally, we complete our dynamical scheme for two spin-$\frac{1}{2}$ particles by introducing electromagneticlike vector interactions in addition to the scalar and timelike vector interactions. Again, the interactions enter the Dirac-like constraints through (spin-dependent) momentum and mass modifications:

$$\mathcal{S}_1=\tilde{\pi}_1\cdot\theta_1+\tilde{M}_1\theta_{51}\ , \quad (155)$$

$$\mathcal{S}_2=\tilde{\pi}_2\cdot\theta_2+\tilde{M}_2\theta_{52}\ . \quad (156)$$

We are guided in our search for acceptable $\tilde{\pi}_i$ and $\tilde{M}_i$'s by the corresponding spin-zero, spin-$\frac{1}{2}$ forms given in (127) and (128) as well as the results for $\tilde{E}_i$ and $\tilde{M}_i$ given in (150). Accordingly, we take

$$\tilde{\pi}_1^\mu=\tilde{E}_1\hat{P}^\mu+Gp^\mu+i\theta_1\cdot\partial G\,\theta_{1\perp}^\mu+i\theta_2\cdot\partial G\,\theta_{2\perp}^\mu\ , \quad (157a)$$

$$\tilde{\pi}_2^\mu=\tilde{E}_2\hat{P}^\mu-Gp^\mu-i\theta_1\cdot\partial G\,\theta_{1\perp}^\mu-i\theta_2\cdot\partial G\,\theta_{2\perp}^\mu\ , \quad (157b)$$

where

$$\tilde{E}_1=E_1-iG\frac{\partial E_1}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1+iG\frac{\partial E_1}{E_2}\cdot\theta_2\hat{P}\cdot\theta_2$$
$$+G^2\theta_1\cdot\partial\frac{\theta_2\cdot\partial E_1}{E_1E_2}\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2\ , \quad (158a)$$

$$\tilde{E}_2=E_2-iG\frac{\partial E_2}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1+iG\frac{\partial E_2}{E_2}\cdot\theta_2\hat{P}\cdot\theta_2$$
$$+G^2\theta_2\cdot\partial\frac{\theta_1\cdot\partial E_2}{E_1E_2}\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2\ . \quad (158b)$$

The corresponding spin-dependent mass potentials then become

$$\tilde{M}_1=M_1+iG\frac{\partial M_1}{M_1}\cdot\theta_1\theta_{51}-iG\frac{\partial M_1}{M_2}\cdot\theta_2\theta_{52}$$
$$+G^2\theta_1\cdot\partial\frac{\theta_2\cdot\partial M_1}{M_1M_2}\theta_{51}\theta_{52}\ , \quad (159a)$$

$$\tilde{M}_2=M_2+iG\frac{\partial M_2}{M_1}\cdot\theta_1\theta_{51}-iG\frac{\partial M_2}{M_2}\cdot\theta_2\theta_{52}$$
$$+G^2\theta_2\cdot\partial\frac{\theta_1\cdot\partial M_2}{M_1M_2}\theta_{51}\theta_{52}\ . \quad (159b)$$

We substitute these into the constraints (155) and (156) to obtain

$$\mathcal{S}_1=G\left[\theta_1\cdot p+\frac{M_1\theta_{51}+E_1\hat{P}\cdot\theta_1}{G}+i\partial\ln G\cdot\theta_2\theta_{1\perp}\cdot\theta_{2\perp}\right.$$
$$\left.+i\frac{\partial E_1}{E_2}\cdot\theta_2\hat{P}\cdot\theta_2\hat{P}\cdot\theta_1-i\frac{\partial M_1}{M_2}\cdot\theta_2\theta_{52}\theta_{51}\right]\ , \quad (160)$$

$$\mathcal{S}_2=G\left[-\theta_2\cdot p+\frac{M_2\theta_{52}+E_2\hat{P}\cdot\theta_2}{G}-i\partial\ln G\cdot\theta_1\theta_{2\perp}\cdot\theta_{1\perp}\right.$$
$$\left.-i\frac{\partial E_2}{E_1}\cdot\theta_1\hat{P}\cdot\theta_1\hat{P}\cdot\theta_2+i\frac{\partial M_2}{M_1}\cdot\theta_1\theta_{51}\theta_{52}\right]\ . \quad (161)$$

For the full system, one also finds that (see Ref. 45)

$$\mathcal{H}_1=\frac{1}{i}\{\mathcal{S}_1,\mathcal{S}_1\}\approx\tilde{\pi}_1{}^2+\tilde{M}_1{}^2\ , \quad (162)$$

$$\mathcal{H}_2=\frac{1}{i}\{\mathcal{S}_2,\mathcal{S}_2\}\approx\tilde{\pi}_2{}^2+\tilde{M}_2{}^2\ , \quad (163)$$

with the appropriate tilde forms (157a), (157b), (159a), and (159b). Once again, $\mathcal{H}_1-\mathcal{H}_2=P\cdot p$, so that these forms incorporate the third-law condition. The proofs that these four constraints are compatible carry through in the usual way. (Repeated use of the appropriate pseudoclassical analog of the product rule (56) [with (139)] is essential[51].)

When all the interactions are present, canonical quantization of (160) and (161) produces the compatible two-body Dirac equations

$$\mathcal{S}_1\psi=(\tilde{\pi}_1\cdot\theta_1+\tilde{M}_1\theta_{51})\psi=0\ , \quad (164)$$

$$\mathcal{S}_2\psi=(\tilde{\pi}_2\cdot\theta_2+\tilde{M}_2\theta_{52})\psi=0 \quad (165)$$

(with appropriately Hermitized terms). Notice that one quantizes the Grassmann product forms $\tilde{\pi}_i\cdot\theta_i$ and $\tilde{M}_i\theta_{5i}$ after all internal Grassmann multiplications have been performed. One does not first quantize $\tilde{\pi}_i$ and $\tilde{M}_i$ and then multiply them by the operator θ's (γ matrices). This would not only lead to Dirac operators that were not compatible, but would lead even in the ordinary one-body case to the incorrect description of a spin-$\frac{1}{2}$ particle in an external potential. One demonstrates the compatibility of the quantum operators $\mathcal{S}_1$ and $\mathcal{S}_2$ corresponding to the pseudoclassical constraints (160) and (161) by a process that is isomorphic to the pseudoclassical proof.[50] This isomorphism is made possible by the correspondence between the pseudoclassical brackets and the quantum (anti)commutators [particularly the quantum product rule (56)].

VI. TWO-BODY DIRAC EQUATIONS

Now that we have constructed two compatible Dirac equations that describe the quantum mechanics of two interacting spin-$\frac{1}{2}$ particles, we shall write them out in full in three useful forms: (i) manifestly covariant; (ii) center-of-mass rest frame; (iii) reduced Pauli form (in c.m. rest frame[52]). Along the way we review the various claims made about them in the Introduction. In covariant form our two-body Dirac equations are

$$\mathcal{S}_1\psi=\gamma_{51}[\gamma_1\cdot(p_1-\tilde{A}_1)+m_1+\tilde{S}_1]\psi=0\ , \tag{1a}$$

$$\mathcal{S}_2\psi=\gamma_{52}[\gamma_2\cdot(p_2-\tilde{A}_2)+m_2+\tilde{S}_2]\psi=0\ , \tag{1b}$$

in which

$$\tilde{A}_1=\left[(\epsilon_1-E_1)-i\frac{G}{2}\gamma_2\cdot\left[\frac{\partial E_1}{E_2}+\partial\ln G\right]\gamma_2\cdot\hat{P}\right]\hat{P}+(1-G)p-\frac{i}{2}\partial G\cdot\gamma_2\gamma_2\ , \tag{166}$$

$$\tilde{A}_2=\left[(\epsilon_2-E_2)+i\frac{G}{2}\gamma_1\cdot\left[\frac{\partial E_2}{E_1}+\partial\ln G\right]\gamma_1\cdot\hat{P}\right]\hat{P}-(1-G)p+\frac{i}{2}\partial G\cdot\gamma_1\gamma_1\ , \tag{167}$$

$$\tilde{S}_1=M_1-m_1-\frac{i}{2}G\gamma_2\cdot\frac{\partial M_1}{M_2}\ , \tag{168}$$

$$\tilde{S}_2=M_2-m_2+\frac{i}{2}G\gamma_1\cdot\frac{\partial M_2}{M_1}\ . \tag{169}$$

Note that (as claimed in the Introduction) the relative time is regulated in a covariant manner through the dependence of these potentials on the variable $x_\perp$. These compatible wave equations (1a) and (1b) take the following forms in the c.m. system (where $P\cdot p\psi=0$ implies $\theta_i\cdot p\psi=-\gamma_{5i}\gamma_i/2\cdot\mathbf{p}\psi$):

$$\mathcal{S}_1\psi=\gamma_{51}G\left[\boldsymbol{\gamma}_1\cdot\mathbf{p}+\frac{M_1-E_1\gamma_1^0}{G}+\frac{i}{2}\boldsymbol{\gamma}_2\cdot\nabla\ln G\boldsymbol{\gamma}_2\cdot\boldsymbol{\gamma}_1+\frac{i}{2}\boldsymbol{\gamma}_2\cdot\frac{\nabla E_1}{E_2}\gamma_2^0\gamma_1^0-\frac{i}{2}\boldsymbol{\gamma}_2\cdot\frac{\nabla M_1}{M_2}\right]\psi=0\ , \tag{170}$$

$$\mathcal{S}_2\psi=\gamma_{52}G\left[-\boldsymbol{\gamma}_2\cdot\mathbf{p}+\frac{M_2-E_2\gamma_2^0}{G}-\frac{i}{2}\boldsymbol{\gamma}_1\cdot\nabla\ln G\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_2-\frac{i}{2}\boldsymbol{\gamma}_1\cdot\frac{\nabla E_2}{E_1}\gamma_1^0\gamma_2^0+\frac{i}{2}\boldsymbol{\gamma}_1\cdot\frac{\nabla M_2}{M_1}\right]\psi=0\ . \tag{171}$$

In Ref. 2 we showed how an exact reduction of these coupled 16-component wave equations to four decoupled (with diagonal γ_i^0) 4-component Schrödinger-like equations can be carried out. The details of this procedure for the more general case of (170) and (171) are similar to those outlined in that paper. First one computes $\mathcal{H}_i\equiv[\mathcal{S}_i,\mathcal{S}_i]_+$. This gives a wave equation with two varieties of off-diagonal coupling terms: singly odd (such as $\boldsymbol{\gamma}_i\cdot\mathbf{p}=\gamma_{5i}\gamma_i^0\boldsymbol{\sigma}_i\cdot\mathbf{p}$) and doubly odd (such as $\boldsymbol{\gamma}_1\cdot\mathbf{r}\boldsymbol{\gamma}_2\cdot\mathbf{r}=\gamma_{51}\gamma_1^0\gamma_{52}\gamma_2^0\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{r}$). The singly odd terms are brought to diagonal form by using the appropriate Dirac equation [$\mathcal{S}_i\psi=0$ to bring the term $\boldsymbol{\gamma}_i\cdot(\mathbf{p}-\nabla\ln G)$ to a diagonal form, analogous to that found in (136) and (138)]. This produces diagonal (γ_i^0) spin-orbit and Darwin terms and more doubly odd terms. One then diagonalizes the doubly odd terms through rearrangements that use both the $\mathcal{S}_1\psi=0$ and $\mathcal{S}_2\psi=0$ Dirac equations (see Appendix B). This procedure (with the additional scale transformation $\psi=\sqrt{\chi_1\chi_2}\,\Psi$) results in the following four decoupled (with diagonal γ_i^0's) four-component Schrödinger-like equations:

$$(\mathbf{p}^2+\Phi_{SI}+\Phi_{SS}+\Phi_{SO}+\Phi_T+\Phi_{DO})\Psi=b^2\Psi\ , \tag{172}$$

where

$$\Phi_{SI}=2m_wS+S^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2\ ,$$

$$\Phi_{SS}=-\nabla^2\ln(\chi_1\chi_2G^{1-2\sigma\cdot\sigma_2/3})/2+[\nabla\ln(\chi_1\chi_2G^{1-2\sigma_1\cdot\sigma_2/3})]^2/4+(\nabla\ln G)^2(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)/18\ ,$$

$$\Phi_{SO}=-(\partial\ln\chi_1/\partial r\mathbf{L}\cdot\boldsymbol{\sigma}_1+\partial\ln\chi_2/\partial r\mathbf{L}\cdot\boldsymbol{\sigma}_2)/r\ ,$$

$$\Phi_T=S_T[-(r\partial^2\ln G/\partial r^2-\partial\ln G/\partial r)/r+\nabla\ln G\cdot\nabla\ln(\chi_1\chi_2)]/6\ ,$$

$$\begin{aligned}\Phi_{DO}=(\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{E}})^2/4-\{&\epsilon_2\boldsymbol{\sigma}_1\cdot\nabla[\boldsymbol{\sigma}_2\cdot(\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{E}})]/w+\epsilon_1\boldsymbol{\sigma}_2\cdot\nabla[\boldsymbol{\sigma}_1\cdot(\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{E}})]/w-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\nabla\ln G\cdot(\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{E}})\\&-\boldsymbol{\sigma}_1\cdot\nabla\ln(\chi_1)\boldsymbol{\sigma}_2\cdot(\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{E}})-\boldsymbol{\sigma}_2\cdot\nabla\ln(\chi_2)\boldsymbol{\sigma}_1\cdot(\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{E}})\}(-)^s(\Phi_{SI}-b^2)/(2\chi_1\chi_2)\ ,\end{aligned}$$

with $(-)^s=1$ for spin-singlet and -1 for spin-triplet cases, $\chi_i=(E_i\gamma_i^0+M_i)/G$,

$$\boldsymbol{\mathcal{M}}=\nabla(M_1{}^2+M_2{}^2)/4M_1M_2\ ,$$

and

$$\boldsymbol{\mathcal{E}}=\nabla(E_1{}^2+E_2{}^2)/4E_1E_2\gamma_1^0\gamma_2^0\ .$$

We make no approximation in going from the two-body Dirac equations (170) and (171) to the Pauli form (172). This Pauli form is the two-body analog of the reduced form of the standard Dirac equation involving two coupled two-component equations with the characteristic form $\gamma^0(E-V)-m-S$ appearing in the denominators of the $\mathbf{L}\cdot\mathbf{S}$ and Darwin terms. That potential energy and energy-dependent denominator structure appears in our equations through the X forms.

The notation we have used in labeling the various parts of the "quasipotential" Φ is motivated by correspondence with the standard semirelativistic expansions of atomic physics. Φ_{SI} is the spin-independent piece not including the Darwin interactions. The Darwin interactions are contained along with the spin-spin interactions in Φ_{SS}. [Many relativistic extensions of the quark model based on replacing the Schrödinger operator by

$$(\mathbf{p}_1^2+m_1^2)^{1/2}+(\mathbf{p}_2^2+m_2^2)^{1/2}+V$$

leave out these important Darwin terms.] Even though the Darwin terms are spin independent, they do not affect all angular-momentum levels to the same degree. In fact they provide a very short-ranged interaction. Hence, leaving them out tends to distort the splittings between the $l=0$ and the other angular-momentum levels. Φ_{SO} is the spin-orbit term. Through X it contains both magnetic and Thomas precession parts. We symbolize the main tensor terms by Φ_T. Φ_{DO} contains those terms (spin-independent, Darwin-like, spin-spin, and tensor terms) that arise primarily from those terms in $\mathcal{H}_i$ $(=1/i[\mathcal{S}_i,\mathcal{S}_i]_+)$ that are doubly odd in γ_1 and γ_2 [which couple the upper-upper and the lower-lower components of the 16-component spinor (see Appendix B)]. Equation (172) is a spin-dependent elaboration of (50). Its momentum structure is as simple as that which appears in the nonrelativistic Schrödinger equation. Notice that all of the spin-dependent and Darwin terms contain denominator forms that temper their singular nature into quantum-mechanically legal operators. Thus, our Pauli forms make quantum-mechanical sense in the strong potential nonperturbative regime where relativistic effects of the wave operator on ψ are not negligible. This property is easiest to demonstrate when $S=0$, $\mathcal{V}=0$. We compare the main spin-spin, tensor, and spin-orbit terms of our equation with the corresponding terms of the Breit equation (two-body Dirac form$\to$Breit form):

$$-\tfrac{1}{6}\sigma_1\cdot\sigma_2\nabla^2\ln\left[1-\frac{2\mathcal{A}}{w}\right]\mapsto\frac{1}{3}\frac{\sigma_1\cdot\sigma_2\nabla^2\mathcal{A}}{m_1+m_2}\,, \tag{173}$$

$$-\tfrac{1}{12}S_T\left[\frac{1}{r}\frac{\partial}{\partial r}-\frac{\partial^2}{\partial r^2}\right]\ln\left[1-\frac{2\mathcal{A}}{w}\right]\mapsto\tfrac{1}{6}S_T\frac{[(1/r)(\partial\mathcal{A}/\partial r)]-\partial^2\mathcal{A}/\partial r^2}{m_1+m_2}\,, \tag{174}$$

$$-\frac{1}{4}\frac{\partial}{\partial r}\left\{\ln\left[[E_1(\mathcal{A})+m_1]\left[1-\frac{2\mathcal{A}}{w}\right]^{1/2}\right]L\cdot\sigma_1+\ln\left[[E_2(\mathcal{A})+m_2]\left[1-\frac{2\mathcal{A}}{w}\right]^{1/2}\right]L\cdot\sigma_2\right\}$$
$$\mapsto\frac{1}{r}\frac{\partial\mathcal{A}}{\partial r}\left[\frac{2m_1+m_2}{2m_1(m_1+m_2)}L\cdot\sigma_1+\frac{2m_2+m_1}{2m_2(m_1+m_2)}L\cdot\sigma_2\right]. \tag{175}$$

For $\mathcal{A}$'s that have singular short-range behavior like $-\alpha/r$ (QED) and $8\pi/27r\ln r$ (QCD) the weak $\mathcal{A}$ form on the right-hand sides can only be used perturbatively. Notice that the weak potential forms of our spin-dependent interaction terms are the same as the corresponding spin-dependent interaction terms of the $O(1/c^2)$ Breit Hamiltonian. (As we have shown earlier[4] the spin-independent semirelativistic $[O(1/c^2)]$ terms (not shown here) are canonically equivalent to the Darwin interaction.) However, unlike the Breit forms (which can only be used perturbatively[53]), our forms (on the left-hand side) can be used even when the effect of this term on the wave function is not that of a small perturbation. The logarithmic terms appearing in our Pauli forms provide a natural smoothing mechanism that avoids the necessity for extra singularity softening parameters in phenomenological applications.

As shown in a future paper,[32] for weak potentials, the upper-upper components of these equations, (172), reduce to the Todorov equations for scalar and vector interactions. Its equivalence to the Breit Hamiltonian follows by making a further slow-motion expansion. There we shall also show how, in its nonperturbative form, (172) becomes exactly soluble for singlet positronium. The resulting spectrum is correct through order α^4 (Ref. 54).

As mentioned in the Introduction, Eqs. (170) and (171) reduce to the ordinary Dirac equation in the limit that either particle becomes infinitely massive. For example, when $m_2\to\infty$, $G\to 1$ and the derivative terms at the end of Eq. (170) (particle one's equation) vanish, so that we are left with the Dirac equation for a single spin-$\frac{1}{2}$ particle in external scalar and vector potentials $[M_1\to m_1+S,\ E_1\to\epsilon_1-\mathcal{A}$ (for $\mathcal{V}=0$)]. This static limit feature carries through to the Pauli form of these equations. For example, when $\mathcal{V}=0$ (172) becomes the standard static limit form of two decoupled two-component equations similar to Eq. (138) with $\mathcal{V}=0$, $\epsilon_w\to\epsilon_1$, $m_w\to m_1$, and $G\to 1$.

VII. CONCLUSION

The most important results of this paper are the compatible wave equations (1a) and (1b) for two spin-$\frac{1}{2}$ particles interacting mutually through electromagneticlike

$$\mathcal{H}_2=\pi_2{}^2+M_2{}^2\approx 0\ , \tag{186}$$

leads to the effective system Klein-Gordon equation

$$[\mathbf{p}^2-(\epsilon_w-\mathcal{A})^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+(m_w+S)^2+\tfrac{1}{2}\nabla^2\ln G+\tfrac{1}{4}(\nabla\ln G)^2]\phi=0\ . \tag{187}$$

For a spin-zero, spin-$\frac{1}{2}$ system, the Pauli form of the two-body equation turns out to be

$$[\mathbf{p}^2-(\epsilon_w-\mathcal{A})^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+(m_w+S)^2-\nabla^2\ln\mathcal{X}_1/2+(\nabla\ln\mathcal{X}_1)^2/4-\partial\ln\mathcal{X}_1/\partial r\mathbf{L}\cdot\sigma_1]\Psi=0\ , \tag{188}$$

where $\mathcal{X}_1=(E_1\gamma_1^0+M_1)/G$. This wave equation is a spin-dependent elaboration of (187). The wave function Ψ has four components, but in the representation in which γ_1^0 is diagonal, this equation reduces to an uncoupled set of two two-component wave equations. It results from quantization of pseudoclassical constraints (displaying two types of supersymmetry associated with the separate scalar and vector interactions).

For a spin-$\frac{1}{2}$, spin-$\frac{1}{2}$ system, the Pauli form of the two-body Dirac equations given in (170) and (171) turns out to be

$$(\mathbf{p}^2+\Phi_{\rm SI}+\Phi_{\rm SS}+\Phi_{\rm SO}+\Phi_T+\Phi_{\rm DO})\Psi=b^2\Psi\ , \tag{189}$$

where

$$\Phi_{\rm SI}=2m_wS+S^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2\ ,$$

$$\Phi_{\rm SS}=-\nabla^2\ln(\mathcal{X}_1\mathcal{X}_2G^{1-2\sigma_1\cdot\sigma_2/3})/2+[\nabla\ln(\mathcal{X}_1\mathcal{X}_2G^{1-2\sigma_1\cdot\sigma_2/3})]^2/4+(\nabla\ln G)^2(3+\sigma_1\cdot\sigma_2)/18\ ,$$

$$\Phi_{\rm SO}=-(\partial\ln\mathcal{X}_1/\partial r\mathbf{L}\cdot\sigma_1+\partial\ln\mathcal{X}_2/\partial r\mathbf{L}\cdot\sigma_2)/r\ ,$$

$$\Phi_T=S_T[-(r\partial^2\ln G/\partial r^2-\partial\ln G/\partial r)/r+\nabla\ln G\cdot\nabla\ln(\mathcal{X}_1\mathcal{X}_2)]/6\ ,$$

$$\begin{aligned}\Phi_{\rm DO}=(\mathcal{M}-\mathcal{E})^2/4-\{&\epsilon_2\sigma_1\cdot\nabla[\sigma_2\cdot(\mathcal{M}-\mathcal{E})]/w+\epsilon_1\sigma_2\cdot\nabla[\sigma_1\cdot(\mathcal{M}-\mathcal{E})]/w\\&-\sigma_1\cdot\sigma_2\nabla\ln G\cdot(\mathcal{M}-\mathcal{E})-\sigma_1\cdot\nabla\ln(\mathcal{X}_1)\sigma_2\cdot(\mathcal{M}-\mathcal{E})\\&-\sigma_2\cdot\nabla\ln(\mathcal{X}_2)\sigma_1\cdot(\mathcal{M}-\mathcal{E})\}(-)^s(\Phi_{\rm SI}-b^2)/(2\mathcal{X}_1\mathcal{X}_2)\ ,\end{aligned}$$

with $\mathcal{X}_i=(E_i\gamma_i^0+M_i)/G$,

$$\mathcal{M}=\nabla(M_1{}^2+M_2{}^2)/4M_1M_2\ ,$$

and

$$\mathcal{E}=\nabla(E_1{}^2+E_2{}^2)/4E_1E_2\gamma_1^0\gamma_2^0\ .$$

This c.m. form is an equation for a 16-component wave function. Since all terms depend only on the diagonal γ matrices γ_1^0, γ_2^0, this equation reduces to an uncoupled set of four four-component wave equations. It results directly from quantization of pseudoclassical constraints (with broken supersymmetries) as outlined in the conclusion.

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APPENDIX A: MODIFICATIONS OF THE THIRD-LAW CONDITION

Consider the classical compatibility condition on the constraints

$$\begin{aligned}\mathcal{H}_1&=p_1{}^2+m_1{}^2+\Phi_1\approx 0\ ,\\ \mathcal{H}_2&=p_2{}^2+m_2{}^2+\Phi_2\approx 0\ .\end{aligned} \tag{A1}$$

Let

$$P=p_1+p_2,\quad P^2=w^2,\quad \hat{P}=P/w\ , \tag{A2}$$

$$x=x_1-x_2,\quad p=\frac{1}{w}(\epsilon_2p_1-\epsilon_1p_2)\ , \tag{A3}$$

$$\epsilon_1+\epsilon_2=w,\quad \epsilon_i=\epsilon_i(w)\ . \tag{A4}$$

Following Todorov,[17] we define D and Φ by

$$\Phi_1\equiv\Phi+\frac{\epsilon_1}{w}D=\Phi_1(x,p_1,p_2)\ , \tag{A5}$$

$$\Phi_2\equiv\Phi-\frac{\epsilon_2}{w}D=\Phi_2(x,p_2,p_2)\ . \tag{A6}$$

Then compatibility leads to

$$\{\mathcal{H}_1,\mathcal{H}_2\}=2w\hat{P}\cdot\partial\Phi+2p\cdot\partial D+\{D,\Phi\}\approx 0\ . \tag{A7}$$

Let

$$\Phi=\Phi\left[x_\perp{}^2,x_\perp\cdot p_\perp,\frac{p_\perp{}^2}{2},x_\parallel p_\parallel,\frac{p_\parallel{}^2}{2},P^2\right]\ , \tag{A8}$$

$$D=D\left[\frac{x_\parallel{}^2}{2},x_\parallel p_\parallel,\frac{p_\parallel{}^2}{2},P^2\right]\ , \tag{A9}$$

where

$$x_\perp^\mu=(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)x_\nu\ ,$$

$$x_\parallel=-x\cdot\hat{P},\quad p_\parallel=-p\cdot\hat{P}\ .$$

vector, timelike vector, and scalar interactions. The counterparts to these equations for a system of one spin-$\frac{1}{2}$ particle and one spinless particle and for a system of two spinless particles are given in (131), (132), (47a), and (47b), respectively. These three sets of equations are operator versions of the pseudoclassical descriptions given by (160), (161), (123), (125), and (45). (These pseudoclassical constraints generate systems of coupled Lorentz and Bargmann-Michel-Telegdi equations.) The Pauli forms for the wave equations with spin are given in (172) and (138) and are the most useful forms for practical applications (employing simple extensions of nonrelativistic forms). Their spinless counterpart is given in Eq. (50). If we trace the spin structure of the wave equation back to pseudoclassical mechanics, we see that it is rigidly dictated by two ingredients. The first is the dependence of interactions on constituent supersymmetric position variables $\tilde{x}_i$ that are themselves interaction dependent (one supersymmetric variable for the scalar and one for the timelike vector interactions). The second ingredient is an effective Gordon decomposition of the electromagnetic current. These ingredients lead to the spin-dependent potential structures given in (127a), (127b), (128a), and (128b) for the spinless and spin-$\frac{1}{2}$ system and (157)–(159) for the two spin-$\frac{1}{2}$ particle system. The spin dependences arising from these effects are supersymmetric elaborations of the spinless forms contained in (34), where (43a), (43b), (44a), and (44b) are true. The resultant spin-dependent tilde forms of the potential automatically enforce the same third-law forms as appear in the spinless equations. Thus, our procedure takes care of spin complications, reducing compatibility arguments to those appearing in the spinless case. The $\tilde{x}$ variables and their supersymmetries are essential ingredients in the pseudoclassical mechanics that underlies Dirac's own one-body equation with external potentials. Consequently, through proper extension of the tilde variables to the case of two particles (resulting in interaction dependence on $\tilde{x}_\perp$), our procedure ultimately leads to wave equations that stretch both perturbative and nonperturbative structures of Dirac's own one-body equation to the two-body problem. Hence, our equations possess correct relativistic kinematics, heavy-particle limits to relativistic one-body equations, and correct fine-structure as well as nonperturbative quantum-mechanical meaning. In a future paper[32] we describe the applications of these equations to electrodynamic systems and to covariant quark model calculations.

VIII. SUMMARY OF IMPORTANT EQUATIONS AND DEFINITIONS

The dynamical variables most convenient for the constraint description of the relativistic two-body problem are (i) relative position x_1-x_2, (ii) relative momentum $p=(1/w)(\epsilon_2 p_1-\epsilon_1 p_2)$, (iii) total c.m. energy $w=\sqrt{-P^2}$, (iv) total momentum $P=p_1+p_2$, (v) (conserved) constituent c.m. energies

$$\epsilon_1=\frac{w^2+m_1{}^2-m_2{}^2}{2w},\quad \epsilon_2=\frac{w^2+m_2{}^2-m_1{}^2}{2w},$$

(vi) relativistic reduced mass and energy of a fictitious particle of relative motion,

$$m_w=\frac{m_1m_2}{w},\quad \epsilon_w=\frac{w^2-m_1{}^2-m_2{}^2}{2w},$$

and (vii) on-shell value of the relative momentum squared:

$$b^2(w)=\epsilon_w{}^2-m_w{}^2=\epsilon_1{}^2-m_1{}^2=\epsilon_2{}^2-m_2{}^2$$
$$=\frac{1}{4w^2}[w^4-2w^2(m_1{}^2+m_2{}^2)+(m_1{}^2-m_2{}^2)^2]\,.$$

We introduce scalar and vector interactions through constituent mass potentials and minimal substitutions:

$$M_i=m_i+S_i(x_\perp,p_1,p_2),\quad i=1,2\,, \tag{176}$$

$$\pi_1^\mu=p_1^\mu-A_1^\mu=E_1(x_\perp,p_1,p_2)\hat{P}^\mu+G(x_\perp,p_1,p_2)p^\mu\,, \tag{177}$$

$$\pi_2^\mu=p_2^\mu-A_2^\mu=E_2(x_\perp,p_1,p_2)\hat{P}^\mu-G(x_\perp,p_1,p_2)p^\mu\,, \tag{178}$$

where $\hat{P}=(p_1+p_2)/w$, $w^2=-(p_1+p_2)^2$ and

$$x_\perp^\mu=(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)x_\nu\,. \tag{179}$$

The vector potentials A_i^μ are divided into timelike and spacelike parts. The constituent energy potentials E_i are responsible for the timelike vector interactions while G is responsible for the recoil-dependent spacelike vector interactions. In the absence of interactions, $E_i\to\epsilon_i$ and $G\to 1$. Since there are only two independent parts of the vector potential, the three forms E_1, E_2, and G are not independent but related through

$$E_1{}^2(\mathcal{A},\mathcal{V})\approx G^2((\epsilon_1-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2)$$
$$=G^2(\epsilon_1-\mathcal{A}_1)^2\,, \tag{180}$$

$$E_2{}^2(\mathcal{A},\mathcal{V})\approx G^2((\epsilon_2-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2)$$
$$=G^2(\epsilon_2-\mathcal{A}_2)^2\,, \tag{181}$$

$$G^2=\frac{1}{1-2\mathcal{A}/w}\,. \tag{182}$$

Likewise, since there is only one independent scalar interaction the two forms M_1 and M_2 are related through

$$M_1{}^2(\mathcal{A},S)=m_1{}^2+G^2(2m_wS+S^2)=(m_1+S_1)^2\,, \tag{183}$$

$$M_2{}^2(\mathcal{A},S)=m_2{}^2+G^2(2m_wS+S^2)=(m_2+S_2)^2\,. \tag{184}$$

$\mathcal{A}$, $\mathcal{V}$, and S are invariants associated with electromagneticlike vector, timelike vector, and scalar potentials, respectively. Each is a function of $x_\perp{}^2$ and l^2 ($l=x_\perp\times p$). These particular invariants have the virtue of displaying the relation of our covariant formalism to the nonrelativistic and semirelativistic limits on the one hand and the appropriate field theory on the other.

Quantization of the compatible constraints,

$$\mathcal{H}_1=\pi_1{}^2+M_1{}^2\approx 0\,, \tag{185}$$

Let

$$\partial\Phi = x_\perp \Phi_{,1} + p_\perp \Phi_{,2} - p_\parallel \hat{P}\Phi_{,4} ,$$

$$\partial D = -x_\parallel \hat{P} D_{,1} - p_\parallel \hat{P} D_{,2} ,$$

$$\partial_p \Phi = x_\perp \Phi_{,2} + p_\perp \Phi_{,3} - \hat{P} x_\parallel \Phi_{,4} - \hat{P} p_\parallel \Phi_{,5} ,$$

$$\partial_p D = -x_\parallel \hat{P} D_{,2} - p_\parallel \hat{P} D_{,3} .$$

The comma notation means derivative with respect to the argument number. The compatibility condition is

$$\{\mathcal{H}_1,\mathcal{H}_2\} = 2wp_\parallel \Phi_{,4} + 2x_\parallel p_\parallel D_{,1} + 2p_\parallel{}^2 D_{,2} - x_\parallel{}^2 D_{,1}\Phi_{,4} - x_\parallel p_\parallel D_{,4}\Phi_{,5}$$
$$- x_\parallel p_\parallel D_{,2}\Phi_{,4} - p_\parallel{}^2 D_{,2}\Phi_{,5} + x_\parallel p_\parallel D_{,2}\Phi_{,4} + p_\parallel{}^2 \Phi_{,4} D_{,3}$$
$$= \Phi_{,4}(2wp_\parallel - x_\parallel{}^2 D_{,1} + p_\parallel{}^2 D_{,3}) + x_\parallel p_\parallel D_{,1}(2-\Phi_{,5}) + D_{,2}(2p_\parallel{}^2 - p_\parallel{}^2 \Phi_{,5}) .$$

The simplest solution is $\Phi_{,4}=0$ and $\Phi_{,5}=2$, and (A8) becomes

$$\Phi = \Phi\left[\frac{x_\perp{}^2}{2}, x_\perp\cdot p_\perp, \frac{p_\perp{}^2}{2}, P^2\right] + p_\parallel{}^2 \qquad (A10)$$

and

$$D = D\left[\frac{x_\parallel{}^2}{2}, x_\parallel p_\parallel, p_\parallel{}^2, P^2\right] . \qquad (A11)$$

Thus, using $p_1=\epsilon_1\hat{P}+p$, $p_2=\epsilon_2\hat{P}-p$, and differencing the constraints leads to

$$2P\cdot p + D + (\epsilon_2-\epsilon_1)w + m_1{}^2 - m_2{}^2 \approx 0 .$$

Choose

$$\epsilon_1-\epsilon_2 = \frac{m_1{}^2-m_2{}^2}{w}$$

leading to

$$2P\cdot p + D \approx 0 . \qquad (A12)$$

Note that ϵ_1 and ϵ_2 can be interpreted as c.m. constituent energies only if $D=0$, that is,

$$-p_i\cdot\hat{P} = \epsilon_1 - p\cdot\hat{P} \approx \epsilon_1 + \frac{D}{2} ,$$

$$-p_2\cdot\hat{P} = \epsilon_2 + p\cdot\hat{P} \approx \epsilon_2 - \frac{D}{2} .$$

Note also that

$$\mathcal{H}_1 = p^2 - \epsilon_1{}^2 + m_1{}^2 + \Phi + p_\parallel{}^2 + \frac{\epsilon_1}{w}(2P\cdot p + D) \approx 0 ,$$
$$\mathcal{H}_2 = p^2 - \epsilon_2{}^2 + m_2{}^2 + \Phi + p_\parallel{}^2 - \frac{\epsilon_2}{w}(2P\cdot p + D) \approx 0 , \qquad (A13)$$

leading to

$$\mathcal{H} \equiv \frac{\epsilon_2}{w}\mathcal{H}_1 + \frac{\epsilon_1}{w}\mathcal{H}_2 = p_\perp{}^2 - b^2(w) + \Phi \approx 0 , \qquad (A14)$$

as given in (14) or (20), since

$$p^2 + p_\parallel{}^2 = p_\perp{}^2 .$$

APPENDIX B: DIAGONALIZATION OF THE DOUBLY ODD TERMS IN Φ_{DO}

In computing $\mathcal{H}_i \equiv [\mathcal{S}_i, \mathcal{S}_i]_+$, one finds diagonal forms (such as Φ_{SI} and bits and pieces of Φ_{SS}, Φ_{SO}, and Φ_T), singly odd terms, namely,

$$\left[i\frac{\nabla(E_1\gamma_1^0+M_1)}{G}\cdot\gamma_1 - i\frac{\nabla(E_2\gamma_2^0+M_2)}{G}\cdot\gamma_2 \right] , \qquad (B1)$$

and doubly odd terms. Using $\mathcal{S}_1\psi=0$ to reduce the first singly odd term and $\mathcal{S}_2\psi=0$ to reduce the second singly odd term, produces more diagonal pieces [bringing Φ_{SS}, Φ_{SO}, and Φ_T to their final forms below (172) after an appropriate scale transformation that transforms the $\mathbf{r}\cdot\mathbf{p}$ Darwin terms to momentum-independent forms] and more doubly odd terms. The doubly odd terms can be written as

$$[\ \ ]\gamma_{51}\gamma_1^0\gamma_{52}\gamma_2^0\psi , \qquad (B2)$$

where [ ] is the square brackets of terms that multiply

$$(-)^s(\Phi_{SI}-b^2)/(2\mathcal{X}_1\mathcal{X}_2)$$

in Φ_{DO}.

In this appendix, we shall show how the latter factor arises. In particular, we show that

$$\psi = [(-)^s(\Phi_{SI}-b^2)/(\bar{\mathcal{X}}_1\bar{\mathcal{X}}_2)]\gamma_{51}\gamma_1^0\gamma_{52}\gamma_2^0\psi . \qquad (B3)$$

To demonstrate this, we write our two Dirac equations in the forms

$$\bar{\mathcal{X}}_1\psi = (\gamma_1\cdot\mathcal{P}_2 - i\gamma_2\cdot\mathcal{B})\psi , \qquad (B4a)$$

$$\bar{\mathcal{X}}_2\psi = (-\gamma_2\cdot\mathcal{P}_1 + i\gamma_1\cdot\mathcal{B})\psi , \qquad (B4b)$$

where

$$\bar{\mathcal{X}}_i = (E_i\gamma_i^0 - M_i)/G , \qquad (B5)$$

$$\mathcal{P}_i = \mathbf{p} + \frac{1}{2i}\nabla \ln G + \tfrac{1}{2}\boldsymbol{\sigma}_i \times \nabla \ln G \qquad (B6)$$

and

$$\mathcal{B}=\tfrac{1}{2}(\mathcal{M}-\mathcal{E}) \ . \tag{B7}$$

We combine the two Dirac equations (B4a) and (B4b) into the form

$$0=(\bar{\chi}_2\gamma_1\cdot\mathcal{P}_i-\chi_1\gamma_1\cdot\mathcal{B})\psi+(\bar{\chi}_2\gamma_2\cdot\mathcal{P}_1-i\bar{\chi}_2\gamma_2\cdot\mathcal{B})\psi \ . \tag{B8}$$

Bringing $\bar{\chi}_i$ to the right of $\mathcal{P}_i$ and using (B4a) and (B4b) in those parts, leads to (using $\boldsymbol{\gamma}=\gamma^0\gamma_5\boldsymbol{\sigma}$)

$$\left[[\sigma_2\cdot\mathcal{P}_1,\sigma_2\cdot\mathcal{P}_1]+i\nabla\ln G\cdot\sigma_1\sigma_2\cdot\mathcal{P}_1-i\nabla\ln G\cdot\sigma_2\sigma_1\cdot\mathcal{P}_2+i\frac{\chi_1}{\bar{\chi}_2}\sigma_1\cdot\mathcal{B}\sigma_2\cdot\mathcal{P}_1-i\frac{\chi_2}{\bar{\chi}_1}\sigma_2\cdot\mathcal{B}\sigma_1\cdot\mathcal{P}_2\right]\gamma_{51}\gamma_1^0\gamma_{52}\gamma_2^0\psi$$
$$=\left[+i\sigma_1\cdot\mathcal{P}_2\sigma_1\cdot\mathcal{B}-i\sigma_2\cdot\mathcal{P}_1\sigma_2\cdot\mathcal{B}-\left[\frac{\chi_1}{\bar{\chi}_2}-\frac{\chi_2}{\bar{\chi}_1}\right]\mathcal{B}^2\right]\psi \ ,$$

where

$$\chi_i=(E_i\gamma_i^0+M_i)/G \ . \tag{B9}$$

One finds that the commutator term on the left-hand side gives a result that cancels with the next two terms. The last term on the right-hand side vanishes since $\chi_1\bar{\chi}_1=\chi_2\bar{\chi}_2$. Multiplying both sides by $\bar{\chi}_1\bar{\chi}_2$ and simplifying leads to

$$-\frac{\Phi_{\rm SI}-b^2}{\bar{\chi}_1\bar{\chi}_2}i\mathbf{L}\cdot(\sigma_1\times\sigma_2)\gamma_{51}\gamma_1^0\gamma_{52}\gamma_2^0\psi=\mathbf{L}\cdot(\sigma_1-\sigma_2)\psi \ . \tag{B10}$$

Multiply both sides of (B10) by $\mathbf{L}\cdot(\sigma_1-\sigma_2)$ and use

$$[\mathbf{L}\cdot(\sigma_1-\sigma_2)]^2=4\mathbf{L}^2-2\mathbf{L}\cdot(\sigma_1+\sigma_2)-[\mathbf{L}\cdot(\sigma_1+\sigma_2)]^2\equiv\mathcal{O}_a \tag{B11}$$

and

$$\mathbf{L}\cdot(\sigma_1-\sigma_2)i\mathbf{L}\cdot(\sigma_1\times\sigma_2)=-\mathcal{O}_a+2\mathbf{L}^2(1-\sigma_1\cdot\sigma_2)\equiv\mathcal{O}_b \ . \tag{B12}$$

For singlet states, $\sigma_1\cdot\sigma_2=-3$ and $\mathcal{O}_a=\mathcal{O}_b$. For triplet states, $\sigma_1\cdot\sigma_2=+1$ and $\mathcal{O}_a=-\mathcal{O}_b$. Thus,

$$\mathcal{O}_a\psi=-\frac{(-)^s\mathcal{O}_a}{\bar{\chi}_1\bar{\chi}_2}(\Phi_{\rm SI}-b^2)\gamma_{51}\gamma_{52}\gamma_1^0\gamma_2^0\psi \ , \tag{B13}$$

for which Eq. (B3) is a solution.

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will be presented in a separate publication.

[30]One can derive from the constraint approach a relativistic Lippmann-Schwinger equation of the form $T+\Phi+\Phi G T=0$ that automatically satisfies two-body elastic unitarity. In ordinary quantum mechanics this would give the scattering amplitude T from Φ. In the context of the constraint approach, Todorov's inhomogeneous quasipotential equation is a postulate that instead relates the quasipotential Φ to the off-mass-shell field-theoretic scattering amplitude T (so that two-body elastic unitarity is automatically satisfied). In work related to this, Horwitz and Rohrlich have shown that the two constraint equations $(p_i{}^2+m_i{}^2+\Phi)\psi=0$ can combine to give an equation of the form of the Nambu–Schwinger–Bethe-Salpeter (Refs. 55–57) equation

$$[(p_1{}^2+m_1{}^2)(p_2{}^2+m_2{}^2)+\mathcal{V}]\psi=0 \ .$$

What they found, however, is that $\mathcal{V}$, unlike the irreducible Bethe-Salpeter kernel, permits only elastic interactions, with the individual particle energies conserved in the c.m. frame. See L. P. Horwitz and F. Rohrlich, Phys. Rev. D **31**, 932 (1985).

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$$\mathcal{G}\equiv\delta(P\cdot p)/[p^2-b^2(w)-i0] \ .$$

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[33]As we shall see, the ϵ_i's can be interpreted as the c.m. energies of the constituent particles and are given explicitly in Eq. (18).

[34]The form of the variable m_w is related to the proper time of the effective particle of relative motion. For free particles, $p_1=m_1dx_1 r\, d\tau_1$, $p_2=m_2dx_2/d\tau$. Let $d\tau$ be the proper-time increment in the c.m. frame, so that $d\tau/d\tau_1=\epsilon_1/m_1$, $d\tau/d\tau_2=\epsilon_2/m_2$. Then

$$p\equiv\frac{\epsilon_2p_1-\epsilon_1p_2}{w}=\frac{\epsilon_1\epsilon_2}{w}\frac{d(x_1-x_2)}{d\tau} \ .$$

Let $d\tau_e$ be a proper-time increment of the effective particle of relative motion defined so that $d\tau_e \underset{m_2\to\infty}{\longrightarrow} d\tau_1$, $d\tau_e \underset{m_1\to\infty}{\longrightarrow} d\tau_2$ and

$$p=m_w\frac{d(x_1-x_2)}{d\tau_e} \ .$$

Then $d\tau/d\tau_e=\epsilon_1\epsilon_2/m_1m_2$, which has the proper limiting form for either static limit.

[35]The reader who wishes a short summary of the various interaction forms should turn to Sec. VIII.

[36]The same nonrelativistic identification gives the covariant but weak potential form that appears in the Todorov equation, i.e., $2\epsilon_w\mathcal{A}-\mathcal{A}^2$. A possible example that differs from the choice $\mathcal{F}(\mathcal{A})=\mathcal{A}$ is

$$\mathcal{F}(\mathcal{A})=\mathcal{A}/[1+(\mathcal{A}/w)^2] \ .$$

This would give the same $O(1/c^2)$ dynamics as the former choice and would satisfy $2\epsilon_w\mathcal{F}-\mathcal{F}^2=2\epsilon_w\mathcal{A}-\mathcal{A}^2+O(\mathcal{A}^3)$ as well giving the same covariant but weak potential form as in the Todorov equation. Such a different structure for $\mathcal{F}$ may be important in scattering problems where $\mathcal{A}$ is positive. The latter choice of $\mathcal{F}$ would not have an additional singularity for finite r, as in the case for $\mathcal{A}=\alpha/r$, whereas the former choice would. The authors would like to thank R. L. Becker and C. Y. Wong for helpful discussions on this point.

[37]It would be unacceptable in the sense that such terms do not appear in perturbative relativistic field theory, both quantum and classical (e.g., Feynman-Wheeler dynamics).

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$$[(\nabla^2+m_1{}^2)^{1/2}+(\nabla^2+m_2{}^2)^{1/2}+V-w]\psi=0 \ ,$$

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[46]The form(s) on the right are equivalent to the original $\mathcal{H}_i$ only when we restrict ourselves to the hypersurface $\hat{P}\cdot p=0$. These forms are not the original generators, but become equivalent to one another when they are regarded as quantized restrictions on a wave function that already has been subjected to the condition $\hat{P}\cdot p\psi=0$.

[47]Since the equalities in (116) and (117) are weak, we cannot use the Jacobi identity to infer $\{\mathcal{G}_{1M},\mathcal{H}_1\}\approx0$ or $\{\mathcal{G}_{1E},\mathcal{H}_1\}\approx0$.

[48]The situation here is analogous to the partial rotational invariance of the parts of a potential-energy term such as

$$k_1(x^2+z^2)/2+k_2(x^2+y^2)/2 \ .$$

These two terms are separately invariant under rotations generated by L_y and L_z; noncommuting generators. There is no overall rotational symmetry.

[49]If, however, we regard $\mathcal{S}_1$ as a generator of supersymmetry transformations, then, since $\mathcal{S}_1$ has zero brackets with both $\mathcal{H}_1$ and $\mathcal{H}_2$, the system is supersymmetric.

[50]The quantum-mechanical proof of consistency turns out to be strictly isomorphic to its classical counterpart because of (1) the isomorphism between classical and quantum brackets

and (2) the nonappearance of terms proportional to the squares of Grassmann variables.

[51]Note that in this case we do not have strong compatibility between $\mathscr{S}_1$ and $\mathscr{S}_2$ and, therefore, we cannot use the Jacobi condition to show that $\{\mathscr{H}_1,\mathscr{H}_2\}\approx 0$, $\{\mathscr{S}_i,\mathscr{H}_j\}\approx 0$. However, these brackets still vanish, since $\mathscr{H}_1-\mathscr{H}_2 \sim P\cdot p$ [as in (107) and (108)].

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